On uniqueness of submaximally symmetric geometric structures

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(partly based on joint work with Johnson Kessy)

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Classifying submaximal structures

This talk is about (locally) classifying structures via symmetry.

Throughout, let $\mathfrak{M}$ and $\mathfrak{S}$ denote the maximal / submax sym dim.
For the geometries in this talk, $\exists!$ maximally symmetric structure.

Goal: Locally classify all submaximally symmetric structures.

No homogeneity assumption is assumed, but we'll put some conditions to a priori guarantee homogeneity.

How to classify homog. such structures? Equivalent descriptions:

- coordinate
- Lie-theoretic: $\mathfrak{f}/\mathfrak{f}^0$ with $\mathfrak{f}^0$-invariant structure
- Cartan-theoretic: focus on these.

Working in the Cartan setting allows for systematic classification.

Examples

1. (Cartan 1910) For $(2,3,5)$-distributions, $\mathfrak{M} = 14$, and $\mathfrak{S} = 7$ realized on (x,y,p,q,z)-space by $D_f \subset TM$ spanned by

$$\partial_q, \quad \partial_x + p\partial_y + q\partial_p + f\partial_z,$$

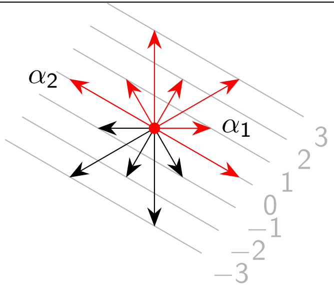
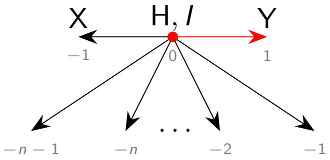
where $f = q^m$ $(m \neq -1, 0, \frac{1}{3}, \frac{2}{3}, 1, 2)$ or $f = \log(q)$.

2. (Lie $\sim$1890) Let $k \geq 4$. For scalar k-th order ODE (mod contact), $\mathfrak{M} = k+4$ (unique), $\mathfrak{S} = \begin{cases} \mathfrak{M} - 2, & k \neq 5, 7; \\ \mathfrak{M} - 1, & k = 5, 7 \end{cases}$ is realized by an ODE locally equivalent to:
   - (a) a linear ODE, or
   - (b) exactly one of:
     - (i) $k = 5$: $9(u_2)^2 u_5 - 45 u_2 u_3 u_4 + 40 (u_3)^3 = 0$.
     - (ii) $k = 7$: $10(u_3)^3 u_7 - 70 (u_3)^2 u_4 u_6 - 49 (u_3)^2 (u_5)^2 + 280 u_3 (u_4)^2 u_5 - 175 (u_4)^4 = 0$.
     - (iii) $k \neq 5, 7$: $(k-1) u_{k-2} u_k - k (u_{k-1})^2 = 0$.

Have equivalent reformulations as (normalized) Cartan geometries $(\mathcal{G} \to M, \omega)$ of type (G, P). We have $\mathfrak{M} = \dim G$.

Structure	G	P	$\mathfrak{M}$	$\mathfrak{S}$	Unique submax model?
2-dim projective	A_2	P_1	8	3	✓
2nd order ODE	A_2	$P_{1,2}$	8	3	×
(2,3,5)-distributions	G_2	P_1	14	7	×
5-dim G_2-contact	G_2	P_2	14	7	✓
3-dim projective	A_3	P_1	15	8	✓
4-dim split-conformal	A_3	P_2	15	9	✓
5-dim Legendrian contact	A_3	$P_{1,3}$	15	8	✓
CR $M^5 \subset \mathbb{C}^3$ /w indef Levi	$\mathrm{SU}(2,2)$	$P_{1,3}$	15	8	✓
CR $M^5 \subset \mathbb{C}^3$ /w def Levi	$\mathrm{SU}(1,3)$	$P_{1,3}$	15	7	×
(3,6)-distributions	B_3	P_3	21	11	✓
57-dim E_8-contact	E_8	P_8	248	147	✓

Curvature fcn κ valued in $\bigwedge^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$.

