M- manifold.

Q.[1] What conditions should we put on M to make it homotopy equivalent to a closed manifold?

2. Let M and N be two manifolds. What are the conditions that tell us that M, N are homeomorphic?

If we fix a closed manifold then Q^1 helps us in finding manifolds that homotopic to it and

Q^2 classifies all these manifolds upto homeomorphism.

The set which contains these classes is called the structure set of the fixed manifold. (Denoted as $S(M)$.)

We shall talk about the structure sets of sphere bundles over spheres.

$$\{\xi : S^i \hookrightarrow E \to S^j\} \longleftrightarrow \pi_{j-1}(SO(i+1))$$

$i+j \geq 5.$

The structure set can be calculated using the surgery exact sequence, defined as:

$$\cdots \to \mathcal{N}_\partial(E \times I) \xrightarrow{\sigma} L_{n+1}(\mathbb{Z}[\pi_1(E)]) \xrightarrow{\rho} \mathcal{S}(E) \xrightarrow{\eta} \mathcal{N}(E) \xrightarrow{\sigma} L_n(\mathbb{Z}[\pi_1(E)])$$

$\mathcal{N}_\partial(E \times I)$ & $\mathcal{N}(E)$ are the Abelian groups of normal invariants;

$L_{n+1}(\mathbb{Z}[\pi_1(E)])$ & $L_n(\mathbb{Z}[\pi_1(E)])$ are L-groups associated to the group ring $\mathbb{Z}[\pi_1(E)]$

The maps between $\mathcal{N}$ & L_n are called the surgery obstruction map (σ)

In our case E being simply connected:

$$L_n(\mathbb{Z}[\pi_1(E)]) = L_n(\mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } n \equiv 0 \pmod 4 \\ \mathbb{Z}_2, & \text{if } n \equiv 2 \pmod 4 \\ 0, & \text{otherwise} \end{cases}$$

$$\mathcal{N}(E) \cong [E; G/TOP]$$

We shall begin with special case of bundles

$$\{\xi : S^{i-1} \hookrightarrow E \to S^i\} \longleftrightarrow \pi_{i-1}(SO(i))$$

$$\pi_{i-1}(SO(i)) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & (i \equiv 0, 4 \pmod 8) \\ \mathbb{Z} \oplus \mathbb{Z}_2 & (i \equiv 2 \pmod 8) \\ \mathbb{Z} & (i \equiv 6 \pmod 8) \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & (i \equiv 1 \pmod 8) \\ \mathbb{Z}_2 & (i \equiv 3, 5, 7 \pmod 8) \end{cases}$$

for $i \neq 1, 2, 3, 7$

So for the bundles of the form

$\left\{ \xi : S^{4k-1} \hookrightarrow E \to S^{4k} \right\} \longleftrightarrow \pi_{4k-1}(SO(4k)) \cong \mathbb{Z} \oplus \mathbb{Z}$

we have plenty of supply of bundles to deal with.

For $k = 1$ we have $\left\{ \xi : S^3 \hookrightarrow E \to S^4 \right\} \leftrightarrow \mathbb{Z} \oplus \mathbb{Z}$

We consider S^3 bundles over S^4 with total space E & structure group $SO(4)$.

The one-to-one correspondence with $\mathbb{Z} \oplus \mathbb{Z}$ is given by :-

Choose two generators of $\pi_3(SO(4))$, ρ & σ such that $\rho(u) = uvu^{-1}$; $\sigma(u)v = u \cdot v$

u, v are quaternions with norm 1.

(We have identified S^3 with unit quaternions)

For these generators and a pair of integer (m, n) we have an element $m \cdot \rho + n \cdot \sigma \in \pi_3(SO(4))$ gives us a vector bundle $\xi_{m,n}$ and thus corresponding sphere bundle $E_{m,n} \to S^4$.

The surgery exact sequence looks like

$$\cdots \to \mathcal{N}_\partial(E_{m,n}\times I) \xrightarrow{\sigma_{n+1}} L_8(\mathbb{Z}\pi) \to \mathcal{S}(E_{m,n}) \to \mathcal{N}(E_{m,n}) \xrightarrow{\sigma_n} L_7(\mathbb{Z}\pi)$$

$$\mathcal{N}_\partial(E_{m,n}\times I) \xrightarrow{\sigma_8} \mathbb{Z} \to \mathcal{S}(E_{m,n}) \to \mathcal{N}(E_{m,n}) \xrightarrow{\sigma_7} 0$$

By a known result σ's are surjective for simply connected spaces, hence we can rewrite seq^n as

$$\mathcal{N}_\partial(E_{m,n}\times I) \to 0 \to \mathcal{S}(E_{m,n}) \to \mathcal{N}(E_{m,n}) \to 0$$

$$\therefore \mathcal{S}(E_{m,n}) \cong \mathcal{N}(E_{m,n}) \cong \mathbb{Z}_n.$$

The next questions should be :-

- What are these elements ?
- Are all elements in $\mathcal{S}(E_{m,n})$ are sphere bundles over spheres (In our case $S^3 \hookrightarrow E_{m,n} \to S^4$).

