Matching van Stockum dust to Papapetrou vacuum

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In commemoration of prof. Jan Horský (1940–2012)

The object of study

= possible shapes of relativistic dust clouds in vacuum

= also known as **matching** dust to vacuum.

Main results

– We show that every van Stockum dust metric can be matched (in the sense of Lichnerowicz) to a 1-parameter family of non-static Papapetrou vacuum metrics, and the converse.

– We obtain dust clouds with prescribed boundaries, including toroidal ones.

– Interpretation of Lichnerowicz matching conditions as first-order contact conditions.

– The role of symmetries and invariants.

– Illustrative examples.

Motivation I

Matching rotating dust to vacuum is a respected problem open for decades:

W.B. Bonnor, Globally regular solutions of Einstein's equations, *Gen. Rel. Grav.* **14** (1982) 807–821.

S. Viaggiu, Rigidly rotating dust solutions depending upon harmonic functions, *Class. Quantum Grav.* **24** (2007) (10) 2755–2760.

T. Zingg, A. Aste and D. Trautmann, Just dust: About the (in)applicability of rotating dust solutions as realistic galaxy models, *Adv. Stud. Theor. Phys.* **1** (2007) 409–432.

N. Gürlebeck, The interior solution of axially symmetric, stationary and rigidly rotating dust configurations, *Gen. Rel. Grav.* **41** (2009) 2687–2696.

Motivation II

A relation between two historically significant classes of metrics.

W.J. van Stockum (1937), The gravitational field of a distribution of particles rotating about an axis of symmetry, *Proc. Roy. Soc. Edinburgh* **57** 135–154.

A. Papapetrou (1953), Eine rotationssymmetrische Lösung in der allgemeinen Relativitätstheorie, *Annalen der Physik* **447** 309–315.

Relevance

Dust clouds have been considered as models of galaxies, galaxy clusters, etc.

Controversy about Papapetrou metrics

Asymptotically flat Papapetrou metrics require a zero-mass source.

J.N. Islam, *Rotating Fields in General Relativity* (Cambridge Univ. Press, Cambridge, 1985) § 2.5.

Yet they may be relevant as limit cases.

Controversy about van Stockum metrics

1) Time machine (closed time-like curves).

2) No global asymptotically flat van Stockum dust solution exist.

A. Caporali, Non-existence of stationary, axially symmetric, asymptotically flat solutions of the Einstein equations for dust, *Phys. Lett. A* **66** (1978) 5–7.

H. Pfister, Do rotating dust stars exist in general relativity?, *Class. Quantum Grav.* **27** (2010) 105016.

Why van Stockum and Papapetrou metrics?

A source of a Papaetrou metric must have a zero mass.

The van Stockum metrics have this property.

Ł. Bratek, J. Jałocha and M. Kutschera, Van Stockum–Bonnor spacetimes of rigidly rotating dust, *Phys. Rev. D* **75** (2007) 107502.

Dust's positive mass is balanced by a negative mass in the singularity.

Negative masses

Google Scholar returns more than 5000 documents containing "negative mass" and "general relativity."

R.L. Forward, Observational search for negative matter in intergalactic voids, in: NASA Breakthrough Propulsion Physics Workshop Proceedings, NASA/CP-1999-208694, 201–203.

Negative masses are taken seriously nowadays.

Known examples of rotating dust–vacuum glued metrics

1937 – rigidly rotating dust cylinder by Lanczos and van Stockum;

1977 – differentially rotating analogue by Vishveshwara and Winicour;

1979 – the constant determinant case by Hoenselaers and Vishveshwara;

1998 – the vacuum exterior to the Gödel metric by Bonnor, Santos and MacCallum;

2003 – the dust interior to the NUT metric by Zsigrai (the dust part due to Lukács, Newman, Sparling and Winicour);

2009 – the vacuum exterior to differentially rotating Maitra's dust cylinder by Bonnor and Steadman.

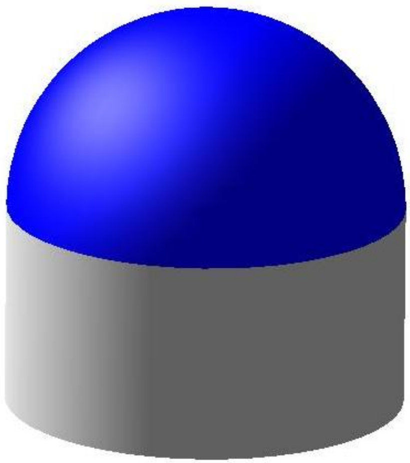
All six possess at least three Killing vectors.

We deal with metrics possessing two Killing vectors.

The matching problem

Given two smooth metrics, glue them along a hypersurface to produce a metric of continuity class C^1.

Example in dimension two



Sphere matched to a flat metric (circular cylinder) along a circle.
The curvature jumps at the boundary.
Hence, cannot be C^2.

For relativistic dust, scalar curvature equals density and jumps at the boundary. Hence, cannot be C^2.

Main difficulty

A boundary problem with free (unknown) boundary.

Various approaches

G. Darmois, Les équations de la gravitation einsteinienne, *Mémorial des sciences mathématiques* XXV (1927).

…the first and second fundamental forms of the boundary hypersurface coincide.

S. O'Brien and J.L. Synge, Jump conditions at discontinuities in general relativity, *Communications of the Dublin Institute for Advanced Studies A* **1952** (1952) (9) (pp 20).

A. Lichnerowicz, *Théories relativistes de la gravitation et de l'électromagnétisme* (Masson, Paris, 1955).

…equality of $\leq$ 1-order derivatives across the boundary.

