Polyakov conjecture and the Pseudosphere

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Mostly derived from results of: O. Hulík, A. Polyakov, T. Prochazka, L. Takhtajan, T. Hartman, A. and Al. Zamolodchikov, P. Zograf

Geometry and Physics

A tale of two monodromy problems:

- Constructing conformal blocks of 2d CFT at large central charge
- The uniformization of the punctured sphere $S^2\backslash\{z_n\}$.

What is known: there exists a unique solution to the Liouville equation that satisfies prescribed boundary conditions at the punctures.

The goal: Discuss an existence criterion for a Liouville solution on the pseudosphere (i.e. Poincaré disk with a boundary condition)

Conformal blocks

2d conformal field theory (CFT): a class of quantum mechanical models of local operators whose correlation functions are invariant under Möbius transformations

$$\langle O_1(z_1)...O_n(z_n)\rangle = \langle O_1'(z_1')...O_n'(z_n')\rangle, \tag{1}$$

where

$$O_i'(z_i') = \left(\frac{\partial z_i'}{\partial z_i}\right)^{-h_i} O_i(z_i), \quad z_n \in \mathbb{C} \cup \{\infty\}. \tag{2}$$

Infinitesimally invariant under (anti-)holomorphic coordinate transformations (generated by two copies of the Virasoro algebra):

$$[L_n, L_m] = L_{n-m} + \frac{c}{12} n(n^2-1)\delta_{n,-m} \tag{3}$$

with central charge c

Conformal blocks

state-operator map: $\exists$ an isomorphism from: states $|\Psi\rangle \to$ local operators $O_\Psi(z)$

$$O_\Psi(0)|0\rangle = |\Psi\rangle \tag{4}$$

Operator product expansion (OPE):

$$O_1(0)O_2(z)|0\rangle = \sum_k c_k(z)|\phi_k\rangle = \sum_k \frac{c_k}{z^{h_1+h_2-h_k}}\phi_k(0)|0\rangle \tag{5}$$

where h_i are eigenvalues under the rescaling generator

Holds as an operator equation

$$O_1(0)O_2(z) = \sum_k \frac{c_k}{z^{h_1+h_2-h_k}}\phi_k(0) \tag{6}$$

Conformal blocks

Operator product expansion (OPE):

$$O_1(0)O_2(z)|0\rangle = \sum_k c_k(z)|\phi_k\rangle = \sum_k \frac{c_k}{z^{h_1+h_2-h_k}} O_k(0)|0\rangle \tag{7}$$

Holds as an operator equation

$$O_1(0)O_2(z) = \sum_k \frac{c_k}{z^{h_1+h_2-h_k}} O_k(0) \tag{8}$$

Compute correlators by subsequent OPEs

$$\langle O_1 O_2 O_3 O_4 \rangle \tag{9}$$

Conformal block: Restrict the sum of the OPE to the states associated to a single Virasoro representation.

$$\mathcal{F}(z_i, h_i|h) = \frac{(z_1-z_3)^{h-h_1-h_3}}{(z_2-z_4)^{h_2+h_4-h}} \sum_{k=0}^{\infty} \beta_k x^k, \quad x = \frac{(z_1-z_4)(z_3-z_2)}{(z_1-z_3)(z_4-z_2)} \tag{10}$$

Null vector decoupling

Conformal blocks at large central charge c:
Put together a null state $\langle \text{null}|\text{null}\rangle = 0$, achieved by

$$|\text{null}\rangle = \left(L_{-2} - \frac{3}{2}(2h_\psi + 1)L_{-1}^2\right)|\psi\rangle \quad c = 2h_\psi \frac{5 - 8h_\psi}{2h_\psi + 1} \tag{11}$$

Inserting the operator $\psi(0)|0\rangle = |\psi\rangle$ yields a **shortening condition**:

$$\psi''(z) + \frac{6}{c}T(z)\psi(z) = 0 \tag{12}$$

where

$$\psi(z) = \frac{\langle \psi(z) O_1 ... O_n\rangle}{\langle O_1 ... O_n\rangle}, \quad T(z) = \frac{\langle T(z) O_1 ... O_n\rangle}{\langle O_1 ... O_n\rangle} \tag{13}$$