In 2001, Diarmuid Crowley & Christine M. Escher answered this question for S^3 bundles over S^4 in their paper "A Classification of S^3 bundles over S^4".

Lemma 1 - There exists a fibre homotopy equivalence $f_j : E_{m+12j, n} \longrightarrow E_{m,n}$ $\forall j \in \mathbb{Z}$.

[Sphere bundles can be considered as spherical fibrations in that case homotopy equivalence is same as fibre homotopy equivalence]

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
0 & \longrightarrow & \pi_3(SO(3)) & \xrightarrow{i_*} & \pi_3(SO(4)) & \xrightarrow{ev|_{SO(4)*}} & \pi_3(S^3) & \longrightarrow & 0 \\
 & & \downarrow i_{3*} & & \downarrow i_{4*} & & \downarrow Id & & \\
0 & \longrightarrow & \pi_3(SF_3) & \longrightarrow & \pi_3(SG(4)) & \xrightarrow{ev_*} & \pi_3(S^3) & \longrightarrow & 0 \\
 & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
0 & \longrightarrow & \mathbb{Z}_{12} & \longrightarrow & \mathbb{Z}_{12} \oplus \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0
\end{array}
$$

$SG(4)$:= Topological monoid of orientation preserving self homotopy equivalence of S^3 which acts as the structure group of spherical fibration with fibre S^3

There is an inclusion $i_4 : SO(4) \hookrightarrow SG(4)$

$$A \longmapsto A|_{(S^3 \hookrightarrow \mathbb{R}^4)}$$

For $\xi, \eta \in \pi_3(SO(4))$, $S(\xi)$ & $S(\eta)$ are fibe homotopy equiv iff $i_{4*}(\xi) = i_{4*}(\eta) \in \pi_3(SG(4))$.

$SF(3) \subset SG(4)$, elements in $SG(4)$ that fix a point $p \in S^3$.

There is an iso $A_{3,3} : \pi_3(SF(3)) \cong \pi_6(S^3)$
&
We have a fibration

$$SF(3) \hookrightarrow SG(4) \xrightarrow{ev} S^3$$
$$f \mapsto f(p)$$

& $ev|_{SO(4)} : SO(4) \longrightarrow S^3$ is a fibration with fibre $SO(3) \subset SF(3)$.

for generators ι_3 & ρ of $\pi_3(S^3)$ & $\pi_3(SO(3))$ & $m, n \in \mathbb{Z}$

~~$S_*($~~

$$s_*(n\iota_3) = \xi_{0,m}, \quad i_*(m\rho) = \xi_{m,0}$$

where $s : S^3 \to SO(4)$ is a section
$$x \mapsto L_x$$
$$L_x(y) = x \cdot y \text{ (quaternion multip.)}$$

if $i_3 : SO(3) \hookrightarrow SF(3)$ is fibre-wise inclusion

then $J_3 := A_{3,3} \circ i_{3*} : \pi_3(SO(3)) \longrightarrow \pi_6(S^3)$ is J-homo.

which is onto $J_{3,3} : \mathbb{Z} \longrightarrow \mathbb{Z}_{12}$

The next lemma connects these equivalences to normal invariants

Lemma :- The fibre homotopy equivalences

~~f_j : M~~

$$f_j : E_{m+12j,n} \longrightarrow E_{m,n} \text{ have}$$

normal invariants $\eta(f_j) = j \in \mathcal{N}^{PL}(E_{m,n}) \cong \mathbb{Z}_n$.

— $(F_j, f_j) : (W_{m+12j,n}, E_{m+12j,n}) \longrightarrow (W_{m,n}, E_{m,n})$.

F_j is disk bundle of f_j

& $i_* : \mathcal{N}(M^k) \longrightarrow \mathcal{N}(\partial M^k)$ in

$$\begin{array}{ccccccc}
L_{k+1}(e \to e) & \longrightarrow & \mathcal{S}(M^k) & \xrightarrow{\eta} & \mathcal{N}(M^k) & \longrightarrow & L_k(e \to e) \\
\downarrow & & \downarrow & & \downarrow i_* & & \downarrow \\
L_k(e) & \longrightarrow & \mathcal{S}((\partial M)^{k-1}) & \xrightarrow{\eta} & \mathcal{N}((\partial M)^{k-1}) & \longrightarrow & L_{k-1}(e)
\end{array}$$

gives us $\eta(f_j) = i_* \eta(F_j)$

So it's enough to prove $\eta(F_j) \in \mathcal{N}(W_{m,n})$ takes on the value $j \in \mathbb{Z}$

For $G/PL \xhookrightarrow{i} BPL \longrightarrow BG$

$$\begin{array}{ccccc}
\mathcal{N}(W_{m,n}) \cong & [W_{m,n}, G/PL] & \xrightarrow{j_*} & [W_{m,n}, BPL] \\
\downarrow & \downarrow i^*_{m,n} & & \downarrow i^*_{m,n} \\
\mathcal{N}(S^4) \cong & [S^4, G/PL] & \xrightarrow{j_*} & [S^4, BPL]
\end{array}$$

The map $J_* : \pi_4(G/PL) \longrightarrow \pi_4(BPL)$ is
$\cong \mathbb{Z}$ $\cong \mathbb{Z}$

multiplication by 24.

