Non-Abelian Gerbes with Connections

and with an explicit cocycle description

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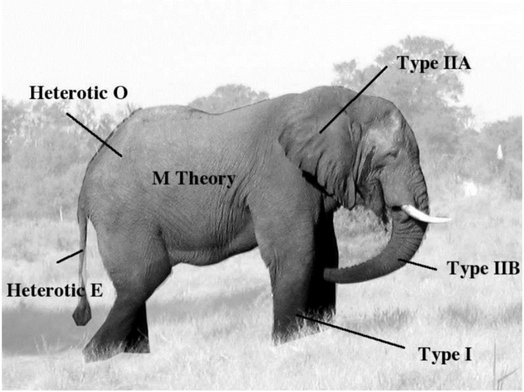
Motivation

Gauge fields are everywhere:

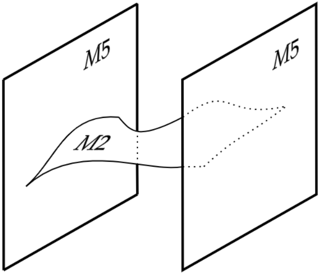
- Gauge fields in the Standard Model and in Yang–Mills theory
- Chern–Simons (matter) theories
  - Importantly, M2-brane models: BLG- and ABJM-model
- Kalb–Ramond B-field in string theory
- B-field and connection in heterotic supergravity
- Tensor hierarchies in gauged supergravity
- T-duality: B-fields on top of circle bundles with connection
- Interacting M5-branes:
  - self-dual strings
  - $(2, 0)$-theory

Motivation: M-theory



"One (superst)ring to rule them all ..."

Motivation: $(2, 0)$-theory



The picture so far:

- M5 branes interact via M2 branes, boundaries = self-dual strings
- 6D SCFT from M-theory with $(2, 0)$ SUSY
- No Lagrangian exists – irreducibly quantum?
- Field content:
  $(2, 0)$ tensor multiplet
  includes 2-form B-field
- This B-field satisfies the self-duality equation:
  $H := \mathrm{d}B = \star H$

Motivation

What we need:

- to go beyond connections on principal bundles
- a framework to describe higher connection forms, e.g. for parallel transport of extended objects

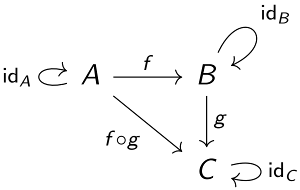
Good news! There is such a framework – higher gauge theory.

Categories, Higher Structures, and All That

What are categories?

A category $\mathscr{C} = (\mathscr{C}_0 \leftleftarrows \mathscr{C}_1)$ consists of

- a collection $\mathscr{C}_0$ of objects,
- a collection $\mathscr{C}_1$ of morphisms between objects such that



When all morphisms are invertible $\implies$ groupoid.
Group = one-object groupoid, $\mathrm{BG} = (\mathrm{G} \rightrightarrows *)$.
Morphisms between categories = functors.

Fun with functors

Principal bundles = functors from $\check{\mathcal{C}}(\mathcal{U})$ to BG.

- Cover $\mathcal{U} = \sqcup_a U_a$ of a manifold M yields Čech groupoid $\check{\mathcal{C}}(\mathcal{U})$:

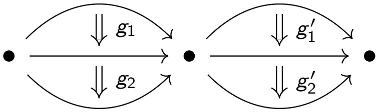
$$(x,a) \xleftarrow{(x,a,b)} (x,b) \xleftarrow{(x,b,c)} (x,c), \qquad (x,a) \xleftarrow{(x,a,c)} (x,c)$$

- $$\begin{array}{ccc} \sqcup_{a,b} U_{ab} & \xrightarrow{\;g_{ab}\;} & G \\ \downdownarrows & & \downdownarrows \\ \sqcup_a U_a & \xrightarrow{\;*\;} & * \end{array}$$

  Transition functions g_{ab},
  cocycle cond. $g_{ab}g_{bc} = g_{ac}$
  cobndries.: $g_{ab}\gamma_b = \gamma_a \tilde{g}_{ab}$

Categories not enough!

Parallel transport of extended objects: replace paths by (higher-dimensional) surfaces.
Consider the parallel transport of one-dimensional strings along a two-dimensional surface



This means that

- $(g_2 g_1)(g'_2 g'_1) = (g_2 g'_2)(g_1 g'_1)$.
- Setting $g_2 = 1$ and $g'_1 = 1$ yields $g_1 g'_2 = g'_2 g_1$.

