On G-algebroids

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Motivation

Definition (partial)

An **algebroid** *is a tuple*

$$(\underbrace{E \to M}_{\text{v. bundle}},\ \underbrace{\rho\colon E \to TM}_{\text{v. bundle map}},\ \underbrace{[\,\cdot\,,\,\cdot\,]\colon \Gamma(E)\times\Gamma(E)\to\Gamma(E)}_{\mathbb{R}\text{-bilinear bracket}},\ \dots) \quad \textit{s.t.}\ \dots$$

Example (Lie algebroids)

- $(TM, \mathrm{id}, [\,\cdot\,,\,\cdot\,]_{\mathfrak{X}})$
- $(\mathfrak{g}, 0, [\,\cdot\,,\,\cdot\,]_{\mathfrak{g}})$

Example (Courant algebroids)

- $TM \oplus T^*M,\ \rho(X+\alpha) = X, \quad \langle X+\alpha, Y+\beta\rangle = \alpha(Y) + \beta(X)$
  $[X+\alpha, Y+\beta] = \mathcal{L}_X Y + \mathcal{L}_X \beta - i_Y d\alpha + H(X, Y, \,\cdot\,), \quad H \in \Omega^3_{cl}(M)$
- $\mathfrak{g}, \quad \rho = 0,\ [\,\cdot\,,\,\cdot\,]_{\mathfrak{g}},\ \langle\,\cdot\,,\,\cdot\,\rangle$ invariant inner product

Algebroids in exceptional generalised geometry ($n < 7$)

[Hull '07, Pacheco–Waldram '08, Coimbra–Strickland-Constable–Waldram '14, ...]

Symmetries of 11-dimensional supergravity on n-dimensional space

$E = TM \oplus \wedge^2 T^*M \oplus \wedge^5 T^*M, \quad \rho(X+\sigma_2+\sigma_5) = X,$ $E_{n(n)} \times \mathbb{R}^+$-structure

$$[X+\sigma_2+\sigma_5, X'+\sigma_2'+\sigma_5'] = \mathcal{L}_X(X'+\sigma_2'+\sigma_5') - i_{X'}(d\sigma_2+d\sigma_5) - \sigma_2' \wedge d\sigma_2$$

Type IIA/B supergravity on $n-1$-dimensional space

$$E = TM \oplus T^*M \oplus (\wedge^{\text{even/odd}} T^*M) \oplus \wedge^5 T^*M, \quad \rho(X + \dots) = X$$

$$[\,\cdot\,,\,\cdot\,] = \dots\,, \quad E_{n(n)} \times \mathbb{R}^+\text{-structure}$$

Bosonic field content: $E_{n(n)} \times \mathbb{R}^+ \rightsquigarrow K(E_{n(n)} \times \mathbb{R}^+)$, Ricci $= 0$

Common structure: $E \cong_{loc} \mathcal{U} \times E_0, \quad [u,\,\cdot\,] = \rho(u) - (\rho^t du)_{ad}$

Questions

Question 1: What is the corresponding class of algebroids?

Question 2: How to recognise the previous examples inside that class?[1]

Leibniz parallelisations:

global $E_{n(n)} \times \mathbb{R}^+$-frame e_i with $[e_i, e_j] = c_{ij}^k e_k$ (c_{ij}^k constant)
(correspond to consistent truncations with maximal supersymmetry [Lee–Strickland-Constable–Waldram '17])

Question 3: Which (compact) M admit such parallelisations?

[1] from quite a long way away [Chapman–Cleese–Gilliam–Idle–Jones–Palin '69]

Algebraic prelude

Definition

An **admissible group data set** *is given by a Lie group G, two reps E, N, and equivariant maps $S^2E \to N$, $N \to S^2E$. (Denote by (G, E, N).)*

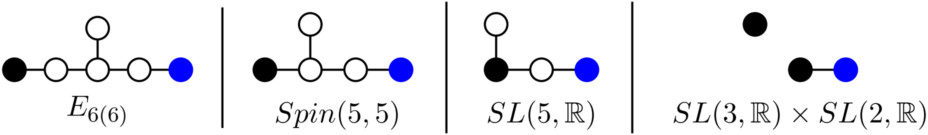
Definition

A vector subspace $V \subset E$ is called

- **isotropic** *if* $(V \otimes V)_N = 0$
- **coisotropic** *if* $(V^{ann} \otimes V^{ann})_{N^*} = 0$

Example

- $(GL(n, \mathbb{R}), \mathbb{R}^n, 0)$
- $(O(n, n), \mathbb{R}^{2n}, \mathbb{R})$
- $(E_{n(n)} \times \mathbb{R}^+, E, N)$



G-algebroids

Fix an admissible group data set.