& For any compact X, we may regard

[X, BPL] as formal differences of stable PL-bundles

$\therefore j_*(\eta(F_j)) = \nu(W_{m,n}) - F_j^{-1*}(\nu(W_{m+12j,n}))$

& $\nu(W_{m,n}) = \pi^*_{m,n}(\nu_{S^4} \oplus -\xi_{m,n}) = \pi^*_{m,n}(\xi_{m,n})$

$\Rightarrow i^*_{m,n}(j_*(\eta(F_j))) = i^*_{m,n}(\nu(W_{m,n}) - F_j^{-1*}(\nu(W_{m+12j,n})))$

$= i^*_{m,n}(\pi^*_{m,n}(-\xi + \xi_{m+12j,n}))$

$= \xi_{m+12,n} - \xi_{m+n}$

$= 2(m+12j)+n - (2(m)+n)$

$= 24j \in \mathbb{Z}$

$\therefore \eta(F_j) = 24j/24 \in \mathcal{N}(W_{m,n})$ so $\eta(F_j) \in \mathcal{N}^{PL}(E_{m,n})$.

For $S^7 \hookrightarrow E \longrightarrow S^8$

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}\oplus\mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow \cong & & \downarrow \cong & & \downarrow & & \\
0 & \longrightarrow & \pi_7(SO(7)) & \longrightarrow & \pi_7(SO(8)) & \longrightarrow & \pi_7(S^7) & \longrightarrow & 0 \\
 & & \downarrow i_{7*} & & \downarrow i_{8*} & & \downarrow Id & & \\
0 & \longrightarrow & \pi_7(SF(7)) & \longrightarrow & \pi_7(SG(8)) & \longrightarrow & \pi_7(S^7) & \longrightarrow & 0 \\
 & & \downarrow \cong & & \downarrow & & \downarrow \cong & & \\
0 & \longrightarrow & \mathbb{Z}_{120} & \longrightarrow & \mathbb{Z}\oplus\mathbb{Z}_{120} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0
\end{array}
$$

&

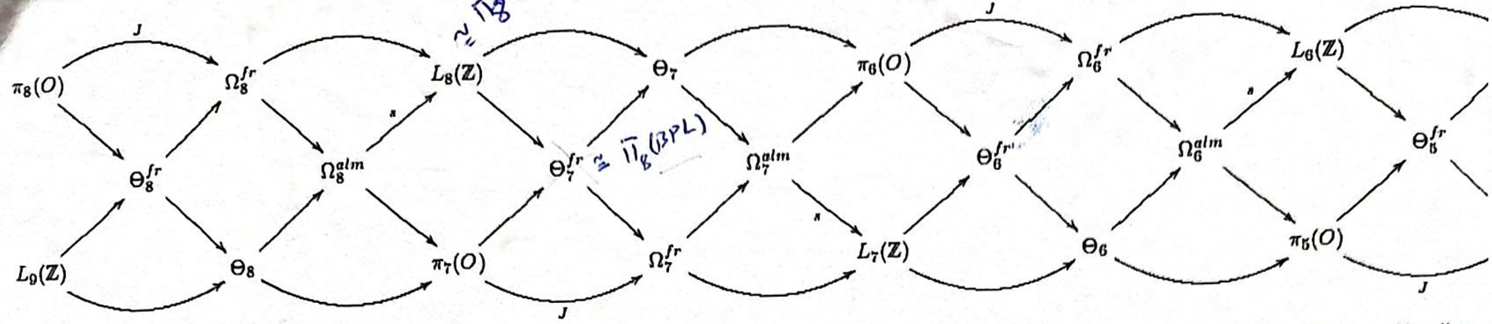
$$
\begin{array}{ccc}
\pi_8(G/PL) & \longrightarrow & \pi_8(BPL) \\
\text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} \\
\mathbb{Z} & & \mathbb{Z}\oplus\mathbb{Z}_4 \\
x & \longmapsto & (60x, x).
\end{array}
$$

(Using Kervaire Milnor Braid)

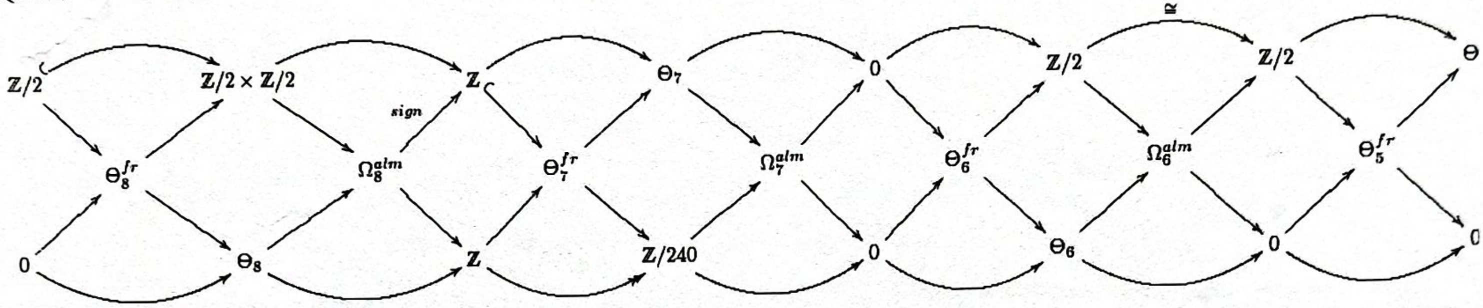
Talk:Kervaire-Milnor Braid (Ex)

