Category-colored operads

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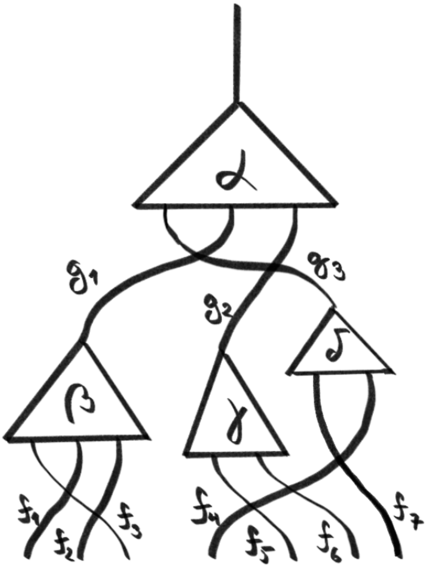
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Interpretation:

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- Symmetry isomorphisms

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The diagram says what to do with elements $x_1, \dots, x_7 \in X$ (of certain types) to produce a new element of X (of a certain type).

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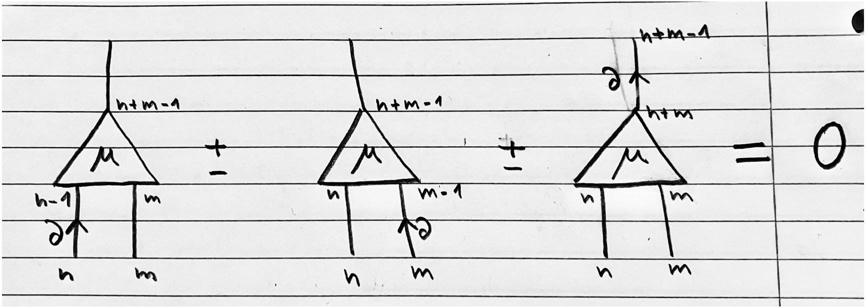
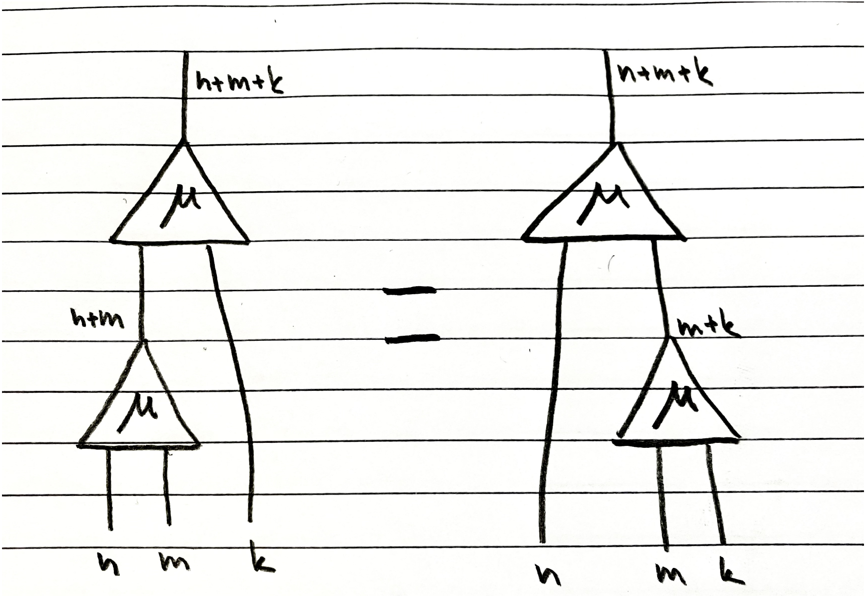


DGAs



DGAs

DGAs are algebras of a D-colored operad, where D is a (linear) category

$$\cdots \xrightarrow{\partial} (n+1) \xrightarrow{\partial} (n) \xrightarrow{\partial} (n-1) \xrightarrow{\partial} \cdots,$$

with $\partial\partial = 0$.

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- If an algebraic structure has unary operations, we can hide them into the coloring category and resolve the remaining operations.
- Goal = homotopy operads:

Classical (non-unital) operads are algebras of a (binary quadratic) category-colored operad.