<table>
<tr><th>$(2,3,5)$</th><th>$(n+1)$-st order ODE</th></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>$\mathfrak{g} = \mathfrak{g}_{-3} \oplus \ldots \oplus \overbrace{\mathfrak{g}_0 \oplus \ldots \oplus \mathfrak{g}_3}^{\mathfrak{p}}$</td><td>$\mathfrak{g} = \mathfrak{g}_{-n-1} \oplus \ldots \oplus \overbrace{\mathfrak{g}_0 \oplus \mathfrak{g}_1}^{\mathfrak{p}}$</td></tr>
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⚠ Let $\mathfrak{g}^i := \bigoplus_{j \geq i} \mathfrak{g}_j$. The $\mathfrak{p}$-inv filtration $\mathfrak{g} \supset \ldots \supset \mathfrak{g}^i \supset \mathfrak{g}^{i+1} \supset \ldots$ is important! (Grading is auxilliary.) Have $[\mathfrak{g}^i, \mathfrak{g}^j] \subset \mathfrak{g}^{i+j}$.

Harmonic curvature κ_H valued in $H_2(\mathfrak{p}_+, \mathfrak{g})^1 \cong_{\mathfrak{g}_0} H^2_+(\mathfrak{g}_-, \mathfrak{g})$.

Symmetry gaps

- Kruglikov & T. (2014): For parabolic geometries:
  - proved $\mathfrak{S} \leq \mathfrak{U}$, where $\mathfrak{U}$ is defined via rep theory.
  - have adapted versions: $\mathfrak{S}_{\mathbb{U}} \leq \mathfrak{U}_{\mathbb{U}}$ for G_0-irrep $\mathbb{U} \subset H^2_+(\mathfrak{g}_-, \mathfrak{g})$.
  - when G is complex or split-real simple with $\mathrm{rank}(G) \geq 3$, we have $\mathfrak{S} = \mathfrak{U}$. (Also for some rank 2 cases.)
  - have efficient Dynkin diagram recipes to find $\mathfrak{U}$.
  - Kostant's theorem for $H^2(\mathfrak{g}_-, \mathfrak{g})$ is crucial.
- Kessy & T. (2022): For ODEs (mod contact):
  - similarly established $\mathfrak{S} \leq \mathfrak{U}$. ($\exists$ analogous harmonic theory)
  - scalar case: modern proof of $\mathfrak{S}$, indep. of classification of Lie algebras of vector fields in the plane.
  - vector case: established $\mathfrak{S} = \mathfrak{M} - 2$ and various $\mathfrak{S}_{\mathbb{U}}$ results.
  - Used the effective part $\mathbb{E} \subset H^2_+(\mathfrak{g}_-, \mathfrak{g})$, which was described by Doubrov (2001), Medvedev (2010) & Doubrov–Medvedev (2014).

⚠ When $\mathfrak{S} = \mathfrak{U}$, any submax sym structure is locally homog. in a nbd of a point where κ_H is nonzero.

Q: How to classify submax homogeneous structures efficiently?

Submaximally symmetric parabolic geometries

Theorem (T. 2021)

Let G be complex simple, $P \leq G$ parabolic. Suppose $\mathrm{rank}(G) \geq 3$ or $(G, P) = (G_2, P_2)$. A regular, normal Cartan geometry $(\mathcal{G} \to M, \omega)$ of type (G, P) with:

1. κ_H *valued in a P-irrep* $\mathbb{U} \subset H_2(\mathfrak{p}_+, \mathfrak{g})^1$, *and*
2. *submaximal symmetry dimension* $\mathfrak{S}_{\mathbb{U}}$

is locally unique near $u \in \mathcal{G}$ with $0 \neq \kappa_H(u) \in \mathbb{U}$. If G were split-real, there is one of at most two possibilities: it is unique iff $\exists g \in P$ s.t. $g \cdot \phi_0 = -\phi_0$ for $\phi_0 \in \mathbb{U}$ a l.w.v.

Classically, Cartan reduction of full str eqns can be applied case-by-case. The proof of the above result circumvents this tedious story.

Vector ODEs & the C-class

Vector ODE $\mathbf{u}_{n+1} = \mathbf{f}(t, \mathbf{u}, \dots, \mathbf{u}_n)$, where $\mathbf{u} = (u^1, \dots, u^m)$, $m \geq 2$.