From Manifold Atlas

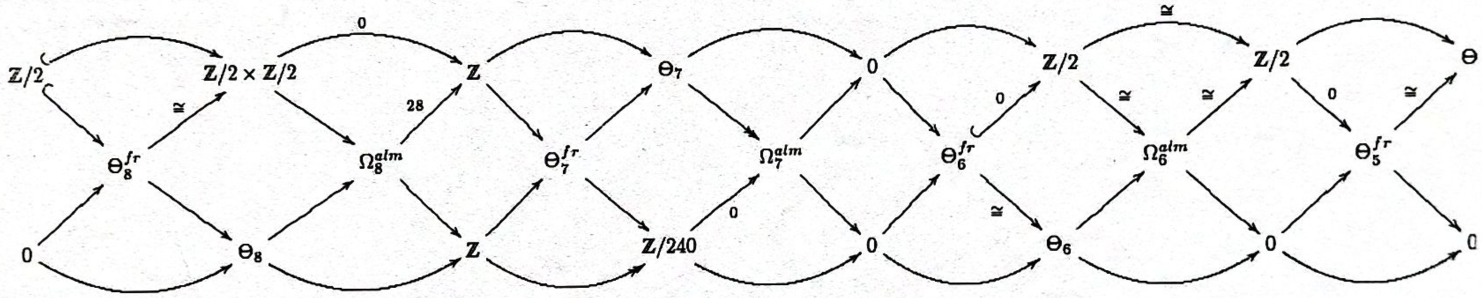
For starters the braid looks as follows:



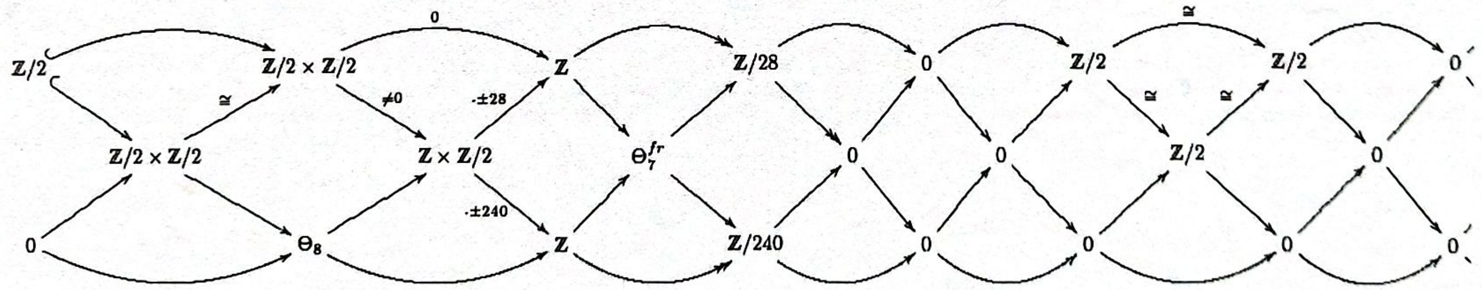
Filling in the L-groups (which were computed by Kervaire and Milnor), the homotopy groups of the orthogonal group (which are given by Bott periodicity), the framed bordism groups (which were shown to be the stable stems by Pontryagin and in low dimensions computed by Serre), and the J-homomorphism (which was computed by Adams and Quillen, but can be computed by hand in low dimensions) we arrive at



where we have used κ to denote the map induced by the Kervaire invariant $\Omega_6^{fr} \to \mathbb{Z}/2$. Using that this is surjective ($S^3 \times S^3$ with its Lie-group framing has Kervaire-invariant 1) and the fact that the signature of an almost framed 8-manifold is divisible by 28 (this proved in the chapter on exotic spheres in Lück's book) and 28 is actually the signature of ???, we obtain the following maps:



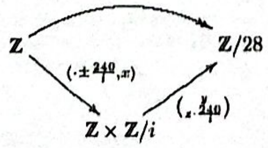
The extension at Ω_8^{alm} is split since it surjects onto $240\mathbb{Z}$ (right lower map) which is free. Clearing this and the obvious 0s, we find:



