Compact and Non-compact Parabolic Space Forms

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Let G be a Lie group and H a Lie subgroup.

Definition

A Cartan geometry $(\mathcal{G}, M, \omega)$ of type (G, H) is a principal H-bundle $\mathcal{G} \to M$ endowed with a Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$, i.e.:

(i) ω is H-equivariant;

(ii) $\omega(\zeta_Y) = Y$, $\forall Y \in \mathfrak{h}$, where $\zeta_Y(u) = \frac{d}{dt}\big|_{t=0} u \cdot \exp(tY)$;

(iii) $\omega_u : T_u\mathcal{G} \to \mathfrak{g}$ is a linear isomorphism $\forall u \in \mathcal{G}$.

The curvature of ω is $d\omega + \frac{1}{2}[\omega, \omega] \in \Omega^2(\mathcal{G}, \mathfrak{g})$. The isomorphism (iii) identifies this 2-form with the curvature function $\kappa : \mathcal{G} \to \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$, which is horizontal, so $\kappa : \mathcal{G} \to \Lambda^2(\mathfrak{g}/\mathfrak{h})^* \otimes \mathfrak{g}$.

A Cartan geometry is flat if $\kappa \equiv 0$.

Parabolic subgroups

Let G be a (real or complex) simple Lie group with Lie algebra $\mathfrak{g}$.

Definition

A minimal parabolic subalgebra $\mathfrak{b}$ is the normalizer of a non-abelian maximal nilpotent subalgebra of $\mathfrak{g}$. In the complex case, the minimal parabolic is called the Borel subalgebra, and is a maximal solvable subalgebra of $\mathfrak{g}$.

Definition

A parabolic subalgebra $\mathfrak{p}$ is a proper subalgebra of $\mathfrak{g}$ such that $\mathfrak{b} \subset \mathfrak{p}$ (for some $\mathfrak{b}$). A parabolic subgroup is any (not necessarily connected) subgroup with $\mathrm{Lie}(P) = \mathfrak{p}$.

Parabolic subalgebras are (up to conjugacy) in bijective correspondance with:

1. The positive parts of nontrivial $\mathbb{Z}-$gradings $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_k \oplus \cdots \oplus \mathfrak{g}_1$.
2. Markings on the Dynkin (or Satake) diagram of $\mathfrak{g}$.

More on markings

Complex Borel subalgebras can be found by picking a hyperplane in the dual space $\mathfrak{h}^*$ to a Cartan subalgebra $\mathfrak{h}$, which does not pass through any roots. Label the roots in the two half-spaces as positive and negative. Then $\mathfrak{b}$ is generated by all positive root vectors together with $\mathfrak{h}$.

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The complex parabolic subalgebras $\mathfrak{p}$ can be classified by which negative root vectors are not included. This is described by crossing nodes on Dynkin diagrams.

- In the real case, the parabolics are those which admit a real slice with the real form in the complexification of $\mathfrak{g}$. This is described by crosses on the Satake diagram, according to some rules.

Parabolic Geometries

A parabolic geometry is a Cartan geometry modelled on G/P where G is a semi-simple Lie group and P is a parabolic subgroup. We say that this is

1. regular, if $\kappa(\mathfrak{g}^i, \mathfrak{g}^j) \subset \mathfrak{g}^{i+j+1}$, $\forall i, j < 0$;
2. normal, if $\partial^* \kappa = 0$, where ∂^* is the Kostant codifferential.

The main invariant of a (normal, regular) parabolic geometry is the harmonic curvature

$$\kappa_H \in C^\infty(\mathcal{G}, H^2(\mathfrak{g}_-, \mathfrak{g}))$$

We have that dim G is always the maximal automorphism dimension, and G/P is the maximal model, also called the flat model as it's locally characterized by $\kappa_H = 0$.

Let G be a real or complex simple Lie group, and P a parabolic subgroup.

Definition (Sharpe)

A (parabolic) Cartan geometry $(M, \mathcal{G}, \omega)$ of type (G, P) will be called a space form if it is complete and flat.

In the setting of parabolic geometries, there are two distinguished choices of space forms (with fixed $(\mathfrak{g}, \mathfrak{p})$-data):

- The universal space form $M = \widetilde{G/P}$, i.e. the simply connected cover of all other space-forms,
- The (real or complex) algebraic model, given by taking $G = \mathrm{Int}(\mathfrak{g}_{\mathbb{C}}) \cap \mathrm{Aut}(\mathfrak{g})$, $P = N_G(\mathfrak{p})$, $M = G/P$, called the *generalized flag manifold* of (G, P).