Definition

G-**algebroid** *is a tuple*

$$(\underbrace{E \to M}_{\text{assoc.v.b.}},\ \underbrace{N \to M}_{\text{assoc.v.b.}},\ \underbrace{\rho\colon E \to TM}_{\text{v.b.map}},\ [\,\cdot\,,\,\cdot\,] \text{ on } E,\ \mathcal{D}\colon \Gamma(N) \to \Gamma(E)) \text{ s.t.}$$

$$[u,[v,w]] = [[u,v],w] + [v,[u,w]] \qquad [u,fv] = f[u,v] + (\rho(u)f)v$$

$$[u,v] + [v,u] = \mathcal{D}(u \otimes v)_N \qquad \mathcal{D}(fn) = f\mathcal{D}n + (\rho^t df \otimes n)_E$$

$$[u,\,\cdot\,] \text{ preserves the } G\text{-structure}$$

Example

- $(GL(n,\mathbb{R}), \mathbb{R}^n, 0) \rightsquigarrow$ Lie algebroid
- $(O(n,n), \mathbb{R}^{2n}, \mathbb{R})$ & $\mathcal{D} = \rho^t \circ d \rightsquigarrow$ Courant algebroid
- $(E_{n(n)} \times \mathbb{R}^+, E, N) \rightsquigarrow$ **exceptional algebroid**

Exactness

Chain complex: $T^*M \otimes N \xrightarrow{\rho^*} E \xrightarrow{\rho} TM \to 0$

Definition

A G-algebroid is **exact** *if this is an exact sequence.*

Example

- A Lie algebroid is exact iff $E \cong TM$.
- There is a unique exact Lie algebroid over M.

Example

- A Courant algebroid is exact iff it is exact in the sense of Ševera ($E \cong TM \oplus T^*M$).
- Exact CA's classified by $H^3(M)$ [Ševera '98].

Exact exceptional algebroids

Theorem ([Bugden–Hulik–V–Waldram '21, Bugden–Hulik–V–Waldram '22, Hulik–V '22])

An exact exceptional algebroid is locally of one of the two forms

$$\dim M = n \quad \rightsquigarrow \quad TM \oplus \wedge^2 T^*M \oplus \wedge^5 T^*M \qquad \textit{(M-theory)}$$

$$\dim M = n-1 \quad \rightsquigarrow \quad TM \oplus T^*M \oplus (\wedge^{\mathrm{odd}} T^*M) \oplus \wedge^5 T^*M \quad \textit{(IIB)}$$

A $\underbrace{\text{minimally nonexact}}_{\dim H^1=1,\, \dim H^2=0}$ algebroid is locally of the form

$$\dim M = n-1 \quad \rightsquigarrow \quad TM \oplus T^*M \oplus (\wedge^{\mathrm{even}} T^*M) \oplus \wedge^5 T^*M \quad \textit{(IIA)}$$

Global:

- Problem with global identification as bundles with G-structures, e.g. $E \cong TM \oplus \wedge^2 T^*M \oplus \wedge^5 T^*M$ globally.
- If assume the identification, can do full classification, bracket can be deformed by:

Deformations of the bracket

M-theory $(\dim M \leq 6)$

$$F_1 \in \Omega^1 \quad F_4 \in \Omega^4$$

$$dF_1 = 0 \qquad dF_4 + F_1 \wedge F_4 = 0$$

Type IIB $(\dim M \leq 5)$

$$F_1 \in \Omega^1 \otimes \mathfrak{gl}(2,\mathbb{R}) \quad F_3 \in \Omega^3 \otimes \mathbb{R}^2 \quad F_5 \in \Omega^5$$

$$dF_1 + \tfrac{1}{2}[F_1, F_1] = 0 \qquad dF_3 + F_1 \wedge F_3 = 0$$

Type IIA $(\dim M \leq 5)$ (cf. [Romans '86, Howe–Lambert–West '98])

$$F_0 \in \Omega^0 \quad G_0 \in \Omega^0 \quad F_1 \in \Omega^1 \quad G_1 \in \Omega^1 \quad F_2 \in \Omega^2 \quad G_3 \in \Omega^3 \quad F_4 \in \Omega^4$$

$$F_0 G_0 = 0 \qquad dF_0 - 2F_0 G_1 - F_0 F_1 = 0 \qquad dG_0 + G_1 G_0 = 0$$

$$dF_1 - F_2 G_0 = 0 \qquad dG_1 = 0 \qquad dF_2 - F_2 \wedge G_1 - F_0 G_3 = 0$$

$$dG_3 - G_3 \wedge F_1 - F_4 G_0 + G_1 \wedge G_3 = 0 \qquad dF_4 + F_4 \wedge F_1 - F_2 \wedge G_3 = 0$$