$\mathbf{u}_{n+1} = \mathbf{0}$ has sym alg $\mathfrak{g} \cong (\mathfrak{sl}_2 \times \mathfrak{gl}_m) \ltimes (\mathbb{V}_n \otimes \mathbb{R}^m)$, so $\mathfrak{M} = \dim \mathfrak{g}$.

Cartan (1938): a contact-invariant class of ODE is a *C-class* if for each ODE in this class, all differential invariants are first integrals. (Utility: generic C-class ODE can be solved w/o any integration.)

Čap–Doubrov–T. (2017):

- modern formulation of C-class: the canonical Cartan geometry associated to the ODE descends to a Cartan geometry over the solution space. (Equiv: κ satisfies a verticality condition.)
- characterization: An ODE is of C-class iff it is Wilczynski-flat, i.e. all generalized Wilczynski invariants $\mathcal{W}_r$ vanish.

Harmonic curvature decomposition

Medvedev (2010), Doubrov–Medvedev (2014): For vector ODEs, κ_H of the associated Cartan geometry is valued in an "effective part" $\mathbb{E} \subset H^2(\mathfrak{g}_-, \mathfrak{g})$, which decomposes into irreducibles $\mathbb{U} \subset \mathbb{E}$:

Type	n	$\mathbb{U}$	Bi-grade	$\mathfrak{sl}_m$-module str.
Wilczynski	≥ 2	$\mathbb{W}_r^{\mathrm{tf}}$ $(2 \leq r \leq n+1)$	$(r, 0)$	$\mathfrak{sl}_m$
	≥ 2	$\mathbb{W}_r^{\mathrm{tr}}$ $(3 \leq r \leq n+1)$	$(r, 0)$	$\mathbb{R}\,\mathrm{id}_m$
C-class	2	$\mathbb{B}_4$	$(2, 2)$	$S^2(\mathbb{R}^m)^*$
	≥ 2	$\mathbb{A}_2^{\mathrm{tf}}$	$(1, 1)$	$(S^2(\mathbb{R}^m)^* \otimes \mathbb{R}^m)_0$
	≥ 2	$\mathbb{A}_2^{\mathrm{tr}}$	$(1, 1)$	$(\mathbb{R}^m)^*$

Kessy–T. (2023): For each C-class irrep $\mathbb{U} \subset \mathbb{E}$, we gave explicit realizations of lowest weight vectors as harmonic 2-cochains.

Submaximally symmetric vector ODEs of C-class

Theorem (Kessy & T. 2023)

Over $\mathbb{R}$, below is the complete local classification (up to point-equivalence) of vector ODE $\mathbf{u}_{n+1} = \mathbf{f}(t, \mathbf{u}, \ldots, \mathbf{u}_n)$ of C-class of order $n+1 \geq 3$ that are submaximally symmetric.

n	*Irreducible C-class module* $\mathbb{U} \subset \mathbb{E}$	$\mathfrak{S}_{\mathbb{U}}$	*ODE of C-class with* $0 \not\equiv \mathrm{img}(\kappa_H) \subset \mathbb{U}$ *realizing* $\mathfrak{S}_{\mathbb{U}}$
2	$\mathbb{B}_4$	$\mathfrak{M} - m$	$\underset{(1 \leq a \leq m)}{u_3^a = \dfrac{3u_2^1 u_2^a}{2u_1^1}}$ *or* $\underset{(1 \leq a \leq m)}{u_3^a = \dfrac{3u_1^1 u_2^1 u_2^a}{1+(u_1^1)^2}}$
≥ 3	$\mathbb{A}_2^{\mathrm{tr}}$	$\mathfrak{M} - m - 1$	$\underset{(1 \leq a \leq m)}{u_{n+1}^a = \dfrac{(n+1)u_n^1 u_n^a}{n u_{n-1}^1}}$
≥ 2	$\mathbb{A}_2^{\mathrm{tf}}$	$\mathfrak{M} - 2m + 1 + \delta_2^n$	$\underset{(1 \leq a \leq m)}{u_{n+1}^a = (u_n^2)^2 \delta_1^a}$

Over $\mathbb{C}$, the two 3rd order models in the $\mathbb{B}_4$ branch are equivalent. (Rmk: When $m=1$, these have $\mathfrak{so}(2,2)$ and $\mathfrak{so}(1,3)$ symmetry.)