Remark

Other parabolic space forms are also sometimes referred to as generalized flag manifolds.

Complex Parabolic Space Forms

Let G be a complex connected simple Lie group with parabolic subgroup P. We have that

- The P-principal bundle $G \to G/P$ with the Maurer-Cartan form ω is a space form.
- The homogeneous space $M = G/P$ is a compact Kähler manifold and a projective algebraic variety.

In particular every complex parabolic space form is compact.

Real Parabolic Space Forms

Let G be a real connected simple Lie group with parabolic subgroup P.

- The P-principal bundle $G \to G/P$ with the Maurer-Cartan form ω is a space form.
- The universal space form is not necessarily compact.
- The generalized flag manifold is compact and a real projective variety.

Example

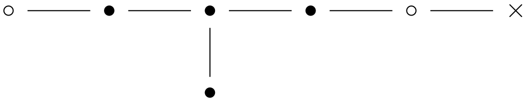
$G = Sp(2n, \mathbb{R})$, $n \geq 1$, $P = P_n$. The space forms: Lagrangian Grassmannian $\Lambda_n = G/P$, its oriented version $\Lambda_n^+ = G/P_o$ and the universal cover $\tilde{\Lambda}_n$. The former two are compact while the latter is non-compact. The isomorphism $H^1(\Lambda_n) = \pi_1(\Lambda_n) \simeq \mathbb{Z}$ is known as the Maslov index.

Note that if the universal space form is compact, all space forms are compact.

Theorem (Main theorem)

Let G be a simple real Lie group and $P = P_I$ its parabolic subgroup. Then the universal space form $\widetilde{G/P}$ is non-compact only in the following cases:

- $G = SU(n,n)$, $n-1 \in I$ for $1 < n$;
- $G = SO(2,n)$, $1 \in I$ for $2 < n$;
- $G = Sp(2n,\mathbb{R})$, $n \in I$ for $1 \leq n$;
- $G = SO^*(2n)$, $n \in I$ for $2 < n \in 2\mathbb{Z}$;
- $G = E_7^{(-25)}$, $7 \in I$.



For each group, there is just one node which determines compactness.

From now on we denote by G the real algebraic Lie group introduced before and by $\rho : \widetilde{G} \to G$ its universal cover. Let $\hat{P} = \rho^{-1}(P)$ and let $\hat{P}_o$ be its connected component of unity. Let $\Gamma = \pi_1(G)$.

Lemma

We have $\widetilde{G/P} = \widetilde{G}/\hat{P}_o$.

Proof.

The equality $\pi_1(\widetilde{G}/\hat{P}) = |\hat{P}/\hat{P}_o|$ follows from the long exact homotopy sequence:

$$0 = \pi_1(\widetilde{G}) \to \pi_1(\widetilde{G}/\hat{P}) \stackrel{\sim}{\to} \pi_0(\hat{P}) \to \pi_0(\widetilde{G}) = 0.$$

If we change $\hat{P}$ to $\hat{P}_o$ here, we get the required claim. $\square$

Lemma

We have $|\mathcal{Z}(\widetilde{G})| < \infty \Leftrightarrow |\Gamma| < \infty$.

Proof.

This follows from the fact that for algebraic groups G, $\mathcal{Z}(G)$ is finite, and the following short exact sequence:

$$0 \to \pi_1(G) \to \mathcal{Z}(\widetilde{G}) \to \mathcal{Z}(G) \to 0$$

□

Let φ be the map of fundamental groups induced by the inclusion $P_o \subset G$, which is a part of the long exact sequence

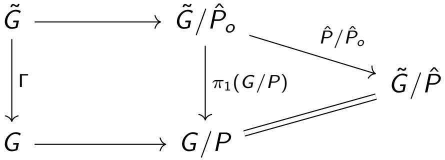
$$\pi_1(P) = \pi_1(P_o) \xrightarrow{\varphi} \pi_1(G) \longrightarrow \pi_1(G/P_o) \to \pi_0(P_o) = 0.$$

Lemma

We have that $\widetilde{G/P}$ is compact $\Leftrightarrow [\Gamma : \varphi(\pi_1(P))] < \infty$.

Proof.

This follows from the following commutative diagram of coverings:



$\widetilde{G}/\hat{P}_o$ is compact $\Leftrightarrow [\hat{P} : \hat{P}_o] < \infty \Leftrightarrow |\pi_1(G/P)| < \infty \Leftrightarrow [\Gamma : \varphi(\pi_1(P))]$ is finite. $\square$

Candidate groups

By Lemma 2, groups with noncompact spaceforms must have $|\pi_1(G)| = \infty$. Since $\pi_1(G) = \pi_1(K)$, we essentially need to look for groups G where $S^1 \subset K$ is normal in K. This can be found from the list of non-compact symmetric spaces.