This in particular implies the smooth Poincaré conjecture in dimension 5, which is not covered by the usual collary of the h-cobordism theorem. The possibilities for Θ_7^{fr} are $\mathbb{Z}, \mathbb{Z} \times \mathbb{Z}/2$ and $\mathbb{Z} \times \mathbb{Z}/4$ since it is simultaneously an extension of $\mathbb{Z}$ by $\mathbb{Z}/28$ and $\mathbb{Z}/240$ whose common factors are 1, 2 and 4. We now claim the following facts from algebra, which we prove after giving the final braid: In an extension of the form

$$\mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}/a \to \mathbb{Z}/ab$$

the map $\mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}/a$ is always given by multiplication with $\pm b$ in the first factor. Looking now at the triangle above Θ_7^{fr}, we find (for some integers x, y, z)



which is only possible for $i = 4$ since the upper map cannot possibly be surjective in the other cases (28 having a common factor with both 60 and 120). With this as input we claim that there is a unique isomorphism class of braids left and it looks as follows (proof given below):

Geometry and Topology of Smooth,Topological Manifolds and CW-complexes

Minimal Thesis Defence

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July 6, 2022

Outline

1 Introduction and Basic Notions

2 L-Groups

3 The Structure Set

4 The Surgery Exact Sequence

5 Spectral Sequences

6 Project of Dissertation

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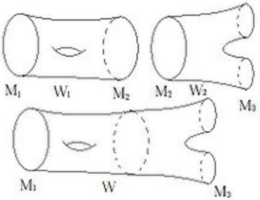
Introduction and Basic Notions

The structure set of a manifold classifies up to homeomorphism the manifolds which are homotopy equivalent to the given manifold. Our aim is to calculate the structure sets of the sphere bundles using methods in surgery theory.

Definition 1 (Cobordism)

Let M and N be two closed n-manifolds. A (Unoriented) cobordism between M and N is a compact $(n+1)$-manifold W with boundary ∂W as the disjoint union $\partial W = M \sqcup N$.



Definition 2 (Surgery)

Suppose M is an n-manifold and $\phi : S^k \times D^{n-k} \to M$ an embedding. Define

$$M' := (D^{k+1} \times S^{n-k-1}) \cup_{\phi|_{S^k \times S^{n-k-1}}} (M - (\mathrm{int}(\mathrm{im}\ (\phi)))).$$

We say the manifold M' is obtained from M by performing k-surgery on M along ϕ.

$$W := (D^{k+1} \times D^{n-k}) \cup_\phi M \times [0, 1]$$

is called the trace of surgery. It is a cobordism between M and M'.

L-Groups

Definition 3

Let R be an associative ring with unit and involution. Two non-degenerate $(-1)^k$-quadratic forms (F, ψ) and (F', ψ') where F and F' are finitely generated free R-modules are equivalent if and only if there exist integers $u, u' \geq 0$ such that

$$(F, \psi) \oplus H_\epsilon(R)^u \cong (F', \psi') \oplus H_\epsilon(R)^{u'}$$

Definition 4 (Even Dimensional L-Group)

For $n = 2k$, $L_n(R)$ is the set of equivalence classes $[(F, \psi)]$ of non-degenerate $(-1)^k$-quadratic forms (F, ψ) with respect to above equivalence relation.

Similarly we have definition of odd dimensional L-groups.

Example 5

For $R = \mathbb{Z}$ the L-groups are given as follows :

$$L_n(\mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } n \equiv 0 (\text{mod } 4); \qquad \text{(signature)}/8 \\ 0, & \text{if } n \equiv 1 (\text{mod } 4); \\ \mathbb{Z}_2, & \text{if } n \equiv 2 (\text{mod } 4); \qquad \text{Arf Invariant} \\ 0, & \text{if } n \equiv 3 (\text{mod } 4). \end{cases}$$

The manifolds we are interested in are mostly simply connected hence knowing the L-groups of $\mathbb{Z}$ will be sufficient.

The Structure Set

Definition 6

Let M_0, M_1, X be closed oriented n-manifolds. The orientation preserving homotopy equivalences $(f_i) : (M_i) \to (X)$ for $i = 0, 1$ are called *equivalent* if :

- There exists a compact manifold $(W, \partial W)$ of dimension $(n + 1)$ with an orientation preserving homotopy equivalence of pairs
$$(F, \partial F) : (W, \partial W) \to (X \times [0, 1], \partial(X \times [0, 1])).$$
- The boundary is the disjoint union $\partial W = \partial_0 W \amalg \partial_1 W$;
- $\partial F : \partial W \to \partial(X \times [0, 1]) = (X \times \{0, 1\})$ induces orientation preserving homotopy equivalences $\partial_i F : \partial_i W \to X \times \{i\}$ for $i = 0, 1$;
- For $i = 0, 1$ there are orientation preserving diffeomorphisms $u_0 : M_0 \to \partial_0 W$ and $u_1 : M_1^- \to \partial_1 W$ satisfying $\partial_i F \circ u_i = j_i \circ f_i$ where $j_i : X \to X \times \{i\}$ sends x to (x, i) for $i = 0, 1$.

Definition 7 (The Structure Set)

The structure set $\mathcal{S}_n^h(X)$ of X is the set of equivalence classes of orientation preserving homotopy equivalences, $M \to X$, from a closed oriented manifold M to X where equivalence relation is as defined above.

