On the restricted three-body problem

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Setup. Three objects: Earth (E), Moon (M), Satellite (S) with masses m_E, m_M, m_S, under gravitational interaction.

Classical assumptions:

1. **(Restricted)** $m_S = 0$, i.e. S is *negligible.*
2. **(Circular)** The *primaries* E and M move in circles around their center of mass.
3. **(Planar)** S moves in the plane containing E and M.

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Spatial case: drop the planar assumption.

Goal: Study motion of S.

Spatial circular restricted three-body problem

In rotating coordinates where $E = (\mu, 0, 0)$, $M = (-1 + \mu, 0, 0)$ are fixed, the Hamiltonian is autonomous and so is conserved:

$$H : \mathbb{R}^3 \backslash \{E, M\} \times \mathbb{R}^3 \to \mathbb{R}$$

$$H(q, p) = \frac{1}{2}\|p\|^2 - \frac{\mu}{\|q - M\|} - \frac{1 - \mu}{\|q - E\|} + p_1 q_2 - p_2 q_1,$$

where we normalize so that $m_E + m_M = 1$, and $\mu = m_M$.

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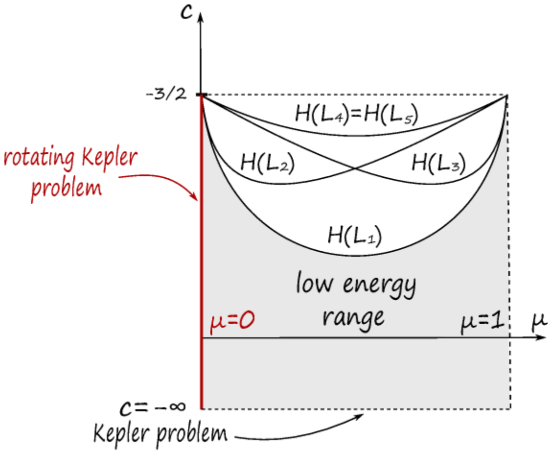
Two parameters: μ, and $H = c$ Jacobi constant.

Lagrangian points

H has five critical points: $L_1, \ldots, L_5$ called *Lagrangians*.



The critical values of H.

Integrable limit cases

If $\mu = 0 \rightsquigarrow H = K + L$, where

$$K(q,p) = \frac{1}{2}\|p\|^2 - \frac{1}{\|q\|}$$

is the *Kepler energy* (two-body problem), and

$$L = p_1 q_2 - p_2 q_1$$

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We have $\{H, K\} = \{H, L\} = \{K, L\} = 0$ and so

$$\phi_t^H = \phi_t^H \circ \phi_t^L.$$

If $T(K) = \frac{\pi}{2(-K)^{3/2}}$ is the period of a Kepler ellipse of energy $K < 0$ (Kepler's 3rd law), then closed orbits iff K satisfies the *resonance condition*

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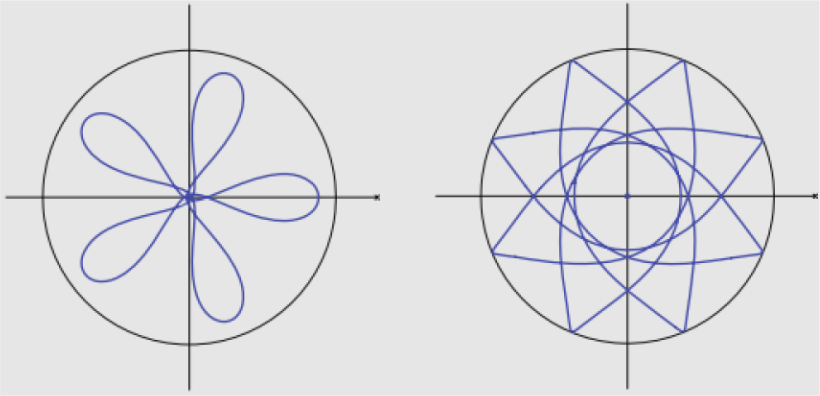
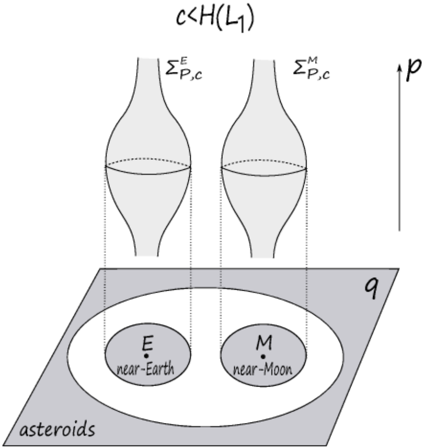
Fact: $c \to -\infty \rightsquigarrow$ Kepler problem (after regularization).