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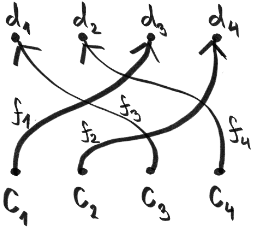

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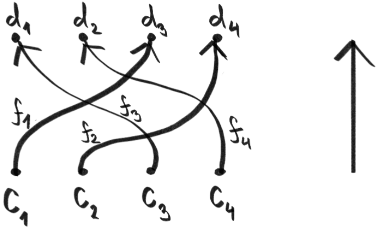

Definition

A C-collection is a functor

$$(\mathbb{S}C)^{op} \times C \xrightarrow{P} Vect_{\mathbb{K}}$$

Define C-operads

A C-collection is:

- Spaces

$$P_n\binom{c}{c_1 \cdots c_n}$$

- symmetric actions

$$P\binom{c}{c_1 \cdots c_n} \xrightarrow[\cong]{(-)\sigma} P\binom{c}{c_{\sigma(1)} \cdots c_{\sigma(n)}},$$

- C-actions

$$P\binom{c}{c_1 \cdots c_n} \xrightarrow{P\binom{f}{f_1 \cdots f_n}} P\binom{d}{d_1 \cdots d_n},$$

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A C-operad is a C-collection P with composition maps

$$P\begin{pmatrix} & c & \\ c_1 \cdots & c_i & \cdots c_n \end{pmatrix} \otimes P\begin{pmatrix} & c_i & \\ d_1 & \cdots & d_m \end{pmatrix} \xrightarrow{\circ_i} P\begin{pmatrix} & & c & & \\ c_1 \cdots & d_1 & \cdots & d_m & \cdots c_n \end{pmatrix},$$

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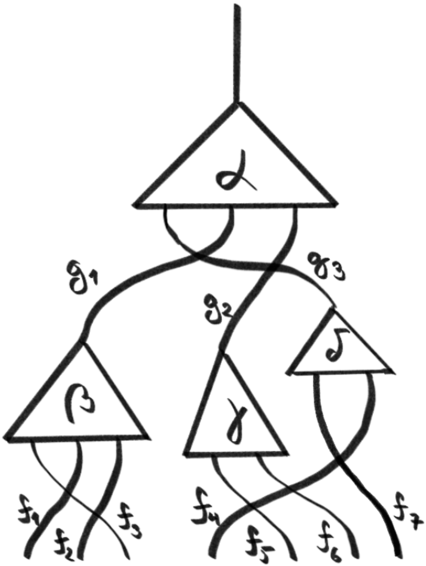
- 'natural' in the connecting variable c_i
- natural in $c_1, \ldots, d_1 \ldots, d_m, \ldots, c_n, c$
- associative
- respect symmetric actions



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$$P\begin{pmatrix} c \\ c_1 \cdots c_n \end{pmatrix} \otimes X(c_1) \otimes \cdots \otimes X(c_n) \xrightarrow{\alpha} X(c),$$

which respect the actions and composition of P.

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- (Binary quadratic) category-colored operad H, whose algebras are classical (non-unital) operads. The construction works in the context of operadic categories - can describe other operad-like structurtes.
- Characterisation of C-operad as *internal operads* in a specific categorical operad of functors (more category theory).

Result I

Classical operad:

spaces $O(n)$ with symmetric actions $O(n) \xrightarrow{(-)\sigma} O(n)$,
copmositions $O(n) \times O(m) \xrightarrow{\circ_i} O(m+n-1)$,
equivariant:

$$\circ_i \cdot ((-)\sigma \otimes (-)\tau) = (\sigma \circ_i \tau) \cdot \circ_{\sigma(i)},$$

associative:

$$\circ_i \cdot (\circ_j \otimes 1) = \begin{cases} \circ_{j+k-1} \cdot (\circ_i \otimes 1) & \text{if } 1 \leq i < j \leq n, \\ \circ_j \cdot (1 \otimes \circ_{i-j+1}) & \text{if } j \leq i < j+m, \\ \circ_j \cdot (\circ_{i-m+1} \otimes 1) & \text{if } j+m \leq i \leq n+m-1. \end{cases}$$

$\Rightarrow$ we want to abstract this data

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- The coloring category is Σ - objects natural numbers, morphisms bijections
- Formal operations

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- $H = F(G)/As$

$$[*_i] \circ_1 [*_j] \sim \tau([*_{j+k-1}] \circ_1 [*_i])$$
$$[*_i] \circ_1 [*_j] \sim [*_j] \circ_2 [*_{i-j+1}]$$
$$[*_i] \circ_1 [*_j] \sim \tau([*_j] \circ_1 [*_{i-m+1}])$$

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