L. Bel and A. Hamoui, Les conditions de raccordement en relativité générale, *Ann. Inst. H. Poincaré A: Phys. Théor.* **7** (1967) 229–244.

We follow the Lichnerowicz approach.
Requires shared (admissible) coordinates.

C^k-continuity and gluing

The pasting lemma. Two C^k-continuous functions $f^{(\mathrm{I})}$ and $f^{(\mathrm{II})}$, glued along a boundary B, produce a C^k-continuous function, if and only if

$$f^{(\mathrm{I})}_{,i_1\cdots i_l}|_B = f^{(\mathrm{II})}_{,i_1\cdots i_l}|_B \quad \text{for all } i_1,\ldots,i_l \text{ and all } 0 \le l \le k.$$

C^k-continuity and kth-order contact

The condition is equivalent to saying that $f^{(\mathrm{I})}$ and $f^{(\mathrm{II})}$ have a contact of order k along B.

We write $f^{(\mathrm{I})} \equiv^k_B f^{(\mathrm{II})}$.

The Lichnerowicz matching condition

There exist coordinates such that

$$g^{(\mathrm{I})}_{ij} \equiv^1_B g^{(\mathrm{II})}_{ij}.$$

Congruence property of $\equiv_B^k$

A folklore in jet theory (a k-jet prolongation of a C^k-function).

Proposition. $\equiv_B^k$ is a congruence of the algebra of C^k-continuous functions with C^k-continuous operations.

Proof. Let $f_1^{(\mathrm{I})} \equiv_B^k f_1^{(\mathrm{II})} \ldots, f_m^{(\mathrm{I})} \equiv_B^k f_m^{(\mathrm{II})}$. Let $F(f_1, \ldots, f_m)$ be a C^k-continuous function in a neighbourhood of the image $f_1^{(\mathrm{I})}B \times \cdots \times f_m^{(\mathrm{I})}B = f_1^{(\mathrm{II})}B \times \cdots \times f_m^{(\mathrm{II})}B$.
Then

$$F(f_1^{(\mathrm{I})}, \ldots, f_m^{(\mathrm{I})}) \equiv_B^k F(f_1^{(\mathrm{II})}, \ldots, f_m^{(\mathrm{II})})$$

by the chain rule and induction. Q.E.D.

I. Kolář, P.W. Michor and J. Slovák, *Natural Operations in Differential Geometry* (Springer, Berlin, 1993).

Shared coordinates

Both van Stockum and Papapetrou metrics admit two commuting and orthogonally transitive Killing vectors, one time-like and one space-like.

Both can be written in the Lewis–Papapetrou form

$$\mathbf{g} = g_{ij}(t^1, t^2)\,\mathrm{d}t^i\,\mathrm{d}t^j + h_{kl}(t^1, t^2)\,\mathrm{d}z^k\,\mathrm{d}z^l,$$
$$i, j = 1, 2, \quad k, l = 1, 2.$$

The Killing vectors are $\xi_{(i)} = \partial/\partial z^i$ and linear combinations.

The first summand $g_{ij}\,\mathrm{d}t^i\,\mathrm{d}t^j$ can be identified with the *orbit metric* = metric on the orbit space (= the space of the Killing orbits).

$h_{kl} = \mathbf{g}(\xi_{(k)}, \xi_{(l)})$ are functions on the orbit space.

Since one of the Killing vectors is time-like, $\det h < 0$ and $\det g > 0$. Hence, the orbit metric is Riemannian.

Transformations

Metrics $\mathbf{g} = g_{ij}(t^1, t^2)\,\mathrm{d}t^i\,\mathrm{d}t^j + h_{kl}(t^1, t^2)\,\mathrm{d}z^k\,\mathrm{d}z^l$ are preserved under

$$\bar{t}^i = \Phi^i(t^1, t^2), \qquad \bar{z}^k = A^k_l\, z^l.$$

$\Phi^i(t^1, t^2)$ are local coordinate transformations in the orbit space, $A = (A^m_n) \in \mathrm{GL}_2$ are constant matrices acting by

$$\bar{h}_{kl} = A^m_k h_{mn} A^n_l$$

An action of GL_2 on the Killing vectors.

Symmetry reduction

The boundary hypersurface projects to a curve in the two-dimensional orbit space.
The problem reduces to finding that curve.

First-order invariants

M. Marvan and O. Stolín, On local equivalence problem of spacetimes with two orthogonally transitive commuting Killing fields, *J. Math. Phys.* **49** (2008) 022503 (pp 17).

Lewis–Papapetrou metrics possess four first-order scalar invariants, preserved under admissible coordinate transformations.

Given a Lewis–Papapetrou metric

$$\mathbf{g} = \mathrm{e}^{p}\,(\mathrm{d}x^2 + \mathrm{d}y^2) + h_{kl}\,\mathrm{d}z^k\,\mathrm{d}z^l,$$

the scalar invariant $Q_\chi(\mathbf{g})$ is defined to be

$$Q_\chi(\mathbf{g}) = \frac{\det\chi}{\mathrm{e}^{2p}}, \quad \chi = \frac{1}{\det h}\begin{vmatrix} \mathrm{d}h_{11} & \mathrm{d}h_{12} \\ \mathrm{d}h_{21} & \mathrm{d}h_{22} \end{vmatrix}.$$

Isothermal coordinates I

Constructing shared coordinates in the orbit space.

The orbit space being two-dimensional, the Korn–Lichtenstein theorem ensures the existence of local isothermal coordinates under the assumption of the Hölder $C^{0,\alpha}$-continuity of degree $0 < \alpha \leq 1$ (Lipschitz continuity when $\alpha = 1$).

A. Korn, Zwei Anwendungen der Methode der sukzessiven Annäherungen, *Mathematische Abhandlungen Hermann Amandus Schwarz* (1914) 215–229.