Fix $(\mathfrak{g}, \mathfrak{p})$ as before. How to systematically classify homog. str.?

Definition (Cartan-theoretic description)

An algebraic model $(\mathfrak{f}; \mathfrak{g}, \mathfrak{p})$ is a Lie algebra $(\mathfrak{f}, [\cdot,\cdot]_{\mathfrak{f}})$ s.t.:

M1: *$\mathfrak{f} \subset \mathfrak{g}$ is a filtered subspace, with filtrands $\mathfrak{f}^i := \mathfrak{f} \cap \mathfrak{g}^i$, and $\mathfrak{s} := \mathrm{gr}(\mathfrak{f})$ satisfying $\mathfrak{s}_- = \mathfrak{g}_-$. (Thus, $\mathfrak{f}/\mathfrak{f}^0 \cong \mathfrak{g}/\mathfrak{p}$.)*

M2: *$\mathfrak{f}^0$ inserts trivially into $\kappa(x, y) := [x, y] - [x, y]_{\mathfrak{f}}$. (Thus, $\kappa \in \bigwedge^2 (\mathfrak{f}/\mathfrak{f}^0)^* \otimes \mathfrak{g} \cong \bigwedge^2 (\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g} \cong \bigwedge^2 \mathfrak{p}_+ \otimes \mathfrak{g}$.)*

M3: *κ is regular / normal, i.e. $\kappa \in \ker(\partial^*)^1$.*

Given (G, P), let $\mathcal{M}$ be the set of all algebraic models $(\mathfrak{f}; \mathfrak{g}, \mathfrak{p})$.

- $\mathcal{M}$ is partially ordered: $\mathfrak{f} \leq \mathfrak{f}'$ iff $\mathfrak{f} \hookrightarrow \mathfrak{f}'$ as Lie algs.
- $\mathcal{M}$ admits a P-action: i.e. $p \cdot \mathfrak{f} = \mathrm{Ad}_p \mathfrak{f}$. Classify!

Necessary constraints

Proposition

Let $(\mathfrak{f};\mathfrak{g},\mathfrak{p})$ be an algebraic model. Then

1. $(\mathfrak{f},[\cdot,\cdot]_{\mathfrak{f}})$ *is a filtered Lie alg &* $\mathfrak{s}=\mathrm{gr}(\mathfrak{f})\subset\mathfrak{g}$ *is a graded Lie subalg.*
2. $\boxed{\mathfrak{f}^0\cdot\kappa=0}$: $[z,\kappa(x,y)]=\kappa([z,x],y)+\kappa(x,[z,y])$, $\forall x,y\in\mathfrak{f}$, $\forall z\in\mathfrak{f}^0$.
3. $\boxed{\mathfrak{s}\subset\mathfrak{a}^{\kappa_H}}$, *i.e.* $\mathfrak{f}$ *is a "filtered sub-deformation" of* $\mathfrak{a}^{\kappa_H}$. *Thus,* κ_H *constrains "leading parts" of* $\mathfrak{f}$. *(Next goal: Find "filtration tails".)*

Here, $\mathfrak{a}^\phi$ is defined below:

Definition (Extrinsic Tanaka prolongation)

Let $\mathfrak{g}$ *be a graded Lie alg with* $\mathfrak{g}_{-1}$ *generating* $\mathfrak{g}_-$. *Given* ϕ *in a* $\mathfrak{g}_0$*-rep, let* $\mathfrak{a}:=\mathfrak{a}^\phi\subset\mathfrak{g}$ *be the graded Lie subalg with* $\mathfrak{a}_{\leq 0}:=\mathfrak{g}_-\oplus\mathfrak{ann}(\phi)$ *and*

$$\mathfrak{a}_k:=\{x\in\mathfrak{g}_k:[x,\mathfrak{g}_{-1}]\subset\mathfrak{a}_{k-1}\},\qquad\forall k>0.$$

The aforementioned upper bound is $\boxed{\mathfrak{U}_{\mathbb{U}}=\max\{\dim\mathfrak{a}^\phi:0\neq\phi\in\mathbb{U}\}}$.

