Correspondences of Quantum L_∞ Algebras

Joint work with Branislav Jurčo, Ján Pulmann

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Winter School of Geometry and Physics

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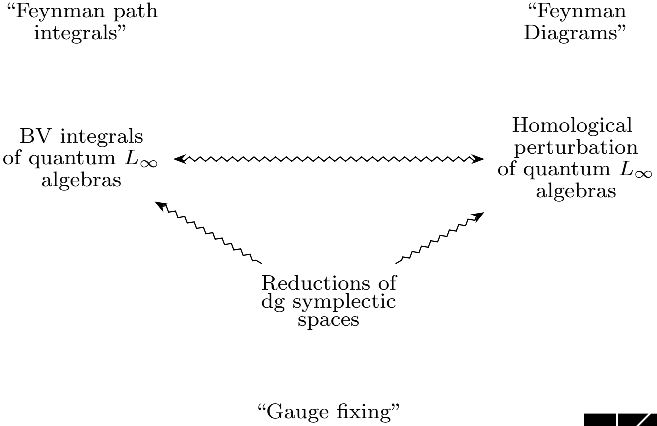

Objects: A **dg symplectic space** is a $\mathbb{Z}$-graded degree-wise finite real vector space $(V = \oplus_{k\in\mathbb{Z}} V_k, Q, \omega)$ st. $|Q| = 1$, $Q^2 = 0$, $|\omega| = -1$, ω is graded-skew and

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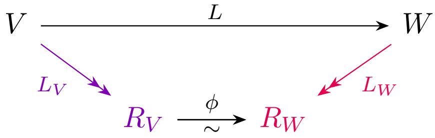
Def: L is a (**dg**) **reduction**, $V \xrightarrow{L} R$, if $\exists C \subset V$ coisotropic ($\Leftrightarrow I \equiv C^\omega \subseteq C$ isotropic) st.

$$R \simeq C/I, \quad (\text{and } I \cap \operatorname{Ker} Q = 0).$$

Proposition: Relations decompose (essentialy uniquely) into (co)reductions and linear symplectomorphisms with

$$R_V \simeq C/I \equiv (L \cap V)\,/\,\mathrm{Ker}\,L, \quad R_W \simeq (L \cap W)\,/\,\mathrm{Ker}\,L^T.$$



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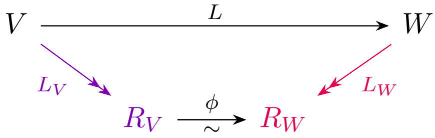
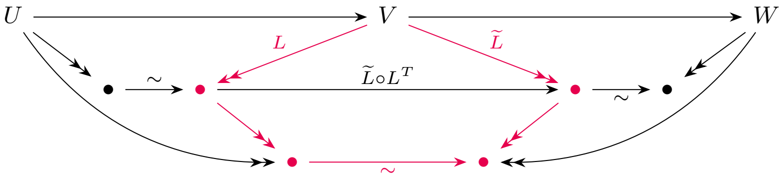


Def: Let $\widehat{\mathcal{S}}(V) \coloneqq \prod_{k \geq 0} (V^*)^{\odot k}$. Then we say L is a **correspondence** between $f_V \in \widehat{\mathcal{S}}(V)$ and $f_W \in \widehat{\mathcal{S}}(W)$ if:

$$\int_I \iota_C^*(f_V) \equiv \int_V \delta_{L_V} f_V = \phi^* \int_W \delta_{L_W} f_W$$

Proposition: Correspondences can be composed if the "neighbouring reductions commute".



Lemma: Given a basis of V, the pentagon commutes if I, $\widetilde{I}$ are given by choices of mutually ω-orthogonal generators.

Lemma: For any dg reductions, the generators of the isotropics are ω-orthogonal.

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Def: A **quantum L_∞ algebra** on (V, Q, ω) is given by $S = S_{\text{free}} + S_{\text{int}} \in \widehat{\mathcal{S}}(V)[\![\hbar]\!]$ st.

1. the quadratic genus zero (ie. $\hbar$-independent) part S_{free} is specified by $S_{\text{free}} = \omega(\bullet, Q\bullet)$,
2. S_{int} is in degree ≥ 3,
3. the **quantum master equation** is satisfied:

$$\hbar\boldsymbol{\Delta}S + \frac{1}{2}\{S, S\} = 0 \quad \left(\Leftrightarrow \boldsymbol{\Delta}\left(e^{S/\hbar} = 0\right)\right)$$

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Lemma [Zwiebach]: A quantum L_∞ algebra S is equivalent to a *loop homotopy Lie algebra* given by a collection of "higher brackets" $\{l_k^g\}_{k\geq 1, g\geq 0}$ via: $S_k^g(\bullet, \ldots, \bullet) = \pm\omega\left(\bullet, l_k^g(\bullet, \ldots, \bullet)\right)$.

Proposition: Let $V = V' \oplus V'' \xrightarrow{L} V'$ be a dg reduction given by an isotropic $I \subset V''$. Then the *homological perturbation lemma* produces:

- An **effective observable**

$$P_{\text{eff}} : \widehat{\mathcal{S}}(V)[\![\hbar]\!] \longrightarrow \widehat{\mathcal{S}}\left(V'\right)[\![\hbar]\!]$$

  defined by a *perturbative series* in powers of (roughly)

$$\hbar \boldsymbol{\Delta} \left(-Q\big|_I\right)^{-1}.$$

- A new *homotopic* qL_∞ algebra on V', $S' = S_{\text{free}}\big|_{V'} + S'_{\text{int}}$ st.

$$e^{S'_{\text{int}}/\hbar} = P_{\text{eff}}\left(e^{S_{\text{int}}/\hbar}\right).$$

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Example: For $V' = H^\bullet_Q$, the “effective action” $S' \equiv W$ defines the *minimal model* of the qL_∞-algebra. [DJP17]

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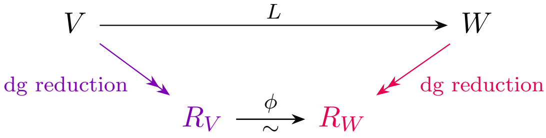
Observation: $e^{S'/\hbar} = \int_V \delta_L e^{S/\hbar}$. Thus a **correspondence of qL_∞ algebras** defined by integrals of Feynman weights along



is an equality (up to pullback along ϕ) of perturbative effective Feynman weights:

$$e^{S'_V/\hbar} = \phi^* \left(e^{S'_W/\hbar} \right).$$

- **Conclusion**: The interplay between finite-dimensional "path integrals" and homological perturbation theory extends to from the minimal model to a Weinstein-like odd symplectic category.

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- **Generalization**: Extension to a global setting of graded manifolds should be feasible using the Wehrheim-Woodward category.
- **Question**: Can we use correspondences to describe a natural model category structure on qL_∞ algebras?