Bounds for the cup-length of some oriented Grassmann manifolds

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Introduction

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In differential topology, one well-known example of such an invariant is the Lusternik-Schnirelman category of a space.

Definition

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For a closed manifold this invariant is the strict lower bound for the number of critical points of a smooth function on X.

In algebraic topology, there is the following closely related invariant.

Definition

For a path connected space X, we may define its $\mathbb{Z}_2$–cup-length $\mathrm{cup}_{\mathbb{Z}_2}(X)$ as the greatest number r such that there exist cohomology classes $x_1, \dots, x_r \in H^*(X;\mathbb{Z}_2)$ in positive dimensions such that $x_1 \cdots x_r \neq 0$.

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The cup-length is related to the Lusternik-Schnirelman category by the inequality

$$\mathrm{cat}(X) \geq \mathrm{cup}_{\mathbb{Z}_2}(X).$$

Introduction

For the Grassmann manifold $G_{n,k} \cong O(n)/(O(k) \times O(n-k))$ we have a simple decription of its cohomology ring.

Cohomology ring of $G_{n,k}$

The cohomology ring of the Grassmann manifold $G_{n,k}$ is

$$H^*(G_{n,k}; \mathbb{Z}_2) = \mathbb{Z}_2[w_1, w_2, \ldots, w_k]/(\bar{w}_{n-k+1}, \ldots, \bar{w}_n),$$

where each indeterminate w_i is a representative of the ith Stiefel-Whitney class $w_i(\gamma_{n,k})$ of the canonical k–plane bundle $\gamma_{n,k}$ over $G_{n,k}$ and polynomials $\bar{w}_i$ correspond to the dual Stiefel-Whitney classes.

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However the same is not true for the oriented Grassmann manifold $\widetilde{G}_{n,k}$ of oriented k–dimensional vector subspaces in $\mathbb{R}^n$, the space $SO(n)/(SO(k) \times SO(n-k))$.

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However the same is not true for the oriented Grassmann manifold $\widetilde{G}_{n,k}$ of oriented k–dimensional vector subspaces in $\mathbb{R}^n$, the space $SO(n)/(SO(k) \times SO(n-k))$.
There is another homotopy invariant that is useful in providing bounds for the cup-length of $\widetilde{G}_{n,k}$.

Upper bounds for $\mathrm{cup}_{\mathbb{Z}_2}(\widetilde{G}_{n,k})$

The notion of characteristic rank quantifies the degree up to which the cohomology ring is generated by Stiefel-Whitney classes.

Definition

Let X be a connected, finite CW–complex and ξ a vector bundle over X. The *characteristic rank* of the vector bundle ξ, denoted $\mathrm{charrank}(\xi)$, is the greatest integer q, $0 \leq q \leq \dim(X)$, such that every cohomology class in $H^j(X;\mathbb{Z}_2)$ for $0 \leq j \leq q$ can be expressed as a polynomial in the Stiefel–Whitney classes $w_i(\xi)$ of ξ.

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For the oriented Grassmann manifold $\widetilde{G}_{n,k}$ and its canonical bundle $\widetilde{\gamma}_{n,k}$ we have

$$\mathrm{cup}_{\mathbb{Z}_2}(\widetilde{G}_{n,k}) \leq 1 + \frac{k(n-k) - \mathrm{charrank}(\widetilde{\gamma}_{n,k}) - 1}{2}.$$

Lower bounds for $\mathrm{cup}_{\mathbb{Z}_2}(\widetilde{G}_{n,k})$

There is a covering projection $p\colon \widetilde{G}_{n,k} \to G_{n,k}$, which induces homomorphism $p^*\colon w_j \mapsto \widetilde{w}_j = w_j(\widetilde{\gamma}_{n,k})$ appearing in the exact sequence

$$\xrightarrow{\psi} H^{j-1}(G_{n,k}) \xrightarrow{w_1} H^j(G_{n,k}) \xrightarrow{p^*} H^j(\widetilde{G}_{n,k}) \xrightarrow{\psi} H^j(G_{n,k}) \xrightarrow{w_1} H^{j+1}(G_{n,k}) \rightarrow$$

where $H^j(G_{n,k}) \xrightarrow{w_1} H^{j+1}(G_{n,k})$ is given by the cup product with w_1.

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where $H^j(G_{n,k}) \xrightarrow{w_1} H^{j+1}(G_{n,k})$ is given by the cup product with w_1. From the exactness we see that $\widetilde{w}_1 = p^*(w_1) = 0$ and more generally $\mathrm{Im}(p^*) = \mathbb{Z}_2[\widetilde{w}_2, \dots, \widetilde{w}_k]/p^*(g_{n-k+1}, \dots, g_n)$, where $g_i \in \mathbb{Z}_2[w_2, \dots, w_k]$ is the reduction of the polynomial $\bar{w}_i$ modulo w_1.