$\Longrightarrow$ Forces G to be abelian = Eckmann-Hilton argument.

Problem!

Categories not enough!

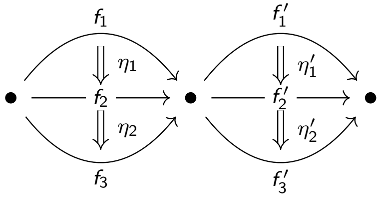
String/M- theory requires the framework of higher category theory.

2-categories to the rescue

Recall a category $\mathscr{C} = (\mathscr{C}_1 \rightrightarrows \mathscr{C}_0)$ has $\mathscr{C}_1$ as the collection of morphisms. To construct a (strict) 2-category

- take $\mathscr{C}_1$ to be a category itself, i.e. introduce morphisms between morphisms = 2-morphisms,
- introduce associative horizontal composition $\otimes$ between 2-morphisms (the other being vertical $\circ$) such that the following interchange law holds

$$(\eta_2 \circ \eta_1) \otimes (\eta_2' \circ \eta_1') = (\eta_2 \otimes \eta_2') \circ (\eta_1 \otimes \eta_1')$$



$\Longrightarrow$ relaxes Eckmann-Hilton!

Morphisms of 2-categories = 2-functors.

Constructing higher structures

A mathematical structure ("Bourbaki-style") consists of

- Sets
- Structure Functions
- Structure Equations

Categorification: "*adding morphisms between morphisms*"

Sets $\rightarrow$ Categories

Structure Functions $\rightarrow$ Structure Functors

Structure Equations $\rightarrow$ Structure Isomorphisms

Example: Group(oid) $\rightarrow$ 2-Group(oid)

Note: Process not unique, variants: weak/strict/...

Strict 2-groups

Only need strict Lie 2-groups = crossed modules of Lie groups:

- a pair of Lie groups (G, H) with an automorphism action $\rhd$ of G on H
- a morphism of Lie groups $\mathsf{t} : \mathsf{H} \to \mathsf{G}$ such that

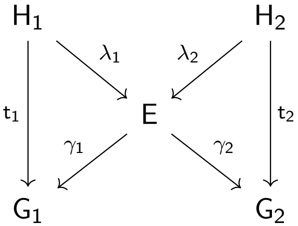
$$\mathsf{t}(g \rhd h_1) \;=\; g\mathsf{t}(h_1)g^{-1} \quad \text{and} \quad \mathsf{t}(h_1) \rhd h_2 \;=\; h_1 h_2 h_1^{-1}$$

  for all $g \in \mathsf{G}$ and for all $h_1, h_2 \in \mathsf{H}$.
- Written as $\mathcal{G} := (\mathsf{H} \xrightarrow{\mathsf{t}} \mathsf{G}, \rhd)$.
- Play the role of structure groups for principal 2-bundles a.k.a. gerbes.

Examples:

- $(\mathsf{U}(1) \longrightarrow 1) \cong \mathsf{BU}(1)$ underlies an abelian gerbe.
- $(1 \hookrightarrow \mathsf{G}, \mathrm{id})$ underlies a principal G-bundle (flat?)
- $\mathcal{L}\mathsf{G} := (L_0\mathsf{G} \hookrightarrow P_0\mathsf{G}, \mathrm{Ad})$

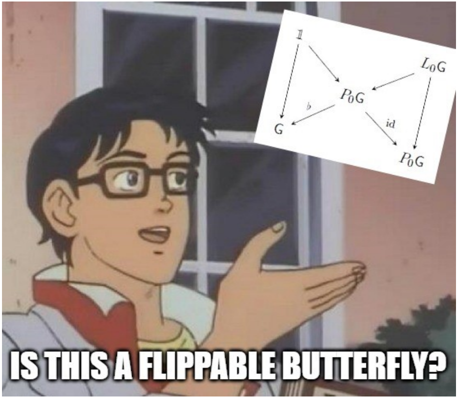
Equivalences between crossed modules given by flippable butterflies:

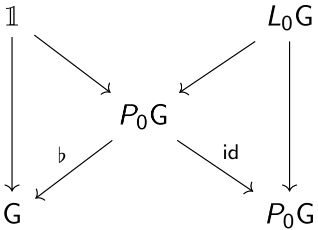


where E is a Lie group, λ_i, γ_i are morphisms of Lie groups and both diagonals are short exact sequences.

Butterflies

Recall crossed modules $\mathcal{L}\mathrm{G} = (L_0\mathrm{G} \hookrightarrow P_0\mathrm{G}, \mathrm{Ad})$ and $(\mathbb{1} \hookrightarrow \mathrm{G}, \mathrm{id})$.
Are they equivalent?