L. Lichtenstein, Beweis des Satzes, daß jedes hinreichend kleine, im wesentlichen stetig gekrümmte, singularitätenfreie Flächenstück auf einen Teil einer Ebene zusammenhängend und in den kleinsten Teilen ähnlich abgebildet werden kann, *Berl. Abh.* (1911) 1–49.

Shiing-Shen Chern, An elementary proof of the existence of isothermal parameters on a surface. *Proc. Amer. Math. Soc.* **6** (1955) 771–782.

Isothermal coordinates II

The C^1-continuity required by the Lichnerowicz conditions is stronger than the Hölder $C^{0,\alpha}$-continuity required by the Korn–Lichtenstein theorem.

Consequently, the glued orbit space admits local isothermal coordinates x, y.

Thus, the glued orbit metric can be written as

$$\mathrm{e}^{p(x,y)}\,(\mathrm{d}x^2 + \mathrm{d}y^2).$$

Consequently, the glued space-time metric can be written locally in the form

$$\mathbf{g} = \mathrm{e}^{p(x,y)}\,(\mathrm{d}x^2 + \mathrm{d}y^2) + h_{kl}(x,y)\,\mathrm{d}z^k\,\mathrm{d}z^l.$$

Matching vacuum to van Stockum dust

Proposition. All van Stockum metrics $\mathbf{g}$ satisfy

$$Q_\chi(\mathbf{g}) = 0.$$

Proof. Since $h_{22} = -1 = \text{const}$, we have $\chi = (\mathrm{d}h_{12})^2/\det h$ and then $\det \chi = 0$.

Corollary. If a vacuum Lewis–Papapetrou metric $\mathbf{g}$ matches a van Stockum metric, then $Q_\chi(\mathbf{g}) = 0$ on the boundary.

Proof. Being a first-order invariant, $Q_\chi(\mathbf{g})$ is continuous for every C^1-metric $\mathbf{g}$.

Matching conditions in isothermal coordinates

There is a remaining freedom to transform z^1, z^2
$=$ the GL_2-action on the Killing vectors.

In shared isothermal coordinates, the matching conditions for two Lewis–Papapetrou metrics

$$\mathbf{g}^{(\mathrm{I})} = \mathrm{e}^{p^{(\mathrm{I})}}(\mathrm{d}x^2 + \mathrm{d}y^2) + h^{(\mathrm{I})}_{kl}\,\mathrm{d}z^{(\mathrm{I})k}\,\mathrm{d}z^{(\mathrm{I})l},$$

$$\mathbf{g}^{(\mathrm{II})} = \mathrm{e}^{p^{(\mathrm{II})}}(\mathrm{d}x^2 + \mathrm{d}y^2) + h^{(\mathrm{II})}_{kl}\,\mathrm{d}z^{(\mathrm{II})k}\,\mathrm{d}z^{(\mathrm{II})l}$$

are

$$p^{(\mathrm{I})} \equiv^1_B p^{(\mathrm{II})}, \quad h^{(\mathrm{I})}_{kl} \equiv^1_B A^m_k h^{(\mathrm{II})}_{mn} A^n_l.$$

Here $A \in \mathrm{GL}_2$ is an unknown constant matrix.

Locating the boundary

$$Q^{\mathrm{v}}_{\chi}|_B = 0.$$

The van Stockum dust part I

The energy-momentum tensor for dust is $\mathbf{T}^{ab} = \mu^{\mathrm{d}} \mathbf{U}^a \mathbf{U}^b$.

$\mathbf{U}^a$ is the 4-velocity and μ^{d} is the dust density.

Van Stockum's dust is isometrically flowing, meaning that the 4-velocity $\mathbf{U}^a$ is a Killing vector.

Choose $z^1 = \phi$, $z^2 = t$, in such a way that $\mathbf{U}^a = \partial_t$ (comoving coordinates).

Under a particular choice of the variables h_{kl}, the metric can be written in the form

$$\mathrm{e}^p \left(\mathrm{d}x^2 + \mathrm{d}y^2\right) + r^2 \, \mathrm{d}\phi^2 - (f \, \mathrm{d}\phi + \mathrm{d}t)^2.$$

The coefficient at $\mathrm{d}t^2$ equals $\mathbf{g}_{ab} \mathbf{U}^a \mathbf{U}^b$, which is -1 since $\mathbf{U}$ is a 4-velocity.

The van Stockum dust part II

In consequence of the vacuum Einstein equations $\mathbf{R}_{ab} - \frac{1}{2}\mathbf{R}g_{ab} = \mu^{\mathrm{d}}\mathbf{U}_a\mathbf{U}_b$, we have $\mathbf{R}^3{}_3 + \mathbf{R}^4{}_4 = 0$.

This implies $r_{xx} + r_{yy} = 0$ and then r is a harmonic function. If non-constant, r can serve as one of the isothermal coordinate functions (Weyl's canonical coordinates).

H. Weyl, Zur Gravitationstheorie, *Ann. Phys.* **54** (1917) 117–145.

Thus, we can set $r = x$.

With $r = x$, the Einstein equations reduce to

$$f_{xx} + f_{yy} - \frac{f_x}{x} = 0, \quad p_x = \frac{f_y^2 - f_x^2}{2x}, \quad p_y = -\frac{f_x f_y}{x},$$

whereas

$$\mu^{\mathrm{d}} = \frac{f_x^2 + f_y^2}{x^2 p}.$$

For $r = $ const, see Hoenselaers and Vishveshwara (*loc. cit.*).

The Papapetrou vacuum part I

Here we rederive the general Papapetrou solution.

