Parabolic quasi-contact cone structures with an infinitesimal symmetry

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There are well known results for contact geometries with additional structure in this spirit. In particular, work of Čap and Salač relates parabolic contact structures with a transversal infinitesimal symmetry to parabolic conformally symplectic structures on the leaf space.

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The class of parabolic geometries that we consider contains:

1. Conformal structures $(\tilde{M}, [g])$ of signature $(p+1, q+1)$, and a generalization called causal structures
2. $(2, 3, 5)$ distributions
3. $(3, 6)$ distributions

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We call them parabolic quasi-contact cone structures. The construction can be formulated in a unified way.

In this talk I will focus on the case of conformal structures.

Step 1: Lift to projectivized null-cone bundle

$(\tilde{M}^{n+2}, [g])$ conformal manifold of signature $(p+1, q+1)$, $n = p + q$.
Consider the projectivized null-cone bundle

$$\pi : \tilde{\mathcal{C}} \to \tilde{M}, \quad \tilde{\mathcal{C}}_x = \{[v] \in \mathbb{P}(T_x\tilde{M}) : g(v,v) = 0\}.$$

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$\rightsquigarrow$ filtration of growth $(n+1, 2n+1, 2n+2)$ with splitting

$$T^{-1}\tilde{\mathcal{C}} = \tilde{\mathcal{E}} \oplus \tilde{\mathcal{V}} \subset T^{-2}\tilde{\mathcal{C}} = [\tilde{\mathcal{E}}, \tilde{\mathcal{V}}] \subset T^{-3}\tilde{\mathcal{C}} = T\tilde{\mathcal{C}}.$$

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Corank 1 distribution

$$T^{-2}_{[v]}\tilde{\mathcal{C}} = (T_{[v]}\pi)^{-1}(v^{\perp})$$

is quasi-contact (or even contact), i.e. locally, the kernel of a 1-form α
s.t. $d\alpha|_{T^{-2}\tilde{\mathcal{C}}}$ has maximal rank $\implies$ 1-dimensional kernel, the
characteristic line bundle $\tilde{\mathcal{E}} = \ker(d\alpha|_{T^{-2}\tilde{\mathcal{C}}})$, which corresponds to
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Conformal structure $\rightsquigarrow$ normal parabolic geometry $(\tilde{\mathcal{G}} \to \tilde{M}, \tilde{\psi})$ of type $(SO(p+2, q+2), P_1)$. $\tilde{\mathcal{C}}$ can be identified with correspondence space

$$\tilde{\mathcal{C}} = \tilde{\mathcal{G}}/P_{12}.$$

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Underlying structure is given by a bracket generating distribution

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More general structures correspond to causal structures (Makhmali, 2018), where one allows $\tilde{\mathcal{C}}_x \subset \mathbb{P}(T_x\tilde{M})$ to be arbitrary hypersurface with nondegenerate 2nd fundamental form of sig. (p, q).

Step 2: Symmetry Reduction

Transversal infinitesimal symmetry $\xi \in \mathfrak{X}(\tilde{\mathcal{C}})$:

$$\mathcal{L}_\xi \eta \in \Gamma(T^{-1}\tilde{\mathcal{C}}) \ \forall \eta \in \Gamma(T^{-1}\tilde{\mathcal{C}}), \quad \xi_x \notin T_x^{-2}\tilde{\mathcal{C}} \text{ for any } x \in \tilde{\mathcal{C}}.$$

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Orthopath geometry:

- Generalized path geometry:

$$T^{-1}\mathcal{C} = \underset{rank 1}{\mathcal{E}} \oplus \underset{integrable}{\mathcal{V}} \subset T^{-2}\mathcal{C} = T\mathcal{C}, \quad \mathcal{L} : \mathcal{E} \otimes \mathcal{V} \cong T\mathcal{C}/T^{-1}\mathcal{C}$$

  locally: $\mathcal{C}$ open $\mathbb{P}(TM)$ and on M path geometry: family of paths with unique path through each point in each direction ($\mathrm{gr}(T^{-1}\mathcal{C})$ is quotient of $\mathrm{gr}(T^{-1}\tilde{\mathcal{C}})$ by last grading component.)

- conformal class $[h]$, $h \in \Gamma(S^2\mathcal{V}^*)$ of sig. (p, q)

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Such variational orthopath geometries are in 1-1 correspondence with equivalence classes of non-degenerate first order Lagrangians $L(x, y^1, \cdots, y^n, p^1, \cdots, p^n)$, $\det\left(\frac{\partial^2 L}{\partial p^a \partial p^b}\right) \neq 0$ (has interpretation in terms of generalized Finsler structures).

Step 3: Quasi-contactification

Theorem

Given a causal structure with a transversal infinitesimal symmetry ξ. Then the induced orthopath geometry on the leaf space $\mathcal{C}$ is variational. Any variational orthopath geometry can be locally realized as a quotient of a causal structure in this way.