Here, $\flat : P_0\mathrm{G} \to \mathrm{G}$ is the endpoint evaluation map.
They are **equivalent**!

Gerbes without connections

Principal $\mathcal{G}$-bundle $=$ 2-functor from $\check{\mathcal{C}}(\mathcal{U})$ to $B\mathcal{G}$.

- $\mathcal{G} := (\mathsf{H} \xrightarrow{\mathsf{t}} \mathsf{G}, \rhd)$
- Trivially regard $\check{\mathcal{C}}(\mathcal{U})$ as a 2-groupoid by adding identity 2-morphisms
-

$$\begin{array}{ccccc} \sqcup_a U_a & \Leftarrow & \sqcup_{a,b} U_{ab} & \Leftarrow & \sqcup_{a,b} U_{ab} \\ \downarrow & & \downarrow & & \downarrow \\ * & \Leftarrow & \mathsf{G} & \Leftarrow & \mathsf{G} \ltimes \mathsf{H} \end{array}$$

- Transition functions (g_{ab}, h_{abc}) satisfying cocycle conditions:

$$\mathsf{t}(h_{abc}) g_{ab} g_{bc} = g_{ac} \ , \quad h_{acd} h_{abc} = h_{abd} (g_{ab} \rhd h_{bcd})$$

- Equivalent when related by coboundaries (g_a, h_{ab}):

$$g_a g'_{ab} = \mathsf{t}(h_{ab}) g_{ab} g_b \ , \quad h_{ac} h_{abc} = (g_a \rhd h'_{abc}) h_{ab} (g_{ab} \rhd h_{bc})$$

The Local Picture: L_∞-algebras and Higher Connections

Crossed modules of Lie algebras

Differentiating a crossed module of Lie groups yields a crossed module of Lie algebras:

- given by a pair $(\mathfrak{g}, \mathfrak{h})$ of Lie algebras,
- automorphism action $\rhd$ of $\mathfrak{g}$ on $\mathfrak{h}$,
- morphism of Lie algebras $\mathrm{t} : \mathfrak{h} \to \mathfrak{g}$ such that

$$\mathrm{t}(X \rhd Y_1) \;:= [X, \mathrm{t}(Y_1)] \quad \text{and} \quad \mathrm{t}(Y_1) \rhd Y_2 \;:= [Y_1, Y_2]\ .$$

- Can be viewed as a (strict 2-term) L_∞-algebra:
  - $\mathbb{Z}$-graded vector space $\mathfrak{L} = \underbrace{\mathfrak{L}_{-1}}_{\mathfrak{h}} \oplus \underbrace{\mathfrak{L}_0}_{\mathfrak{g}}$,
  - with differential $\mathrm{t} = \mu_1 : \mathfrak{L}_{-1} \to \mathfrak{L}_0$,
  - with graded anti-symmetric Lie bracket $\mu_2 : \mathfrak{L}_i \times \mathfrak{L}_j \to \mathfrak{L}_{i+j}$, $(\alpha_1, \alpha_2) \mapsto [\alpha_1, \alpha_2]$, $(\alpha_1, \beta_1) \mapsto \alpha_1 \rhd \beta_1$,
  - such that μ_1 is a derivation and μ_2 satisfies a graded Jacobi identity + higher maps

L_∞-algebras

- Generalizations of (differential graded) Lie algebras
- Come with their own gauge theory:
  - elements $a \in \mathfrak{L}_1$ of degree 1 are generalized gauge potentials
  - curvature: $f := \mu_1(a) + \frac{1}{2}\mu_2(a,a) + \frac{1}{3!}\mu_3(a,a,a) + \dots \quad \in \mathfrak{L}_2$
    - Homotopy Maurer-Cartan equation: $f = 0$
  - Bianchi identity
    $\mu_1(f) - \mu_2(f,a) + \frac{1}{2}\mu_3(f,a,a) - \frac{1}{3!}\mu_4(f,a,a,a) + \cdots = 0$
  - Elements $\in \mathfrak{L}_0$ parametrize generalized gauge transformations.
- Ordinary gauge theory on manifold M modelled on $\Omega^\bullet(M) \otimes \mathfrak{g}$
- Higher gauge theory modelled on $\Omega^\bullet(M) \otimes \mathfrak{L}$.