Under a particular choice of the variables h_{kl}, the metric can be written as

$$\mathrm{e}^q \left(\mathrm{d}x^2 + \mathrm{d}y^2\right) + \frac{r^2}{v}\,\mathrm{d}\phi^2 - v(w\,\mathrm{d}\phi + \mathrm{d}t)^2$$

Einstein equations imply $\mathbf{R}^3{}_3 + \mathbf{R}^4{}_4 = 0$, giving $r_{xx} + r_{yy} = 0$, i.e., r is a harmonic function again.

We have $\det h = -x^2$ on the dust side and $\det h = -r^2$ on the vacuum side.

The requirement of the first-order contact amounts to the conditions $r = \pm x$, $r_x = \pm 1$, $r_y = 0$ at the boundary.

This leaves us with $r^2 = x^2$ everywhere.

Then the matrix A is restricted to lie in SL_2.

The Papapetrou vacuum part II

The vacuum Einstein equations for the metric

$$\mathrm{e}^{q}\,(\mathrm{d}x^2+\mathrm{d}y^2)+\frac{x^2}{v}\,\mathrm{d}\phi^2-v(w\,\mathrm{d}\phi+\mathrm{d}t)^2$$

are

$$\begin{aligned}
w_{xx}+w_{yy}-\frac{w_x}{x} &= -2\,\frac{v_x w_x+v_y w_y}{v},\\
v_{xx}+v_{yy}+\frac{v_x}{x} &= \frac{v_x^2+v_y^2}{v}-v^3\,\frac{w_x^2+w_y^2}{x^2},\\
q_x &= -\frac{v_x}{v}+\frac{x}{2v^2}(v_x^2-v_y^2)-\frac{v^2}{2x}(w_x^2-w_y^2),\\
q_y &= -\frac{v_y}{v}+\frac{x v_x v_y}{v^2}-\frac{v^2 w_x w_y}{x}.
\end{aligned}$$

The Papapetrou class is determined by

$$v_x w_x+v_y w_y=0.$$

For $w=$ const the metric is static and even flat.

The first integral

Assume $w_y \neq 0$ (results for $w_x \neq 0$ are the same). Denoting

$$c^2 = \frac{x^2 v_x^2}{v^2 w_y^2} + v^2 > 0,$$

one easily sees that $c = \text{const}$ in consequence of the field equations and the Papapetrou condition. Thus, a first integral.

Computing Lie symmetries, we obtain coordinate transformations

	w	v	c	q
$\mathcal{S}_1$	$\mathrm{e}^a w$	$\mathrm{e}^{-a} v$	$\mathrm{e}^{-a} c$	q
$\mathcal{S}_2$	$w + a$	v	c	q
$\mathcal{S}_3$	w	v	c	$\mathrm{e}^a q$

where a denotes the group parameter.

Using coordinate transformation $\mathcal{S}_1$, one can normalise c to 1.

Field equations for normalised Papapetrou metrics

Papapetrou metrics with $c = 1$ are said to be *normalised*.

By substituting

$$v = \frac{1}{\cosh u}, \quad e^q = \frac{e^s}{v} = e^s \cosh u,$$

we rewrite the Einstein equations in the form

$$w_{xx} + w_{yy} - \frac{w_x}{x},$$

$$u_x = -\frac{w_y}{x}, \quad u_y = \frac{w_x}{x}, \quad s_x = -\frac{w_x^2 - w_y^2}{2x}, \quad s_y = -\frac{w_x w_y}{x}.$$

Companion metrics I

Compare the field equations for the van Stockum dust,

$$f_{xx} + f_{yy} - \frac{f_x}{x} = 0, \quad p_x = \frac{f_y^2 - f_x^2}{2x}, \quad p_y = -\frac{f_x f_y}{x},$$

with the field equations for the normalised Papapetrou vacuum,

$$w_{xx} + w_{yy} - \frac{w_x}{x} = 0, \quad s_x = \frac{w_y^2 - w_x^2}{2x}, \quad s_y = -\frac{w_x w_y}{x},$$
$$u_x = -\frac{w_y}{x}, \quad u_y = \frac{w_x}{x}.$$

Definition. A van Stockum metric $\mathbf{g}^{\mathrm{d}}$ determined by field variables f, p and a normalised non-static Papapetrou metric $\mathbf{g}^{\mathrm{v}}$ determined by field variables w, s and u are called *companions* if $w = f$, $s = p$.

For companion metrics, the first three equations of each set are identical.

Companion metrics II

To every normalised non-static Papapetrou vacuum metric there corresponds a unique companion van Stockum dust metric.

To every van Stockum dust metric there corresponds a one-parameter family of companion normalised non-static Papapetrou vacuum metrics.

Indeed, u is determined up to an integration constant, which si the parameter.

The companion dust and vacuum metrics are

$$\mathbf{g}^{\mathrm{d}} = \mathrm{e}^{p}\,(\mathrm{d}x^2 + \mathrm{d}y^2) + x^2\,\mathrm{d}\phi^2 - (f\,\mathrm{d}\phi + \mathrm{d}t)^2$$

and

$$\mathbf{g}^{\mathrm{v}} = \mathrm{e}^{p}\cosh u\,(\mathrm{d}x^2 + \mathrm{d}y^2) + x^2\cosh u\,\mathrm{d}\phi^2 - \frac{(f\,\mathrm{d}\phi + \mathrm{d}t)^2}{\cosh u},$$

respectively.

Locating the boundary

Recall that

$$0 = Q^{\mathrm{v}}_{\chi}|_B = \frac{\mathrm{e}^p \sinh u}{x^4 \cosh^4 u} P(\sinh u, \cosh u, f_x, f_y).$$

Theorem. A dust metric and a normalised Papapetrou metric that are companions match along the boundary placed at $u = 0$.