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Let $\alpha \in \Omega^1(\tilde{\mathcal{C}})$ be quasi-contact form, i.e. $\ker(\alpha) = T^{-2}\tilde{\mathcal{C}}$, s.t. $\alpha(\xi) = 1$.

$$d\alpha(\xi, \eta) = -\alpha([\xi, \eta]) \; \forall \eta \in \Gamma(T^{-2}\tilde{\mathcal{C}}) \implies \iota_\xi d\alpha = 0 \implies \mathcal{L}_\xi d\alpha = 0$$

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$\implies$ $d\alpha$ descends to closed 2-form $\rho \in \Gamma(\ell)$;

Conversely, suppose $\rho = d\beta \in \Gamma(\ell)$ on $U \subset \mathcal{C}$. Define

$$\pi : \tilde{\mathcal{C}} := U \times \mathbb{R} \to \mathcal{C} \quad \text{and} \quad \alpha := \pi^*\beta + dt.$$

Then $T^{-2}\tilde{\mathcal{C}} := \ker(\alpha)$ is quasi-contact structure, the rest of the filtration lifts and has symmetry ∂_t.

Theorem

An orthopath geometry on $\mathcal{C}$ has an associated regular normal Cartan geometry of type $(\mathrm{GL}(2,\mathbb{R}) \times \mathrm{SO}(p,q) \ltimes \mathbb{R}^2 \otimes \mathbb{R}^{p+q}, B \times \mathrm{SO}(p,q))$.

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$$\mathsf{A}_{abc}, \mathsf{T}_{ab}, \mathsf{N}_{ab}, \mathsf{q}.$$

Let $\tilde{\mathcal{C}} \to \mathcal{C}$ be submersion onto the local leaf space of integral curves of the symmetry ξ.

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Let $\tilde{\mathcal{C}} \to \mathcal{C}$ be submersion onto the local leaf space of integral curves of the symmetry ξ.Let F, W the harmonic curvature invariants on $\tilde{\mathcal{C}}$. Then

$$\mathsf{F} = \pi^* \mathsf{A} \quad \text{and} \quad \mathsf{W} = \pi^* \mathsf{T}$$

. We have

- $\mathsf{A} = 0 \iff$ quasi-contactifies to conformal structure.
- $\mathsf{A} = 0$, $\mathsf{q} = 0 \iff$ conformal Killing field ξ is null.
- $\mathsf{A} = 0$, $\mathsf{T} = 0 \iff$ conformal structure is flat (finite parameter family of such structures).

Partially integrable almost CR structure of hypersurface type: $\mathcal{H} \subset TM$ contact distribution $J : \mathcal{H} \to \mathcal{H}$ complex str. s.t. $\mathcal{L}(J\xi, J\eta) = \mathcal{L}(\xi, \eta) \rightsquigarrow$ $\mathcal{L}$ imaginary part of Hermitian form of signature (p, q).

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Has associated regular normal parabolic geometry $(\mathcal{P} \to M, \omega)$ of type $(\mathrm{SU}(p+1, q+1), P)$, $\mathfrak{p} = \mathrm{Lie}(P)$ is non-neg. part in contact grading

$$\mathfrak{su}(p+1, q+1) = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 =$$

$$\left\{ \begin{pmatrix} a & Z & iz \\ X & A & -\mathrm{I}_{pq} Z^* \\ ix & -X^* \mathrm{I}_{pq} & -\bar{a} \end{pmatrix} : x, z \in \mathbb{R}, X \in \mathbb{C}^n, Z \in \mathbb{C}^{n*}, A \in \mathfrak{u}(p, q) \right\}$$

The orthopath geometry of chains

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Family of canonical curves called chains, defined as projections of integral curves of $\omega^{-1}(\mathfrak{g}_{-2}) \subset T\mathcal{P}$ to M.

Given $x \in M$ and a line $l \subset T_x M$ transversal to $\mathcal{H}_x$, there is a unique chain γ through x s.t. $T_x \gamma = l$. $\rightsquigarrow$ path geometry on subset $\mathcal{C} \subset \mathbb{P}(TM)$ of transversal lines (Čap-Žádnik).

Let $S \subset P$ be stabilizer of $\mathfrak{g}_{-2}$ in $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \cong \mathfrak{su}(p+1, q+1)/\mathfrak{p}$ and $\mathfrak{s} = \mathrm{Lie}(S) = \mathfrak{g}_0 \oplus \mathfrak{g}_2$. Then one has a natural identification

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Observation:

$\mathfrak{p}_0 \cong \mathfrak{cu}(p, q)$-invariant conformal class of sig. $(2p, 2q)$ on $\mathfrak{p}_1 \cong \mathbb{C}^{n*} \rightsquigarrow$ $[h] \in \Gamma(\mathrm{Sym}^2\mathcal{V}^*)$

$\mathcal{C}$ equipped with path geometry and $[h] \rightsquigarrow$ orthopath geometry of chains.

Proposition

The orthopath geometry of chains is variational if and only if the almost CR structure is integrable. For these structures $A = q = 0$.

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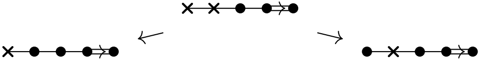
Fefferman's construction also fits with our picture: If we lift ξ to the subset of the projectivized null-cone bundle $\tilde{\mathcal{C}}$ of null-lines *not* in $\xi^{\perp}$ (where it is transversal) and form local leaf space, we recover the orthopath geometry of chains.

Parabolic Geometry Description

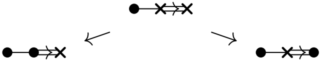
Regular, normal parabolic geometries with homogeneous models as on top of the double fibrations:

1. $(SO(p+2, q+2), P_{12})$ (odd and even)



2. (G_2, P_{12})

3. $(SO(3,4), P_{23})$



On the bottom left we have the models for conformal, $(2,3,5)$ and $(3,6)$
On the bottom right we have homogeneous contact manifolds