L_∞-algebras

The tensor product $\hat{\mathfrak{L}} = \Omega^\bullet(M) \otimes \mathfrak{L}$ is an L_∞-algebra as well

- connection forms = gauge potentials
  $A + B + \cdots \quad \in \hat{\mathfrak{L}}_1 = (\Omega^1(M) \otimes \mathfrak{L}_0) \oplus (\Omega^2(M) \otimes \mathfrak{L}_{-1}) \oplus \cdots$
- higher products $\hat{\mu}_1 = \mathrm{d} + \mu_1$, μ_2, μ_3, ...
- curvatures

$$
\begin{aligned}
F &= \mathrm{d}A + \tfrac{1}{2}\mu_2(A, A) + \mu_1(B) \ \rightarrow \ \text{fake curvature} \\
H &= \mathrm{d}B + \mu_2(A, B) - \tfrac{1}{3!}\mu_3(A, A, A) + \ldots \\
&\vdots
\end{aligned}
$$

- Infinitesimal gauge transformations can be obtained from the definition of curvatures as partially flat homotopies. They are parametrised by
  $\alpha + \Lambda \in \hat{\mathfrak{L}}_0 = (\Omega^0(M) \otimes \mathfrak{L}_0) \oplus (\Omega^1(M) \otimes \mathfrak{L}_{-1}) \oplus \cdots.$

The fake curvature condition

Problem: consistency requires fake flatness
$F = \mathrm{d}A + \frac{1}{2}\mu_2(A, A) + \mu_1(B) = 0$

- For $\mu_3 \neq 0$ infinitesimal gauge transformations do not close: $[\delta_{c_0}, \delta_{c_1}] = \delta_{[c_0, c_1]} + \mu_3(F, c_0, c_1)$.
- For $\mu_3 = 0$ finite gauge transformations do not compose.
- Self-duality $H = \star H$ requires $F = 0$ for covariance.
- Parallel transport requires $F = 0$ for reparametrisation invariance.
- Can gauge away any non-Abelian part $\implies$ theory Abelian.

Problem!

The fake curvature condition

Solution: adjust the definition of curvatures: $H \to H + \kappa(A, F)$ where κ is called the adjustment datum.

Examples:

- $\mathcal{L}\mathfrak{g} = (L_0\mathfrak{g} \hookrightarrow P_0\mathfrak{g})$:
  - connection given by 1- and 2-forms $(A, B) \in \Omega^1(M, P_0\mathfrak{g}) \oplus \Omega^2(M, L_0\mathfrak{g})$.
  - curvature forms

$$
\begin{aligned}
F &:= \mathrm{d}A + \tfrac{1}{2}[A, A] + \mathrm{t}(B) \ \in \ \Omega^2(M, P_0\mathfrak{g}) \ , \\
H &:= \mathrm{d}B + A \rhd B - \kappa(A, F) \ \in \ \Omega^3(M, L_0\mathfrak{g}) \ ,
\end{aligned}
$$

  where $\kappa(A, F) = (1 - \wp \cdot \flat)([A, F])$ for an arbitrary fixed path $\wp \in \mathscr{C}^\infty([0, 1], \mathbb{R})$ such that $\wp(0) = 0$ and $\wp(1) = 1$.
- String Lie 2-algebra: $\mathfrak{string}(\mathsf{G}) = (L_0\mathfrak{g} \oplus \mathfrak{u}(1) \to P_0\mathfrak{g})$
  - $\hat{\kappa}(A, F) = \Big(\kappa(A, F), \frac{\mathrm{i}}{2\pi} \int_0^1 \mathrm{d}r \langle A_r, F \rangle\Big)$

The Global Picture: Non-Abelian Gerbes with Connections

Non-Abelian gerbes with connections

Putting everything together to construct principal 1- and 2-bundles with connections.

- Manifold M with cover $\mathcal{U} = (U_a)$
- Lie 2-group and 2-algebra: crossed modules $\mathsf{H} \to \mathsf{G}$ and $\mathfrak{h} \to \mathfrak{g}$ with adjustment datum κ