Lower bounds for $\mathrm{cup}_{\mathbb{Z}_2}(\widetilde{G}_{n,k})$

The sequence of polynomials g_i associated with $G_{n,k}$ is uniquely determined by k and it satisfies the recurrence formula

Recurrence formula

$g_i = w_2 g_{i-2} + w_3 g_{i-3} + \cdots + w_k g_{i-k}.$

Every $g_i \in \mathbb{Z}_2[w_2, \ldots, w_k]$ is a weighted homogeneous polynomial corresponding to a class in $H^i(G_{n,k})$.

Example

The sequence of polynomials g_i for low values of k is as follows.

$k = 3$	$k = 4$
$g_1 = 0$	$g_1 = 0$
$g_2 = w_2$	$g_2 = w_2$
$g_3 = w_3$	$g_3 = w_3$
$g_4 = w_2^2$	$g_4 = w_2^2 + w_4$

Review of results for $\mathrm{cup}_{\mathbb{Z}_2}(\widetilde{G}_{n,3})$

Theorem (Petrović, Prvulović, Radovanović, 2017)

Let $t \geq 3$. For all n in the interval $2^t - 1 \leq n < 2^t - 1 + \frac{2^t}{3}$ we have

$$\mathrm{cup}_{\mathbb{Z}_2}(\widetilde{G}_{n,3}) = 2^t - 3$$

Theorem

Let $t \geq 3$. We have

$$\mathrm{cup}_{\mathbb{Z}_2}(\widetilde{G}_{2^t+2^{t-1}+1,3}) = 2^t + 2^{t-2},$$
$$\mathrm{cup}_{\mathbb{Z}_2}(\widetilde{G}_{2^t+2^{t-1}+2,3}) = 2^t + 2^{t-2} + 1.$$

Review of results for $\mathrm{cup}_{\mathbb{Z}_2}(\widetilde{G}_{n,4})$

Theorem (Prvulović, Radovanović, 2019)

Let $t \geq 3$*. If* $2^t \leq n-1 \leq 2^t+1$*, then*

$$2^t - 5 \leq \mathrm{cup}_{\mathbb{Z}_2}(\widetilde{G}_{n,4}) \leq 2^t + 2^{t-1} - 5.$$

If $n-1 = 2^t + 2^r + j$ *with* $1 \leq r \leq t-1$, $0 \leq j \leq 2^r - 1$*. If* $r \leq t-3$*,then*

$$2^t + 2^{r+1} + j - 5 \leq \mathrm{cup}_{\mathbb{Z}_2}(\widetilde{G}_{n,4}) \leq 2^t + 2^{t-1} - 5.$$

If $t-2 \leq r \leq t-1$*, then*

$$2^t + 2^{r+1} + j - 5 \leq \mathrm{cup}_{\mathbb{Z}_2}(\widetilde{G}_{n,4}) \leq 2^t + 2^{r+1} + 2j - 3.$$

Review of results for $\mathrm{cup}_{\mathbb{Z}_2}(\widetilde{G}_{n,4})$

Theorem (Prvulović, Radovanović, 2019)

Let $t \geq 3$. *If* $2^t \leq n-1 \leq 2^t+1$, *then*

$$2^t - 5 \leq \mathrm{cup}_{\mathbb{Z}_2}(\widetilde{G}_{n,4}) \leq 2^t + 2^{t-1} - 5.$$

If $n-1 = 2^t + 2^r + j$ *with* $1 \leq r \leq t-1$, $0 \leq j \leq 2^r - 1$. *If* $r \leq t-3$,*then*

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If $t-2 \leq r \leq t-1$, *then*

$$2^t + 2^{r+1} + j - 5 \leq \mathrm{cup}_{\mathbb{Z}_2}(\widetilde{G}_{n,4}) \leq 2^t + 2^{r+1} + 2j - 3.$$

Theorem

We have $\mathrm{cup}_{\mathbb{Z}_2}(\widetilde{G}_{8,4}) = \mathrm{cup}_{\mathbb{Z}_2}(\widetilde{G}_{9,4}) = 6$, $\mathrm{cup}_{\mathbb{Z}_2}(\widetilde{G}_{10,4}) = 7$, *and* $\mathrm{cup}_{\mathbb{Z}_2}(\widetilde{G}_{11,4}) = 9$.

Thank you.