Periodic orbits in the rotating Kepler problem



Some orbits with different resonance.

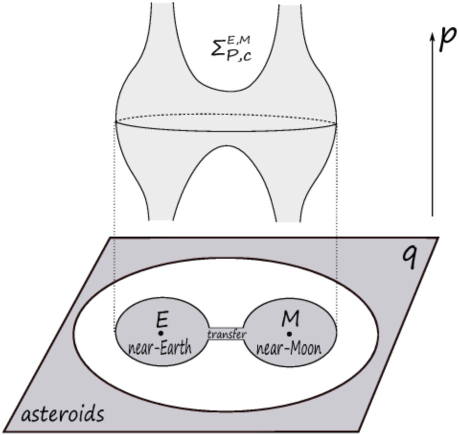
Low energy Hill regions



Morse theory in the three-body problem.

Low energy Hill regions

$$c \in (H(L_1), H(L_1) + \varepsilon)$$



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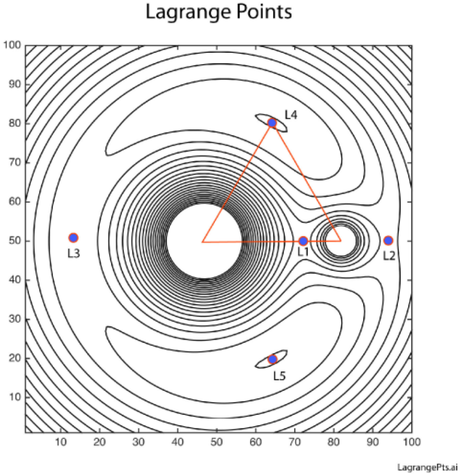
Level sets of potential



The Lagrange points and the level sets of the potential. The Euler points L_1, L_2, L_3 are collinear and unstable, the Lagrange points L_4, L_5 give equilateral triangles and are stable.

Moser regularization

H is singular at *collisions* ($q = E$ ó $q = M \rightsquigarrow p = \infty$).

Moser regularization, near E or M:

$$(q, p) \overset{\text{switch}}{\longmapsto} (-p, q) \overset{\text{estereo. proj.}}{\longmapsto} (\xi, \eta) \in T^*S^3$$

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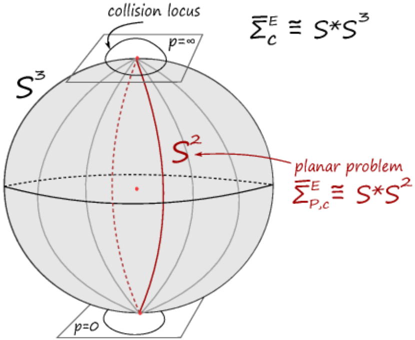
$\rightsquigarrow$ regularized Hamiltonian $Q : T^*S^3 \to \mathbb{R}$, with level set $Q^{-1}(0) = \overline{\Sigma}_c^E \cong S^*S^3 = S^3 \times S^2$.



Contact geometry of the three-body problem

$\overline{\Sigma}_c^E$, $\overline{\Sigma}_c^M$ bounded energy components for $c < H(L_1)$, $\overline{\Sigma}_c^{E,M}$ connected sum bounded component, $c \in (H(L_1), H(L_2))$. Similarly, $\overline{\Sigma}_{P,c}^E$, $\overline{\Sigma}_{P,c}^M$ and $\overline{\Sigma}_{P,c}^{E,M}$ for planar problem.