Submaximal $(2,3,5)$ models

For $(2,3,5)$, $\mathfrak{a}_+^\phi = 0$ when $0 \neq \phi \in H^2(\mathfrak{g}_-, \mathfrak{g}) \cong S^4\mathfrak{g}_1$. Then $\dim \mathfrak{a}^\phi$ is maximized on the GL_2-orbit of l.w.v. ϕ_0. Weight: $+4\alpha_1$.

$$\mathfrak{s} = \mathrm{gr}(\mathfrak{f}) = \mathfrak{a}^{\phi_0} = \mathfrak{g}_- \oplus \mathfrak{a}_0 = \langle f_{31}, f_{32}\rangle \oplus \langle f_{21}\rangle \oplus \langle f_{10}, f_{11}\rangle \oplus \langle Z_2, f_{01}\rangle$$

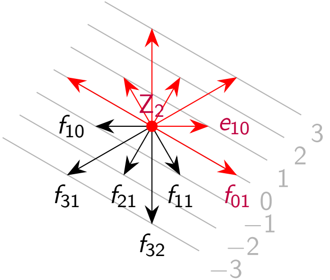
Proposition (T. 2022)

For $(2,3,5)$: Any 7-dim algebraic model is P-equivalent to $(\mathfrak{f}; \mathfrak{g}, \mathfrak{p})$ given below for some $c \in \mathbb{C}$. These are classified by the essential invariant c^2.

$$\mathfrak{f}: \quad \begin{aligned} S &= Z_2, \\ N &= f_{01}, \\ X_1 &= f_{10} + c e_{10}, \\ X_2 &= f_{11}, \\ X_3 &= f_{21}, \\ X_4 &= f_{31}, \\ X_5 &= f_{32}. \end{aligned}$$

$$\kappa = \kappa_H = f_{10}^* \wedge f_{31}^* \otimes f_{01}$$

(Kostant!)



Lie-theoretic structure: $[\cdot,\cdot]_{\mathfrak{f}} = [\cdot,\cdot] - \kappa(\cdot,\cdot)$. The *canonical submax sym model* here is the structure when $c = 0$.

Submaximal parabolic geometries – preparation

- $\mathfrak{g}$: complex simple Lie alg, with highest root λ
- $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{p}$, where $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{p}_+$.
- $\mathbb{V}_\mu$: $\mathfrak{g}_0$-irrep with lowest weight μ.

Kostant (1961): $H^2(\mathfrak{g}_-, \mathfrak{g}) = \bigoplus_{w \in W^{\mathfrak{p}}(2)} \mathbb{V}_\mu$, where $\mu = -w \bullet \lambda$ (mult. 1).

If $w = \sigma_j \circ \sigma_k$, then $\mathbb{V}_\mu$ has lwv $\phi_0 = e_{\alpha_j} \wedge e_{\sigma_j(\alpha_k)} \otimes e_{w(-\lambda)}$.

Kruglikov–T. (2014): $\dim \mathfrak{a}^\phi$ is maximized on the G_0-orbit of $[\phi_0]$.

Classify $(\mathfrak{f}; \mathfrak{g}, \mathfrak{p})$ with $\mathrm{gr}(\mathfrak{f}) = \mathfrak{a}^{\phi_0} =: \mathfrak{a}$. (Str.grp: $\mathrm{Stab}_{G_0}([\phi_0]) \ltimes P_+$)

Using Čap (2005), WLOG pass to the minimal twistor space. Then using Kruglikov–T. (2014), we can assume that $\mathfrak{f}^1 = 0$.

Submaximal parabolic geometries – proof outline

Let $\ker(\mu) := \{h \in \mathfrak{h} : \mu(h) = 0\}$. Note $\ker(\mu) \subset \mathfrak{a}_0 = \mathfrak{ann}(\phi_0)$.

Lemma

Fix $(\mathfrak{g}, \mathfrak{p})$ and $\mu = -w \bullet \lambda$ as before. Let $\ell = \mathrm{rank}(\mathfrak{g}) \geq 3$. Then:

(a) $\mu = \sum_{i=1}^{\ell} m_i \alpha_i$ *has coefficients of opposite sign.*

(b) $\exists H_0 \in \ker(\mu) \subset \mathfrak{h}$ *with* $f(H_0) \neq 0$, $\forall f = \alpha + \beta$ *with* $\alpha \in \Delta^+$ *and* $\beta \in \Delta^+ \cup \{0\}$.