Monodromy problem

At $c >> 1$ and $h_i \sim c$ conformal blocks exponentiate:

$$\mathcal{F}(z_i, h_i|h) = e^{-\frac{c}{12} f(z_i, \frac{h_i}{c}|h)} \tag{14}$$

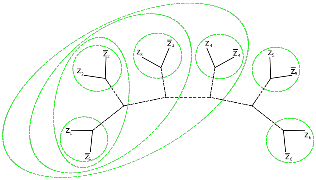
Virasoro Ward identity decrees:

$$T(z) = \sum_{i=1}^{n} \left(\frac{h_i}{(z-z_i)^2} - \frac{\frac{c}{12}\partial_{z_i} f(z_i, h_i|h)}{z - z_i} \right) \tag{15}$$

Our ODE is multi-valued, $\psi'' + \frac{6}{c} T(z)\psi = 0$

Recipe:

- solve second-order ODE
- determine monodromies as function of $\partial_{z_i} f$
- Select out the monodromies project out our restricted OPE
- solve for $\partial_{z_i} f$ and integrate



Uniformization of Riemann surfaces

Theorem

Uniformization theorem: *Any compact oriented genus > 2 surface S is univerally covered by the upper half-plane*

Method of Poincaré: *Construct a line element $ds^2 = e^{-\phi} dz d\bar{z}$ on S that has constant negative scalar curvature everywhere and respects the identifications imposed by non-contractable cycles on S.*

$$R = -1 \quad \Leftrightarrow \quad \partial\bar{\partial}\phi(z,\bar{z}) + e^{-\phi(z,\bar{z})} = 0 \tag{16}$$

Local solution: $e^{-\phi} = \frac{|\partial f(z)|^2}{(1-|f(z)|^2)^2}$ for any holomorphic $f(z)$.

Note: $e^{-\phi}$ invariant under $f \to \frac{af+b}{cf+d}$ where $ad - bc = 1$, $a, b, c, d \in \mathbb{R}$.

uniformization of the punctured sphere

Uniformizable surface: the finitely punctured sphere $S^2 \backslash \{z_n\}$
Boundary conditions at the (elliptic) punctures

$$\phi(z \to z_i) = -2\alpha_i \log|z - z_i| + \mathcal{O}(1) \tag{17}$$

Easily shown: $T(z) \equiv \partial^2\phi - \frac{1}{2}(\partial\phi)^2$ is meromorphic with second-order poles:

$$T(z) = \sum_{i=1}^{n} \left(\frac{\frac{1}{2}\alpha_i(2-\alpha_i)}{(z-z_i)^2} + \frac{c_i}{z-z_i} \right) \tag{18}$$

Classical result: if $f(z)$ solves

$$S[f, z] = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 = \frac{1}{2}T(z) \tag{19}$$

then $\phi(z)$ is a solution to the Liouville equation that satisfies the correct boundary conditions.

Monodromy problem

If $\psi_1(z)$ and $\psi_2(z)$ are linearly independent and solve

$$\psi''(z) + \frac{1}{2}T(z)\psi(z) = 0 \tag{20}$$

then $f(z) = \psi_1(z)/\psi_2(z)$ solves (19) above

The problem: A priori, $f(z)$ is a multi-valued function on the punctured sphere.
Under a closed loop:

$$\psi_1(z) \to a\psi_1(z) + b\psi_2(z), \tag{21}$$
$$\psi_2(z) \to c\psi_1(z) + d\psi_2(z), \tag{22}$$

Translates into

$$f(z) \to \frac{af(z) + b}{cf(z) + d} \tag{23}$$

The metric is only single-valued on $S^2 \backslash \{z_n\}$ if

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}) \tag{24}$$

Polyakov conjecture

Two similar monodromy problems:

- We fix the the conjugacy class of M around a few pairs (CFT)
- We fix all the monodromy matrices to fall within $SL(2,\mathbb{R})$ (Uniformization)