(A) $G = SU(p, q)$, $K = S[U(p)U(q)]$ $(p, q > 0)$,

(B) $G = SO(p, q)$, $K = S[O(p)O(q)]$ $(p = 2 \vee q = 2)$,

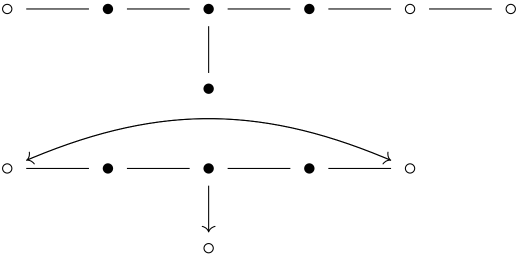
(C) $G = Sp(2n, \mathbb{R})$, $K = U(n)$,

(D) $G = SO^*(2n)$, $K = U(n)$,

(E) $G = E_6^{(-14)}$, $K = SO(2)Spin(10)$ and $G = E_7^{(-25)}$, $K = SO(2)E_6^c$.

In each case $\pi_1(G)/\operatorname{Tors}(\pi_1(G)) = \mathbb{Z}$, and so by Lemma 3, the universal space form is non-compact iff $\varphi(\widetilde{\pi_1(P)})$ is finite. Also note that if $\widetilde{G/P}$ is non-compact and $Q \subset P$ then $\widetilde{G/Q}$ is also non-compact. Similarly, if $\widetilde{G/Q}$ is compact then the same is true for $\widetilde{G/P}$.

Satake diagrams for exceptional candidates



Cases

We will consider the simplest case B) and the most complicated case E).

- Case B)

Let $G = SO(2, q), q > 3$. The parabolic P_1 contracts to $CO(1, q-1)$, which has finite fundamental group.

- For $P = P_1$ and $P = P_{1,2}$, the universal space form $\widetilde{G/P}$ is non-compact.

On the other hand, for $P = P_2$ we have $G_0 = SL(2, \mathbb{R})\mathbb{R}_{\times}SO(n-2)$ and the infinite part of the fundamental group is generated by $SO(2) \subset SL(2, \mathbb{R})$ that is nontrivial in $\pi_1(G)$

- For $P = P_2$, $\widetilde{G/P}$ is compact.

- Case E)

Let $G = E_6^{(-14)}$. We have $K = SO(2)\,\mathrm{Spin}(10)$ and $\pi_1(G) = \mathbb{Z}$. We let $P = P_{1,2,6} = B$, the Borel. This gives $G_0 = \mathbb{R}_\times \mathbb{C}_\times SU(4)$. There is a symmetric space decomposition,

$$\mathfrak{e}_6^{(-14)} = \mathfrak{so}(2) \oplus \mathfrak{so}(10) \oplus \mathbb{S}_{10},$$

and $\mathfrak{so}(2)$ acts as a linear complex structure J on $\mathbb{S}_{10}$.

- If this J has a non-trivial component in $\mathbb{C} \subset \mathfrak{g}_0$, then all space forms are compact.

Branch the grading by $\mathfrak{su}(4)$.

$$\mathfrak{g} = \mathfrak{g}_{-4} \oplus \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3 \oplus \mathfrak{g}_4,$$

$$\mathfrak{g}_0 = \mathbb{R} \oplus \mathbb{C} \oplus \mathfrak{su}(4),\ \mathfrak{g}_{\pm 1} = \mathbb{R}^6 \oplus \mathbb{C}^4,\ \mathfrak{g}_{\pm 2} = \mathbb{R} \oplus \mathbb{C}^4,\ \mathfrak{g}_{\pm 3} = \mathbb{R}^6,\ \mathfrak{g}_{\pm 4} = \mathbb{R}.$$

and the symmetric decomposition:

$$\mathfrak{so}(10)|_{\mathfrak{su}(4)} = \Lambda^2\mathbb{R}^{10}|_{\mathfrak{su}(4)} = 2\mathbb{R}+2\mathbb{C}^4+2\mathbb{R}^6+\mathfrak{su}(4), \quad \mathbb{S}_{10}|_{\mathfrak{su}(4)} = 4\mathbb{R}+2\mathbb{C}^4+2\mathbb{R}^6.$$

- Some reasoning will then give the result that $\mathfrak{u}(1) \subset \mathbb{C} \subset \mathfrak{g}_0$ generates a non-trivial loop in G and so all space forms are compact.