Now normalize to $\mathfrak{f} = \mathfrak{a}$, $\kappa = \phi_0$ (canonical submax sym model):

1. Using P_+, normalize $\mathfrak{f}$ s.t. $H_0 \in \mathfrak{f}^0$.

   Take $H = H_0 + H_r + \ldots \in \mathfrak{f}^0$, $\underset{\text{(minimal)}}{r \geq 1}$, $H_r = \sum_{\alpha \in \Delta(\mathfrak{g}_r)} c_\alpha e_\alpha$. Define $X := \sum_{\alpha \in \Delta(\mathfrak{g}_r)} \frac{c_\alpha}{\alpha(H_0)} e_\alpha \in \mathfrak{g}_r$.

   Then $\mathrm{Ad}_{\exp(X)} H = \exp(\mathrm{ad}_X) H = H + [X, H] + \ldots = H_0 + \underbrace{H_r - [H_0, X]}_{=0} + \ldots$. Inductively, $H_+ = 0$.

2. Observe that $\ker(\mu) \subset \mathfrak{f}^0$.

   Fix $0 \neq H_0' \in \ker(\mu)$, $H' := H_0' + H_+' \in \mathfrak{f}$. So $[H_0, H']_{\mathfrak{f}} = [H_0, H'] = [H_0, H_+'] \in \mathfrak{g}_+ \cap \mathfrak{f} = \mathfrak{f}^1 = 0$.

   Since $\alpha(H_0) \neq 0$, $\forall \alpha \in \Delta^+$, then necessarily $H_+' = 0$.

We have $\mathfrak{a} = \mathfrak{g}_- \oplus \mathfrak{a}_0$.

3. Show that $\boxed{\mathfrak{f} = \mathfrak{a}}$ as vector subspaces of $\mathfrak{g}$.

Write $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$, where $\mathfrak{a}^\perp = \ker(\mu)^\perp \oplus \mathfrak{g}_{0,+} \oplus \mathfrak{g}_+$. Filtration tails are encoded by $\mathfrak{d} \in \mathfrak{a}^* \otimes \mathfrak{a}^\perp$ with $H \cdot \mathfrak{d} = 0$, $\forall H \in \ker(\mu) \subset \mathfrak{f}^0$, so $\mathfrak{d}$ lies in the sum of wt spaces for wts that are multiples of μ.

- Let $\alpha \in \Delta^-$. Since $e_\alpha^* \otimes \mathfrak{a}^\perp$ has non-negative wts, then $\mathfrak{d}(e_\alpha) = 0$.
- For the $\alpha \in \Delta^+(\mathfrak{a}_0)$ case, see my article.

4. Show that $\boxed{\kappa = \kappa_H = \phi_0}$.

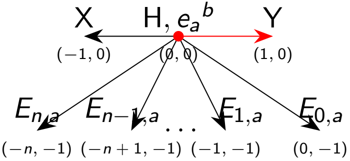
$\kappa \in \ker(\partial^*)^1 \subset \bigwedge^2 \mathfrak{p}_+ \otimes \mathfrak{g}$ and $\mathfrak{f}^0 \cdot \kappa = 0$. Since $\ker(\mu) \subset \mathfrak{f}^0$, the only relevant weights for κ are:

$$\sigma = r\mu = \alpha + \beta + \gamma, \quad \text{where} \quad \alpha, \beta \in \Delta(\mathfrak{p}_+) \text{ distinct}, \quad \gamma \in \Delta \cup \{0\}, \quad \boxed{r \geq 1}.$$

Write $\lambda = \sum_i n_i \alpha_i$, where $n_i > 0$, $\forall i$ (since $\mathfrak{g}$ is simple). We have $-\lambda \leq \gamma \leq \sigma$. If $\{Z_i\}$ are dual to $\{\alpha_i\}$, then $\forall i \neq j, k$:

$$-n_i = Z_i(-\lambda) \leq Z_i(\gamma) \leq Z_i(\sigma) = rZ_i(\mu) = -rn_i. \quad \therefore r \leq 1 \quad \therefore r = 1 \quad \therefore \boxed{\sigma = \mu}.$$

But μ is a l.w. with mult. one (by Kostant), so κ is a nonzero multiple of ϕ_0. By regularity, we can exponentiate the grading element action to normalize to ϕ_0 over $\mathbb{C}$ (or $\pm\phi_0$ over $\mathbb{R}$).