In either case we solve for the residues of the first-order poles

Polyakov conjecture: The Liouville action functional evaluated on the saddle-point is the generating functional for the accessory parameters

$$S_L[\phi] = \frac{1}{2}\int\int_{S^2\setminus\{z_n\}} \left(|\partial\phi|^2 + e^{\phi}\right) dz \wedge d\bar{z} + \frac{i}{2}\sum_{i=1}^{n} \alpha_i \oint_{D_i} \phi \left(\frac{d\bar{z}}{\bar{z}-\bar{z}_i} - \frac{dz}{z-z_i}\right) \quad (25)$$

such that

$$c_i = \frac{1}{2\pi}\frac{\partial S_L[\phi_{\text{saddle}}]}{\partial z_i} \quad (26)$$

Liouville theory on the psuedosphere

Reasonable question: *"Gideon, this is old stuff, where is the new stuff!?"*

Instead of $S^2 \backslash \{z_n\}$ the punctured psuedosphere $UHP \backslash \{z_n\}$
ZZ-boundary condition on the real axis:

$$\phi(z \to \mathrm{Im}(z) = 0) = \log(2\mathrm{Im}(z)) \tag{27}$$

Physical applications:

- CFTs on a background with a reflecting boundary
- Holographic bulk gravity in a 3d asymptotically AdS universe

Doubling trick

Consequence of $\phi(z \to \mathrm{Im}(z) = 0) = \log(2\mathrm{Im}(z))$:

$$T(z)|_{\mathrm{Im}(z)=0} \in \mathbb{R} \tag{28}$$

Continue $T(z)$ to the full (punctured) Riemann sphere by Schwartz reflection

$$T(z) = \bar{T}(\bar{z}) \tag{29}$$

Implies

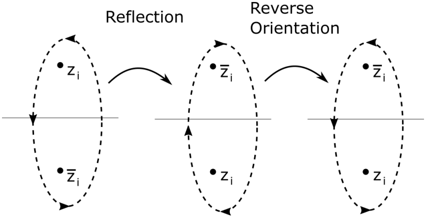
$$c_{z_i} = c_{\bar{z}_i} \quad \Rightarrow \frac{\partial S_L[\phi_{\mathrm{saddle}}]}{\partial z_i} = \frac{\partial S_L[\phi_{\mathrm{saddle}}]}{\partial \bar{z}_i} \tag{30}$$

Hence up to an irrelevant overall constant

$$S_L[\phi_{\mathrm{saddle}}] \in \mathbb{R} \tag{31}$$

Existence criterion

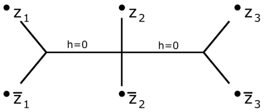
One can show: $T(z) = \bar{T}(\bar{z})$ implies $\exists\, f(z)$ such that $f(z) = \bar{f}(\bar{z})$



Hence $M = M^{-1}$ which implies $M = \mathbb{I}_{2\times 2}$.

Existence criterion

Consequence: the doubled psuedosphere Liouville solution computes a conformal block with exchanged vacuum states



From generic CFT considerations

$$\mathcal{F}(z_i, h_i|h=0) = \frac{(z_1-\bar{z}_1)^{2h_1}}{(z_2-\bar{z}_2)^{2h_2}(z_3-\bar{z}_3)^{2h_3}} \sum_{\{k_1,k_2,k_3\}=0}^{\infty} \beta_{k1,k2,k3} x_1^{k_1} x_2^{k_2} x_3^{k_3}, \tag{32}$$

With the cross ratios

$$x_1 = \frac{(\bar{z}_2-\bar{z}_3)(z_1-\bar{z}_1)}{(\bar{z}_1-\bar{z}_2)(\bar{z}_3-z_1)}, \quad x_2 = \frac{(\bar{z}_2-\bar{z}_3)(z_2-\bar{z}_1)}{(\bar{z}_1-\bar{z}_2)(\bar{z}_3-z_2)}, \quad x_3 = \frac{(\bar{z}_2-\bar{z}_3)(z_3-\bar{z}_1)}{(\bar{z}_1-\bar{z}_2)(\bar{z}_3-z_3)} \tag{33}$$

Burden for reality lies in the cross-ratios

What went wrong

Conclusion: Only when the anharmonic ratios are real do we not obtain a contradiction

Take-away message Existence criterion: while the doubled psuedosphere yields a unique solution to the uniformization problem on the sphere, this solution will only satisfy the ZZ-boundary condition if the anharmonic ratios of the puncture locations are real-valued

Thank you for you attention