Proof. The companion dust and vacuum metrics are

$$\mathbf{g}^{\mathrm{d}} = \mathrm{e}^p\,(\mathrm{d}x^2 + \mathrm{d}y^2) + x^2\,\mathrm{d}\phi^2 - (f\,\mathrm{d}\phi + \mathrm{d}t)^2$$

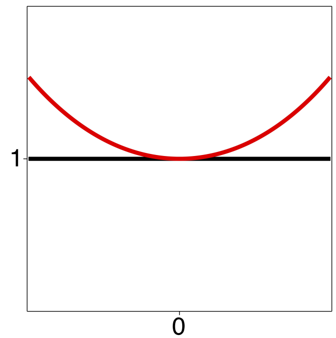
$$\mathbf{g}^{\mathrm{v}} = \mathrm{e}^p \cosh u\,(\mathrm{d}x^2 + \mathrm{d}y^2) + x^2 \cosh u\,\mathrm{d}\phi^2 - \frac{(f\,\mathrm{d}\phi + \mathrm{d}t)^2}{\cosh u},$$

respectively. Obviously,

$$\mathbf{g}^{\mathrm{d}}_{ij} \equiv^1_{\{u=0\}} \mathbf{g}^{\mathrm{v}}_{ij},$$

since

$$1 \equiv^1_{\{u=0\}} \cosh u.$$



Dust clouds of a given shape

Recall that companion metrics are determined by

$$w_{xx} + w_{yy} - \frac{w_x}{x} = 0, \quad s_x = \frac{w_y^2 - w_x^2}{2x}, \quad s_y = -\frac{w_x w_y}{x},$$
$$u_x = -\frac{w_y}{x}, \quad u_y = \frac{w_x}{x}.$$

Eliminating w, we obtains the equivalent system

$$u_{xx} + u_{yy} + \frac{u_x}{x} = 0,$$

$$w_x = x u_y, \quad w_y = -x u_x, \quad s_x = x\frac{u_x^2 - u_y^2}{2}, \quad s_y = x u_x u_y.$$

Thus, u satisfies the cylindrical Laplace equation.

Recall that admissible dust-vacuum boundaries are the levels of u.

Thus, admissible dust-vacuum boundaries are the levels of solutions of the cylindrical Laplace equation.

The electrostatic analogy

The following problems are equivalent:

– van Stockum–Papapetrou dust clouds of a given shape;

– solutions u of the cylindrical Laplace equation

$$u_{xx} + u_{yy} + \frac{u_x}{x} = 0$$

that are constant along a given boundary;

– axisymmetric solutions of the three-dimensional Laplace equation constant along a given axisymmetric boundary;

– axisymmetric electrostatic potentials with a prescribed axisymmetric equipotential surface.

Summarising, the problem can be reduced to a classical potential theory problem.

The corresponding field lines correspond to $f = \text{const}$.

Relation to static Weyl vacuum metrics I

The following proposition is easily verified.

Proposition. The companion dust and vacuum metrics are, respectively, the Ehlers and the Neugebauer–Kramer transform of the static Weyl vacuum metric

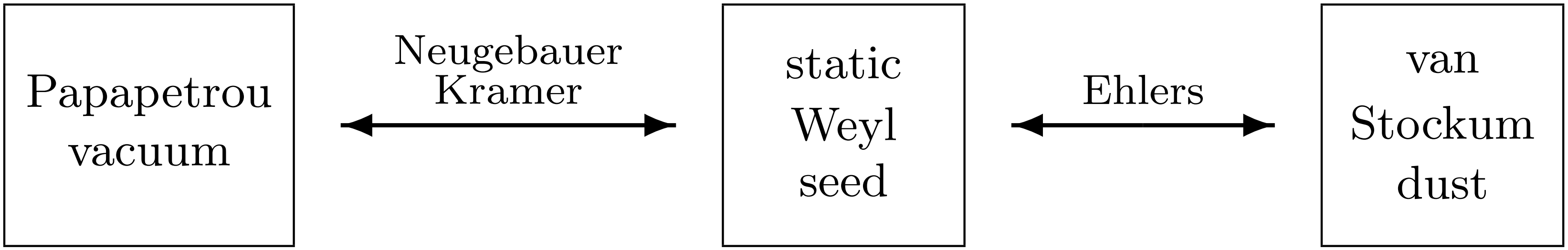
$$\mathrm{e}^{u+p}(\mathrm{d}x^2 + \mathrm{d}y^2) + x^2\mathrm{e}^u\,\mathrm{d}\phi^2 - \mathrm{e}^{-u}\,\mathrm{d}t^2.$$

D. Kramer and G. Neugebauer, Zu axialsymmetrischen stationären Lösungen der Einsteinschen Feldgleichungen für das Vakuum, *Commun. Math. Phys.* **10** (1968) 132–139.

J. Ehlers, Transformations of static exterior solutions of Einstein's gravitational field equations into different solutions by means of conformal mappings, in: *Les théories relativistes de la gravitation*, Proc. Conf. Royaumont, 1959 (Éditions du Centre National de la Recherche Scientifique, Paris, 1962) 275–284.

Relation to static Weyl vacuum metrics II

Summarising, the companion correspondence can be decomposed as



Every static Weyl metric

$$e^{u+p}(dx^2 + dy^2) + x^2 e^u \, d\phi^2 - e^{-u} \, dt^2$$

yields an explicit expression for u and p, sufficient to compute the dust density $\mu = (u_x^2 + u_y^2)/e^p$, curvature invariants, Petrov type, etc.

$$f = \int x u_y \, dx - x u_x \, dy.$$

A closed-form representation is not always available.