We needed to augment the Doubrov–Medvedev classification with realizations of l.w.v. as harmonic 2-cochains, i.e. get an analogue of Kostant's thm.

Theorem (Kessy–T. (2023))

For each C-class irrep $\mathbb{U} \subset \mathbb{E}$, the following are explicit realizations of a l.w.v. $\Phi_{\mathbb{U}} \in \mathbb{U}$ as a harmonic 2-cochain in $C^2(\mathfrak{g}_-, \mathfrak{g})$.

n	$\mathbb{U}$	*Bi-grade*	*Lowest weight vector* $\Phi_{\mathbb{U}} \in \mathbb{U}$
2	$\mathbb{B}_4$	(2, 2)	$E^{2,1} \wedge E^{1,1} \otimes \mathsf{X} - \frac{1}{2} E^{2,1} \wedge E^{0,1} \otimes \mathsf{H} - \frac{1}{2} E^{1,1} \wedge E^{0,1} \otimes \mathsf{Y}$ $+ \sum_{a=1}^m \left(E^{2,1} \wedge E^{0,a} - E^{1,1} \wedge E^{1,a} + E^{0,1} \wedge E^{2,a} \right) \otimes e_a{}^1$
≥ 3	$\mathbb{A}_2^{\rm tr}$	(1, 1)	$\alpha \sum_{i=0}^n \left[\Phi^{2,i} + (\frac{n}{2} - i)\Phi^{1,i} - \frac{1}{2} i(n+1-i)\Phi^{0,i} \right]$ $+\beta \sum_{i=0}^n \left[(n+1-i)(\Phi^{i,0} - \Phi^{0,i}) + \Phi^{i,1} - \Phi^{1,i} \right]$, *where* $\Phi^{i,j} := \sum_{a=1}^m E^{i,1} \wedge E^{j,a} \otimes E_{i+j-1,a}$ *and* $\alpha = \frac{-6(n-1)(m+1)}{mn(n+1)+6} \beta$
≥ 2	$\mathbb{A}_2^{\rm tf}$	(1, 1)	$\sum_{j=0}^n [(n+1-j)\Phi^{0,j} + \Phi^{1,j}]$, *where* $\Phi^{i,j} := E^{i,1} \wedge E^{j,1} \otimes E_{i+j-1,m}$

n	$\mathbb{U}$	$\mathfrak{f}$	κ
2	$\mathbb{B}_4$	$\mathfrak{a}^{\Phi_{\mathbb{U}}}$	$\pm\Phi_{\mathbb{U}}$
≥ 2	$\mathbb{A}_2^{\mathrm{tf}}$	$\mathfrak{a}^{\Phi_{\mathbb{U}}}$	$\Phi_{\mathbb{U}}$
≥ 3	$\mathbb{A}_2^{\mathrm{tr}}$	$\mathfrak{ann}(\Phi_{\mathbb{U}})\oplus$ $\langle E_{n,a}, \ldots, E_{2,a},$ $E_{1,1} + (n-2)\zeta \mathsf{Z}_1, E_{1,b},$ $E_{0,1} + \zeta \mathsf{Y}, E_{0,b}\rangle$ $(b \neq 1)$	$\Phi_{\mathbb{U}} + \kappa_4$

$$\kappa_4 = \mu_1 E^{3,1} \wedge E^{0,1} \otimes \mathsf{X} + \mu_2 E^{2,1} \wedge E^{1,1} \otimes \mathsf{X} - \frac{\mu_1 + \mu_2}{2}\left(E^{2,1} \wedge E^{0,1} \otimes \mathsf{H} + E^{1,1} \wedge E^{0,1} \otimes \mathsf{Y}\right)$$
$$+ \mu_3 \sum_{a=1}^{m} \left(E^{2,1} \wedge E^{0,a} - E^{2,a} \wedge E^{0,1} + E^{1,a} \wedge E^{1,1}\right) \otimes {e_a}^1$$

Jacobi identity $\Rightarrow$ params $\zeta, \mu_1, \mu_2, \mu_3$ are uniquely determined fcns of (n, m).

To match this with aforementioned coordinate models, verify:

- κ_H is in the correct branch (use known invariants), and
- $\dim \mathfrak{f} = \mathfrak{S}$.