Leibniz parallelisations [Inverso '17, BHVW '21, BHVW '22, HV '22]

Recall: global G-frame e_i with $[e_i, e_j] = c_{ij}^k e_k$ (c_{ij}^k constant), M compact

Theorem ([Li-Bland–Meinrenken '08])

Leibniz parallelisable Courant algebroids correspond to pairs $(\mathfrak{g}, \mathfrak{h})$, *with* $\mathfrak{g}$ *a Lie algebra with invariant inner product and* $\mathfrak{h}$ *a coisotropic subalgebra.*

Theorem ([Bugden–Hulik–V–Waldram '21, Bugden–Hulik–V–Waldram '22, Hulik–V '22])

Leibniz parallelisable exceptional algebroids in the M-theory, IIB/A case correspond to pairs (E, V), *with* E *an exceptional algebroid over a point,* $V \subset E$ *a coisotropic subalgebra Lie algebra with*

- $[x, x] \in V, \quad \forall x \in E$
- $\operatorname{codim} V \in \{n-1, n\}$
- $\operatorname{Tr}_E \operatorname{ad}_v = \frac{\lambda}{\lambda+1} \operatorname{Tr}_V \operatorname{ad}_v \quad \forall v \in (V^{ann} \otimes N)_E \subset V \quad (\lambda := \frac{\dim E}{9-n})$

Poisson–Lie U-duality [Sakatani '20, Malek–Thompson '20]**:** $V, V' \subset E$ different

Examples

[Lee–Strickland-Constable–Waldram '17, Hohm–Samtleben '15, Inverso '17]

- E abelian, $V \subset E$ coisotropic subspace $\rightsquigarrow T^n$
- $\mathfrak{k}$ Lie algebra, $E = \mathfrak{k} \oplus \wedge^2 \mathfrak{k}^* \oplus \wedge^5 \mathfrak{k}^*$, $V = \wedge^2 \mathfrak{k}^* \oplus \wedge^5 \mathfrak{k}^* \rightsquigarrow$ group K
- $E = \mathfrak{so}(5)$, $V = \mathfrak{so}(4) \rightsquigarrow S^4$ (M)
- $E = \mathfrak{so}(6) \mathbin{\rightarrowtail} (\mathbf{6} \oplus \mathbf{6})$, $V = \mathfrak{so}(5) \mathbin{\rightarrowtail} (\mathbf{6} \oplus \mathbf{6}) \rightsquigarrow S^5$ (IIB)
- Wigner–İnönü contraction $\rightsquigarrow E = (\mathfrak{so}(5) \ltimes \mathbf{5}) \mathbin{\rightarrowtail} (\mathbf{6} \oplus \mathbf{6})$
  - $V = (\mathfrak{so}(4) \ltimes \mathbf{4}) \mathbin{\rightarrowtail} (\mathbf{6} \oplus \mathbf{6}) \rightsquigarrow S^4 \times \mathbb{R}$ (IIB)
  - $V = \mathfrak{so}(5) \mathbin{\rightarrowtail} (\mathbf{6} \oplus \mathbf{6}) \rightsquigarrow \mathbb{R}^5$ (IIB)
- $E = \mathfrak{su}(2) \mathbin{\rightarrowtail} \mathbf{3}$
  - $V = \mathbf{3} \rightsquigarrow S^3$ (M)
  - $V = \mathfrak{u}(1) \mathbin{\rightarrowtail} \mathbf{3} \rightsquigarrow S^2$ (IIA)

Definition ($\mathfrak{g} \mathbin{\rightarrowtail} R$)

$$[x + r, y + s] := [x, y]_{\mathfrak{g}} + x \cdot s$$

Outlook

- higher rank
  - $E_{7(7)}$-case: breaks $[x,y]+[y,x]=\mathcal{D}(x\otimes y)_N$
  - $E_{8(8)}$-case: (untwisted) bracket not unique
  - $E_{9(9)}$-case: Lie algebra not finite-dimensional
  - $E_{10(10)}$-case: Lie algebra not affine
  - $E_{11(11)}$-case: hard

- global classification, relation to non-abelian gerbes

- non-maximally supersymmetric consistent truncations
  $\rightsquigarrow$ Sasaki–Einstein spaces, ...