Example. The Halilsoy metric I

The electrostatic analogue is the **point charge**.

The static Weyl seed is the Chazy–Curzon metric.

J. Chazy, Sur la champ de gravitation de deux masses fixes dans la théorie de la relativité, *Bull. Soc. Math. France* **52** (1924) 17–38.

H.E.J. Curzon, Cylindrical solutions of Einstein's gravitational equations, *Proc. London Math. Soc.* **23** (1924) 477–480.

In Weyl's coordinates,

$$u = \frac{2}{\sqrt{x^2 + y^2}}, \quad f = \frac{2y}{\sqrt{x^2 + y^2}}, \quad p = -\frac{x^2}{(x^2 + y^2)^2},$$

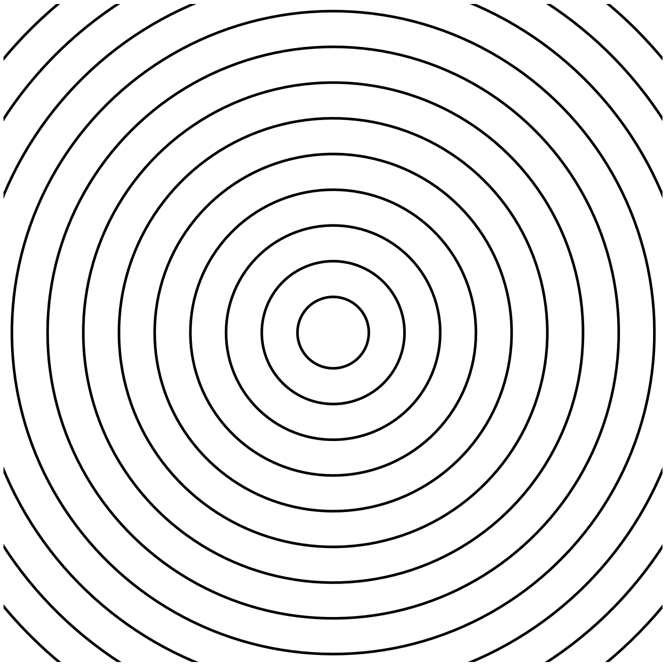
$$\mu = \frac{4}{(x^2 + y^2)^2 \, \mathrm{e}^p}.$$

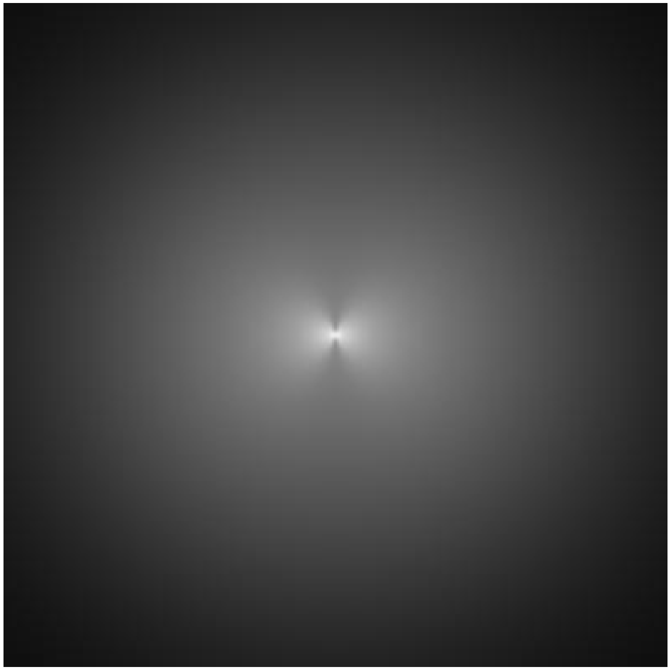
The vacuum part obtained earlier by Halilsoy.

M. Halilsoy, New metrics for spinning spheroids in general relativity, *J. Math. Phys.* **33** (1992) 4225–4230.

Example. The Halilsoy metric II

The boundaries $x^2 + y^2 = \text{const}$ and the dust density





The density blows up at the centre $(x, y) = (0, 0)$.

The curve $x = 0$ is not a regular axis,
but $x = 0$, $y < 0$ and $x = 0$, $y > 0$ are.

Example. The Bonnor metric I

The electrostatic analogue is the **dipole**.

The static Weyl seed is unphysical.

In Weyl's coordinates,

$$u = -\frac{2my}{(x^2+y^2)^{3/2}}, \quad f = \frac{2mx^2}{(x^2+y^2)^{3/2}}, \quad p = \frac{m^2x^2(x^2-8y^2)}{2(x^2+y^2)^4},$$
$$\mu = \frac{4m^2(x^2+4y^2)}{(x^2+y^2)^4 \mathrm{e}^p}.$$

Both the dust and vacuum parts have been studied by Bonnor.