The structure set has a preferred base point, namely, the class of identity id : $X \to X$.

The Surgery Exact Sequence

Theorem 8

Let X be a closed, oriented topological n-manifold where $n \geq 5$. Then there exists a (geometric) surgery exact sequence

$$\mathcal{N}(X \times [0,1], \partial(X \times [0,1])) \xrightarrow{\sigma^h_{n+1}} L^h_{n+1}(\mathbb{Z}\pi(X)) \xrightarrow{\rho^h_{n+1}} \mathcal{S}^h(X) \xrightarrow{\eta^h_n} \mathcal{N}(X) \xrightarrow{\sigma^h_n} L^h_n(\mathbb{Z}\pi(X))$$

which is exact in the following sense

1. *An element $\alpha \in \mathcal{N}(X)$ lies in the image of η^h_n if and only if $\sigma^h_n(\alpha) = 0$;*
2. *Two elements β_1 and β_2 in $\mathcal{S}^h(X)$ have same image under η^h_n if and only if there exists $\omega \in L^h_{n+1}(\mathbb{Z}\pi(X))$ such that $\rho^h_{n+1}(\omega, \beta_1) = \beta_2$;*
3. *For $\omega \in L^h_{n+1}(\mathbb{Z}\pi(X))$ we have $\rho^h_{n+1}(\omega, [id_X]) = [id_X]$ if and only if there is a normal bordism class $\gamma \in \mathcal{N}(X \times [0,1], \partial(X \times [0,1]))$ such that $\sigma^h_{n+1}(\gamma) = \omega$.*

Let us recall few theorems:

Theorem 9

Let M be a closed n-manifold. Then the set $\mathcal{N}(M)$ is non empty and comes with a preferred base point, namely, the identity map and there is canonical bijection

$$[M; G/TOP] \cong \mathcal{N}(M).$$

Theorem 10

Let X be a simply connected compact topological manifold of dimension $n \geq 5$. The surgery obstruction map $\mathcal{N}(X) \xrightarrow{\sigma} L_n(\mathbb{Z}\pi(X))$ is surjective.

Theorem 11

Let X be a simply connected compact topological manifold of dimension $n \geq 5$. Then the map $\mathcal{S}(X) \xrightarrow{\eta} \mathcal{N}(X)$ is injective.

Theorem 12 (Classification of vector bundles over a paracompact base B)

There is a bijection between the sets

$$[B, G_n] \approx \mathit{Vect}^n(B)$$

where $[B, G_n]$ is the set of homotopy classes of maps between B and G_n.

As a special case we have a bijection $[S^{k-1}, O(n)] \rightarrow \text{Vect}^n(S^k)$.
Using **Bott Periodicity Theorem** for orthogonal groups: for $n \geq k+2$ we have :

$$\pi_k(O(n)) = \pi_k(SO(n)) = \begin{cases} 0, & \text{if } k = 2,4,5,6(\bmod 8); \\ \mathbb{Z}_2, & \text{if } k = 0,1(\bmod 8); \\ \mathbb{Z}, & \text{if } k = 3,7(\bmod 8). \end{cases}$$

This implies that for large enough n the homotopy groups are independent of n. These groups give us plenty of vector bundles hence associated sphere bundles over sphere.

Homotopy Groups of $SO(n)$

G	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}
$SO(2)$	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0
$SO(3)$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$
$SO(4)$	$\mathbb{Z}_2$	0	$\mathbb{Z}^{\oplus 2}$	$\mathbb{Z}_2^{\oplus 2}$	$\mathbb{Z}_2^{\oplus 2}$	$\mathbb{Z}_{12}^{\oplus 2}$	$\mathbb{Z}_2^{\oplus 2}$	$\mathbb{Z}_2^{\oplus 2}$	$\mathbb{Z}_3^{\oplus 2}$	$\mathbb{Z}_{15}^{\oplus 2}$
$SO(5)$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	$\mathbb{Z}_{120}$
$SO(6)$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_{120}$
$SO(7)$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2^{\oplus 2}$	$\mathbb{Z}_2^{\oplus 2}$	$\mathbb{Z}_8$
$SO(8)$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}^{\oplus 2}$	$\mathbb{Z}_2^{\oplus 3}$	$\mathbb{Z}_2^{\oplus 3}$	$\mathbb{Z}_8 \oplus \mathbb{Z}_{24}$
$SO(9)$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2^{\oplus 2}$	$\mathbb{Z}_2^{\oplus 2}$	$\mathbb{Z}_8$
$SO(10)$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}$	$\mathbb{Z}_4$

Example 13 (Crowley, Escher 2003)