Object	Principal G-bundle	Principal $(\mathsf{H} \xrightarrow{\mathsf{t}} \mathsf{G})$-bundle
Cochains	(g_{ab}) valued in G	(g_{ab}) valued in G, (h_{abc}) valued in H
Cocycle	$g_{ab}g_{bc} = g_{ac}$	$\mathsf{t}(h_{abc})g_{ab}g_{bc} = g_{ac}$ $h_{acd}h_{abc} = h_{abd}(g_{ab} \rhd h_{bcd})$
Coboundary	$g_a g'_{ab} = g_{ab}g_b$	$g_a g'_{ab} = \mathsf{t}(h_{ab})g_{ab}g_b$ $h_{ac}h_{abc} = (g_a \rhd h'_{abc})h_{ab}(g_{ab} \rhd h_{bc})$
gauge pot.	$A_a \in \Omega^1(U_a) \otimes \mathfrak{g}$	$A_a \in \Omega^1(U_a) \otimes \mathfrak{g}$, $B_a \in \Omega^2(U_a) \otimes \mathfrak{h}$
Curvature	$F_a = \mathrm{d}A_a + \frac{1}{2}[A_a, A_a]$	$F_a = \mathrm{d}A_a + \frac{1}{2}[A_a, A_a] + \mathsf{t}(B)$ $H_a = \mathrm{d}B_a + A_a \rhd B_a - \kappa(A_a, F_a)$
Gauge trafos	$\tilde{A}_a := g_a^{-1}A_a g_a + g_a^{-1}\mathrm{d}g_a$	$\tilde{A}_a := g_a^{-1}A_a g_a + g_a^{-1}\mathrm{d}g_a - \mathsf{t}(\Lambda_a)$ $\tilde{B}_a := g_a^{-1} \rhd B_a + \tilde{A}_a \rhd \Lambda_a + \mathrm{d}\Lambda_a + \frac{1}{2}[\Lambda_a, \Lambda_a] - \kappa(g_a, F_a)$

Cocycle description of gerbes

Consider a surjective submersion $\sigma : Y \to M$ and an adjusted crossed module of Lie groups $\mathcal{G} := (\mathsf{H} \xrightarrow{\mathsf{t}} \mathsf{G}, \rhd, \kappa)$ with the corresponding crossed module of Lie algebras $(\mathfrak{h} \xrightarrow{\mathsf{t}} \mathfrak{g}, \rhd, \kappa)$. An adjusted cocycle for a principal $\mathcal{G}$-bundle over M is given by the data

$$\begin{aligned} h &\in \mathscr{C}^\infty(Y^{[3]}, \mathsf{H}) \,, \\ (g, \Lambda) &\in \mathscr{C}^\infty(Y^{[2]}, \mathsf{G}) \oplus \Omega^1(Y^{[2]}, \mathfrak{h}) \,, \\ (A, B) &\in \Omega^1(Y^{[1]}, \mathfrak{g}) \oplus \Omega^2(Y^{[1]}, \mathfrak{h}) \,, \end{aligned}$$

such that

$$h_{ikl} h_{ijk} = h_{ijl}(g_{ij} \rhd h_{jkl}) \quad \text{and} \quad g_{ik} = \mathsf{t}(h_{ijk}) g_{ij} g_{jk}$$

hold for all appropriate $(i, j, \ldots) \in Y^{[n]}$.

In addition, we have the following cocycle conditions

$$\begin{aligned} \Lambda_{ik} &= \Lambda_{jk} + g_{jk}^{-1} \rhd \Lambda_{ij} - g_{ik}^{-1} \rhd (h_{ijk} \nabla_i h_{ijk}^{-1}) \ , \\ A_j &= g_{ij}^{-1} A_i g_{ij} + g_{ij}^{-1} \mathrm{d} g_{ij} - \mathsf{t}(\Lambda_{ij}) \ , \\ B_j &= g_{ij}^{-1} \rhd B_i + \mathrm{d}\Lambda_{ij} + A_j \rhd \Lambda_{ij} + \tfrac{1}{2}[\Lambda_{ij}, \Lambda_{ij}] - \kappa(g_{ij}, F_i) \end{aligned}$$

for all appropriate $(i, j, \ldots) \in Y^{[n]}$, where κ needs to satisfy the adjustment identity

$$\begin{aligned} (g_2^{-1} g_1^{-1}) \rhd (h^{-1}(X \rhd h)) + g_2^{-1} \rhd \kappa(g_1, X) \\ + \kappa(g_2, g_1^{-1} X g_1 - \mathsf{t}(\kappa(g_1, X))) - \kappa(\mathsf{t}(h) g_1 g_2, X) \ = \ 0 \end{aligned}$$

for all $g_1, g_2 \in \mathsf{G}$, for all $h \in \mathsf{H}$, and for all $X \in \mathfrak{g}$.

Summary

- Higher structures not useless!
- Have now a proper definition of non-Abelian gerbes with connections and explicit examples.
- Can accommodate string structures!

To do:

- understand the origin of the adjustment datum
- construct physically relevant examples

The End

Thank you for your attention!

"It pays off to have the right connections."

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