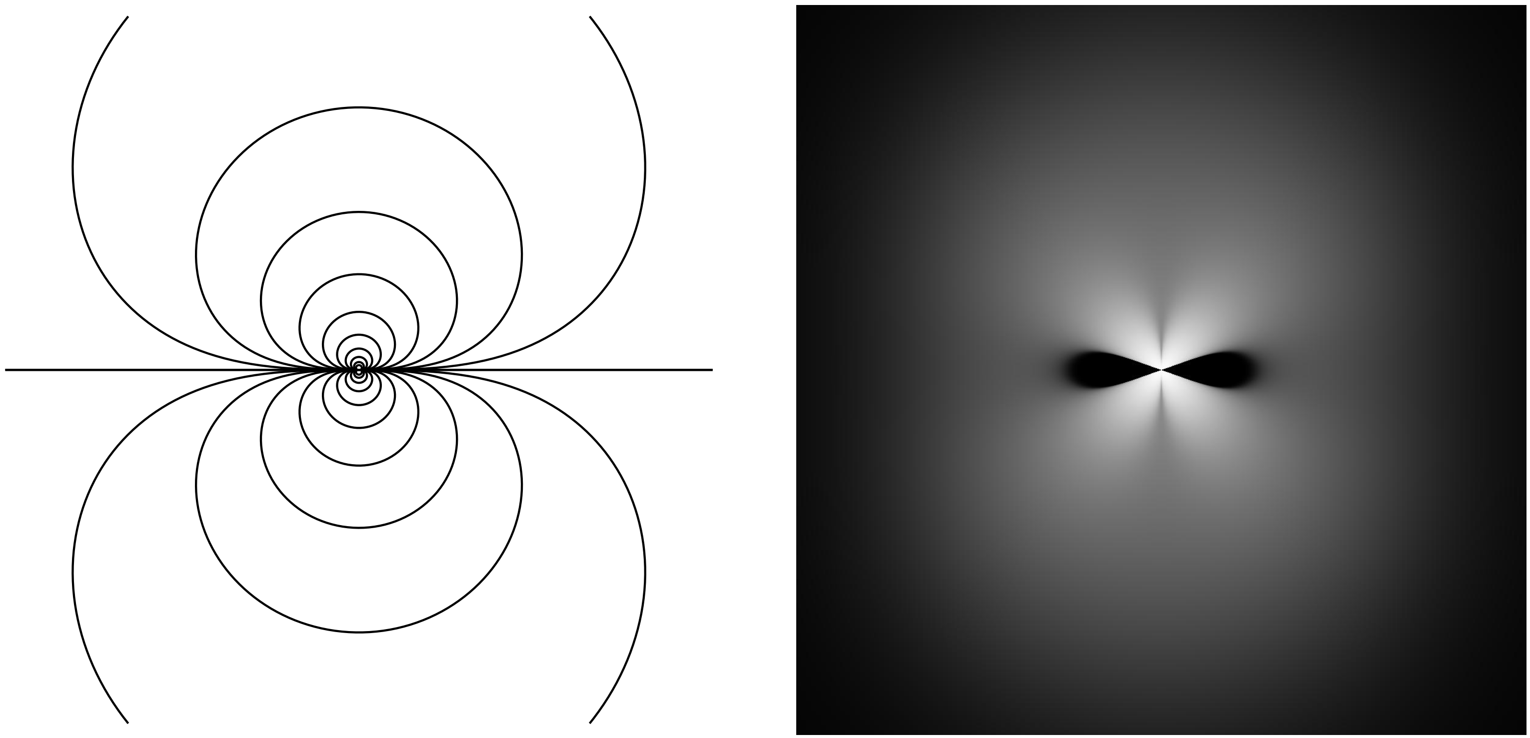
W.B. Bonnor, A rotating dust cloud in general relativity, *J. Phys. A. Math. Theor.* **10** (1977) 1673–1677.

W.B. Bonnor, An exact solution for a rotating body with negligible mass, *Gen. Rel. Grav.* **37** (2005) 1145–1149.

Matching unnoticed.

Example. The Bonnor metric II

Possible boundaries are $2my = u_0(x^2 + y^2)^{3/2}$, $u_0 \in (-\infty, \infty)$.



The density blows up at the centre $(x, y) = (0, 0)$.

Rotationally symmetric.

Example. The Morgan–Morgan disc I

Consider the static Morgan–Morgan disc solution.

T. Morgan and L. Morgan, The gravitational field of a disk, *Phys. Rev.* **183** (1969) 1097–1101.

P.S. Letelier and S.R. Oliveira, Superposition of Weyl solutions: the equilibrium forces, *Class. Quantum Grav.* **15** (1998) 421–433.

In Weyl coordinates,

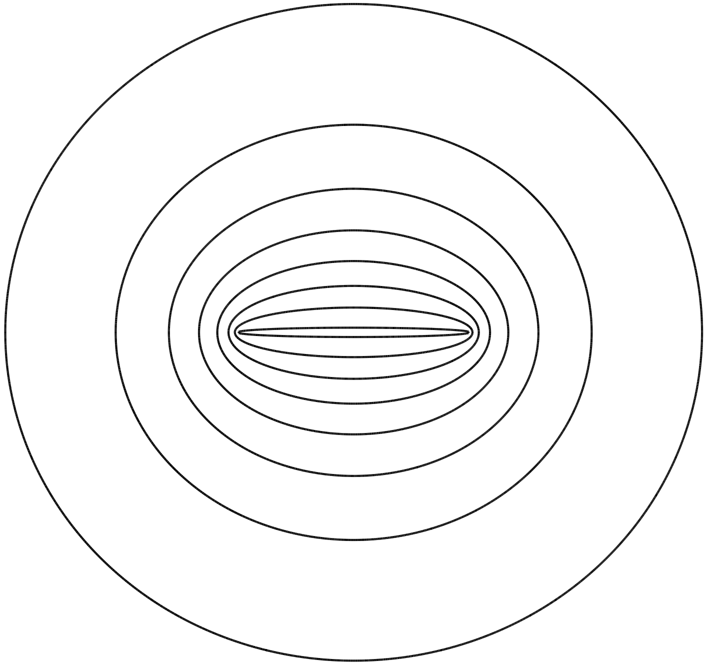
$$u = 2\arctan\left(\frac{\sqrt{2}\,a}{\sqrt{\sqrt{(x^2+y^2-a^2)^2+4a^2y^2}+x^2+y^2-a^2}}\right),$$

$$p = -\ln\left(1+\frac{x^2+y^2+a^2}{\sqrt{(x+a)^2+y^2}\sqrt{(x-a)^2+y^2}}\right),$$

$$\mu = \frac{8a^2}{\big((x+a)^2+y^2\big)\big((x-a)^2+y^2\big)}.$$

A closed-form representation for f is not available.