Consider $p : M \to S^4$ the fibre bundle with fibre S^3 and structure group $SO(4)$. Vect$^n(S^4) \cong \pi_3(SO(4)) \approx \mathbb{Z} \oplus \mathbb{Z}$. The generators ρ and σ of $\pi_3(SO(4))$ with integers m, n gives us a vector bundle $\xi_{m,n} := m \cdot \rho + n \cdot \sigma$ and a sphere bundle $p_{m,n} : S(\xi_{m,n}) = M_{m,n} \to S^4$. The dimension of $M_{m,n}$ is 7 and $\pi = \pi_1(M_{m,n})$ which is trivial. This will give us $L_7(\mathbb{Z}\pi) = \{0\}$ and $L_8(\mathbb{Z}\pi) = \mathbb{Z}$. The surgery exact sequence

$$\mathcal{N}_\partial(M_{m,n} \times I) \xrightarrow{\sigma_{n+1}} L_8(\mathbb{Z}\pi) \xrightarrow{\rho} \mathcal{S}(M_{m,n}) \xrightarrow{\eta} \mathcal{N}(M_{m,n}) \xrightarrow{\sigma_n} L_7(\mathbb{Z}\pi)$$

becomes : $\mathcal{N}_\partial(M_{m,n} \times I) \xrightarrow{\sigma_{n+1}} \mathbb{Z} \xrightarrow{\rho} \mathcal{S}(M_{m,n}) \xrightarrow{\eta} \mathcal{N}(M_{m,n}) \xrightarrow{\sigma_n} \{0\}$
Theorem(10) implies σ_n and σ_{n+1} are surjective. The exactness of the sequence will imply $\mathcal{S}(M_{m,n}) \cong \mathcal{N}(M_{m,n}) \cong \mathbb{Z}_n$.

From Theorem(9) we have $[M; \text{G/TOP}] \cong \mathcal{N}(M)$ and we can calculate $[M; \text{G/TOP}]$ as follows :

We write $X\left[\frac{1}{p}\right]$ for the result of inverting the prime p, the symbol $X_{(p)}$ for localization of the space X at a primes p (i.e. all prime except p are inverted) and the symbol $X_{\mathbb{Q}}$ for rationalization of X (i.e. all primes are inverted). We shall denote by $K(A, l)$ the **Eilenberg-MacLane** space of type (A, l). Computation of homotopy type of G/PL and G/O is due to Sullivan. One obtains the homotopy equivalence

$$G/TOP\left[\tfrac{1}{2}\right] \simeq BO\left[\tfrac{1}{2}\right];$$
$$G/TOP_{(2)} \simeq \prod_{j\geq 1} K(\mathbb{Z}_{(2)}, 4j) \times \prod_{j\geq 1} K(\mathbb{Z}_2, 4j-2).$$

In particular we get for a space X the isomorphisms

$$[X; G/TOP]\left[\tfrac{1}{2}\right] \cong \widetilde{KO}^0(X)\left[\tfrac{1}{2}\right];$$
$$[X; G/TOP]_{(2)} \cong \prod_{j\geq 1} H^{4j}(X; \mathbb{Z}_{(2)}) \times \prod_{j\geq 1} H^{4j-2}(X, \mathbb{Z}_2)$$

where KO^* is K-theory of real vector bundles

Spectral Sequences

To calculate cohomologies we shall use the spectral sequences.

Theorem 14 (Serre Spectral Sequence)

Let $p : E \to B$ be an orientable fibration with B path connected and a fibre F over $b \in B$. Given $A \subset B$, there is a convergent E^2 spectral sequence, with $E^2_{s,t} \cong H_s(B, A; H_t(F; G))$. This spectral sequence is a first quadrant spectral sequence.

Theorem 15 (Atiyah Hirzebruch Spectral Sequence)

For any homology theory h_* and a CW-complex X there is a spectral sequence $\{E^r, d^r\}$ with $E^2_{p,q} \cong H_p(X; h_q(\text{pt}))$.

Note that we have to use cohomology spectral sequences, similar results exist for cohomology case as well.

Project of dissertation

For any bundle $p : M \to S^n$ with fibre some sphere S^m we want to calculate the normal invariant set $\mathcal{N}(M)$. Recall from the discussion so far :

$$\begin{aligned} [M; G/TOP] &\cong \mathcal{N}(M); \\ [M; G/TOP]\left[\frac{1}{2}\right] &\cong \widetilde{KO}^0(M)\left[\frac{1}{2}\right]; \\ [M; G/TOP]_{(2)} &\cong \prod_{j\geq 1} H^{4j}(M; \mathbb{Z}_{(2)}) \times \prod_{j\geq 1} H^{4j-2}(M; \mathbb{Z}_2). \end{aligned} \tag{1}$$

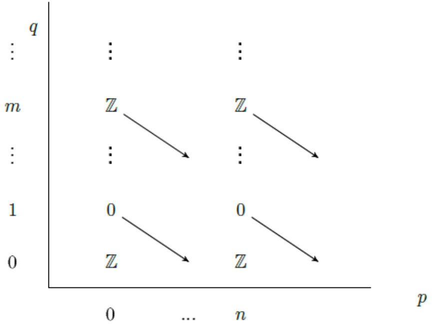
And $\mathcal{N}(M)$ is given by the pullback diagram :

$$\begin{array}{ccc} \mathcal{N}(M) & \longrightarrow & [M; G/TOP]_{(2)} \\ \downarrow & & \downarrow \\ \widetilde{KO}^0(M)\left[\frac{1}{2}\right] & \longrightarrow & \widetilde{KO}^0(M)\left[\frac{1}{2}\right] \otimes \mathbb{Q} \end{array}$$