Theorem ([AFvKP] (planar problem), [CJK] (spatial problem))

We have

$$\overline{\Sigma}_c^E \cong \overline{\Sigma}_c^M \cong (S^*S^3, \xi_{std}), \text{ if } c < H(L_1),$$

$$\overline{\Sigma}_{P,c}^E \cong \overline{\Sigma}_{P,c}^M \cong (S^*S^2, \xi_{std}), \text{ if } c < H(L_1),$$

and

$$\overline{\Sigma}_c^{E,M} \cong (S^*S^3, \xi_{std}) \# (S^*S^3, \xi_{std}), \text{ if } c \in (H(L_1), H(L_1) + \epsilon).$$

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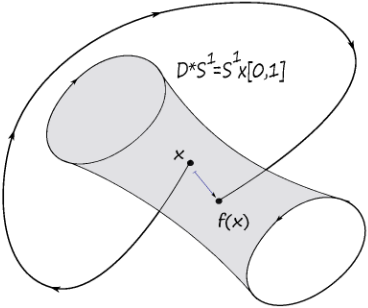
In all above cases, the planar problem is a codimension-2 contact submanifold of the spatial problem. □

Poincaré-Birkhoff and the planar problem

To find closed orbits in the **planar** problem, Poincaré's approach is:

(1) Global surface of section for the dynamics;

(2) Fixed point theorem for the return map.



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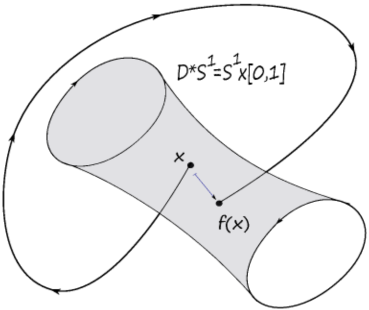

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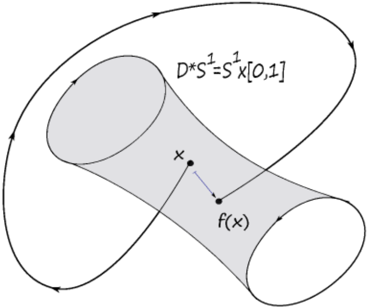

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Goal: Generalize this approach to the **spatial** problem.

Step 1: Global hypersurfaces of section

Open book decompositions

An **OBD** on M is a fibration

$$\pi : M \backslash B \to S^1,$$

with $B \subset M$ codim-2, and $\pi(b, r, \theta) = \theta$ on collar $B \times \mathbb{D}^2$.

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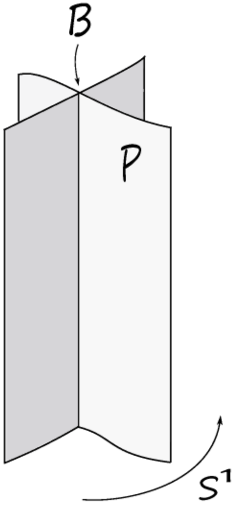
Notation: $M = \mathbf{OB}(P, \phi)$.

- $P = \overline{\pi^{-1}}(pt) =$*page*;
- $B = \partial P =$*binding*;
- $\phi : P \xrightarrow{\cong} P$ *monodromy*, $\phi|_B = id$.



Global hypersurfaces of section

$\varphi_t : M \to M$ flow, then π is **adapted to the dynamics** if B is invariant, and orbits are transverse to the interior of all pages.

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Each page P is a **global hypersurface of section**, i.e.

- P is codimension-1;
- $B = \partial P$ is invariant;
- orbits in $M \backslash B$ meet interior of pages transversely.

$\rightsquigarrow$ Poincaré return map $f : \text{int}(P) \to \text{int}(P)$.