Example. The Morgan–Morgan disc II



The density blows up at the "ring" $(x, y) = (\pm a, 0)$.

Rotational symmetry is not confirmed.

Example. The Bach–Weyl ring I

Consider the static Bach–Weyl ring.

R. Bach and H. Weyl, Neue Lösungen der Einsteinschen Gravitationsgleichungen. B. Explizite Aufstellung statischer axialsymmetrischer Felder, *Math. Z.* **13** (1922) 134–145.

O. Semerák, Static axisymmetric rings in general relativity: How diverse they are, *Phys. Rev. D* **94** (2016) 104021.

In Weyl coordinates,

$$u = \frac{4mK(\Omega)}{\sqrt{(x+a)^2+y^2}}, \qquad \Omega = 2\sqrt{\frac{ax}{(x+a)^2+y^2}},$$

$$p = -\frac{m}{a^2}\left(\frac{x^2+y^2+3a^2}{(x+a)^2+y^2}\,K(\Omega)^2 - 2\,K(\Omega)\,E(\Omega) + \frac{x^2+y^2-a^2}{(x-a)^2+y^2}\,E(\Omega)^2\right),$$

where K, E denote the complete elliptic functions.

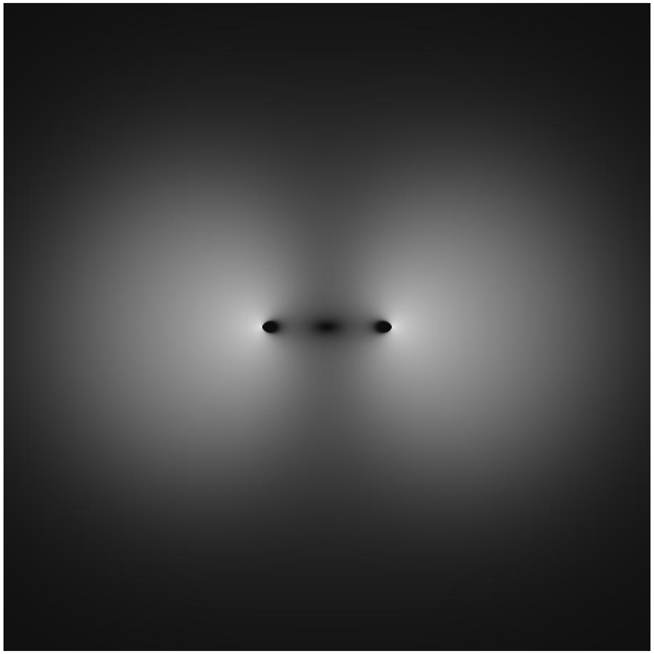
Example. The Bach–Weyl ring II

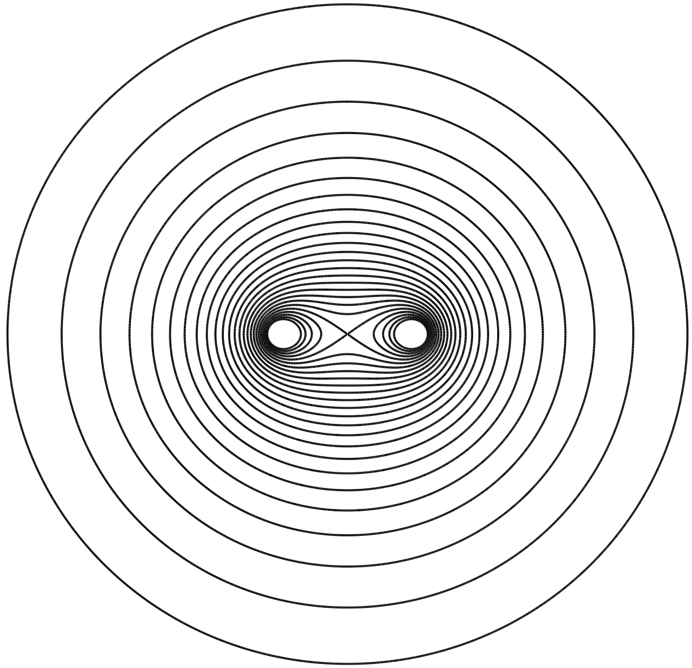
The density is

$$\mu = \frac{4m^2}{x^2\,\mathrm{e}^p}\left(\frac{K(\Omega)^2}{(x+a)^2+y^2} - 2\,(a^2-x^2+y^2)\,\frac{K(\Omega)}{(x+a)^2+y^2}\right.$$
$$\left.\times\,\frac{E(\Omega)}{(x-a)^2+y^2} + \frac{E(\Omega)^2}{(x-a)^2+y^2}\right).$$

A closed-form representation for f is not available.

Example. The Bach–Weyl ring III





The density blows up at the "ring" $(x, y) = (\pm a, 0)$.

Rotational symmetry is not confirmed.

Example. The Lanczos–van Stockum cylinder

K. Lanczos, Über eine stationäre Kosmologie im Sinne der Einsteinschen Gravitationstheorie, *Zeitschrift für Physik* **21** (1924) 73–110.

W.J. van Stockum (1937), The gravitational field of a distribution of particles rotating about an axis of symmetry, *Proc. Roy. Soc. Edinburgh* **57** 135–154.

In Weyl coordinates,

$$u = 2y, \quad f = x^2, \quad p = -x^2, \quad \mu = 4\mathrm{e}^{x^2}.$$

THE END