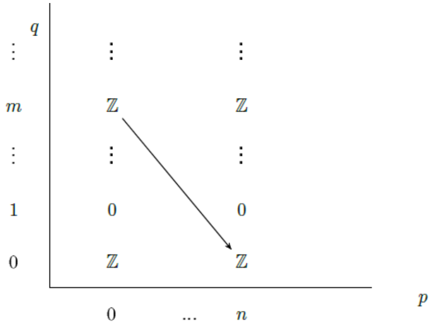
For any sphere bundle $S^m \hookrightarrow M \xrightarrow{p} S^n$. It follows from the Theorem(14) that we have $E_2^{p,q} \cong H^p(S^n; H^q(S^m; \mathbb{Z})$. Since we know that

$$E_2^{p,q} \cong H^p(S^n; H^q(S^m; \mathbb{Z})) = \begin{cases} \mathbb{Z} & \text{if } p = 0, n \text{ and } q = 0, m; \\ 0 & \text{otherwise .} \end{cases}$$

This tells us that only 0-th and the n-th column in the E_2-page can be non-zero :



The only case where the differential will be non-zero if the bidegree of the differential d_r is $(r, 1-r) = (n, -m)$. This will imply $n = m + 1$ and the non-zero differential d_n with bidgree $(n, 1-n)$ and the E_n-page will look like :



So if $n = m + 1$ the differential $d : \mathbb{Z} \to \mathbb{Z}$ is non zero and is determined by the value of $d(1)$.

Let us see the case where $d(1) = \pm 1$. The E_∞-page will be :

q \ p	0	$\dots$	n
$\vdots$	$\vdots$		$\vdots$
m	0		$\mathbb{Z}$
$\vdots$	$\vdots$		$\vdots$
1	0		0
0	$\mathbb{Z}$		0

This will give us the cohomologies as follows :

$$H^i(M;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0, m+n; \\ 0 & \text{otherwise .} \end{cases}$$

Using the Universal Coefficient Theorem we get

$$H^i(M;\mathbb{Z}_{(2)}) = \begin{cases} \mathbb{Z}_{(2)} & \text{if } i = 0, m+n; \\ 0 & \text{otherwise .} \end{cases}$$

and

$$H^i(M;\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{if } i = 0, m+n; \\ 0 & \text{otherwise .} \end{cases}$$

In this case, let us determine what is

$$[M;G/TOP]_{(2)} \cong \prod_{j\geq 1} H^{4j}(M;\mathbb{Z}_{(2)}) \times \prod_{j\geq 1} H^{4j-2}(M;\mathbb{Z}_2) .$$

We have the following cases :

1. If m is even :
   Then $n(= m + 1)$ and $m + n$ are odd. Hence we have :
   $$[M; G/TOP]_{(2)} = \{0\};$$
2. If m is odd :
   Then n is even and $m + n$ is odd. Hence we have :
   $$[M; G/TOP]_{(2)} = \{0\}.$$

For the generalized cohomology theory $h^q(X) = \widetilde{KO}^q(X)$

$$h^q(\text{pt}) = \begin{cases} \mathbb{Z}\left[\frac{1}{2}\right], & \text{if } q = 4l; \\ 0, & \text{otherwise .} \end{cases}$$

The universal Coefficient theorem gives us :

$$H^i(M; h^q(pt)) = \begin{cases} \mathbb{Z}\left[\frac{1}{2}\right] & \text{if } i = 0, m+n \text{ and } q = 4l; \\ 0 & \text{otherwise} \end{cases}$$

Atiyah Hirzebruch spectral sequence gives us :

$$E_2^{p,q} \cong \widetilde{H^p}(M; h^q(\text{pt})) = \begin{cases} \mathbb{Z}\left[\frac{1}{2}\right] & \text{if } p = m+n \text{ and } q = 4l; \\ 0 & \text{otherwise .} \end{cases}$$

This implies that the differential d_2 is zero hence E_∞-page is same as E_2-page. Now since $n = m+1$, $m+n$ is always odd which implies $\widetilde{KO}^0(X)\left[\frac{1}{2}\right] = \{0\}$ in every case. Hence the pullback diagram gives us $\mathcal{N}(M) = \{0\}$.

Ideas for future work

In our PhD project we shall try to :

- calculate Atiyah-Hirzebruch spectral sequences for remaining cases of m and n ;
- calculate the differential of the Serre spectral sequences for the sphere bundles obtained from the identification $\mathrm{Vect}^n(S^k) \cong \pi_k(SO(n))$ i.e. which of the different case do actually exist ?
- calculate $\mathcal{S}(E)$ for $S^m \hookrightarrow E \to \mathbb{C}P^n$;
- calculate $\mathcal{S}(E)$ for $S^n \hookrightarrow E \to B$ where B is the total space of some sphere bundle over a sphere (i.e., for iterated bundles).
- There could be non isomorphic vector bundles with homeomorphic sphere bundles. So one can try to find some relationship in these two different classifications.

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