Step 1: Open books in the spatial three-body problem

$\overline{\Sigma}_c = H^{-1}(c)$ bounded regularized energy surface in the spatial 3BP.

Theorem (M–van Koert)

For $\mu \in (0,1)$,

$$\overline{\Sigma}_c = \begin{cases} \boldsymbol{OB}(\mathbb{D}^* S^2, \tau^2), & c < H(L_1), \\ \boldsymbol{OB}(\mathbb{D}^* S^2 \natural \mathbb{D}^* S^2, \tau_1^2 \circ \tau_2^2), & c \in (H(L_1), H(L_1) + \epsilon), \end{cases},$$

adapted to the dynamics.

- $\tau, \tau_i =$ *Dehn-Seidel twist.*
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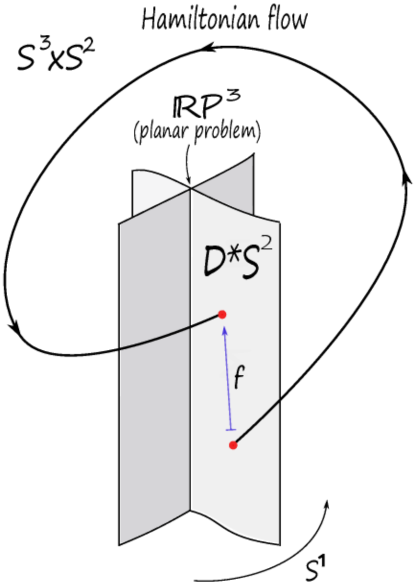
This reduces the dynamics to that of the return map, a Hamiltonian map of $\mathbb{D}^* S^2$. The section is **non-perturbative**, and **explicit** (good for numerics).

Open books



Basic idea

Let $B = \{p_3 = q_3 = 0\}$ (planar problem). Define

$$\pi(q,p) = \frac{q_3 + ip_3}{\|q_3 + ip_3\|} \in S^1, \; d\pi = \frac{p_3 dq_3 - q_3 dp_3}{p_3^2 + q_3^2}.$$

Then

$$d\pi(X_H) = \frac{p_3^2 + q_3^2 \cdot \left(\frac{1-\mu}{\|q-E\|^3} + \frac{\mu}{\|q-M\|^3}\right)}{p_3^2 + q_3^2} > 0,$$

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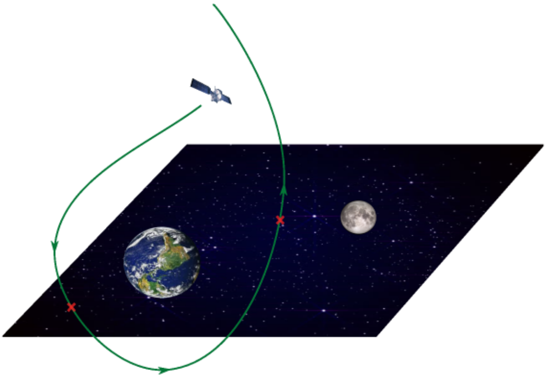
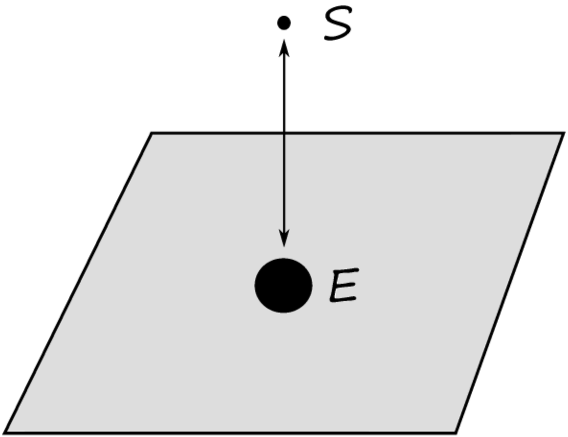
Problem: It does **not** extend to the collision locus $q = E, q = M$.

Physical interpretation



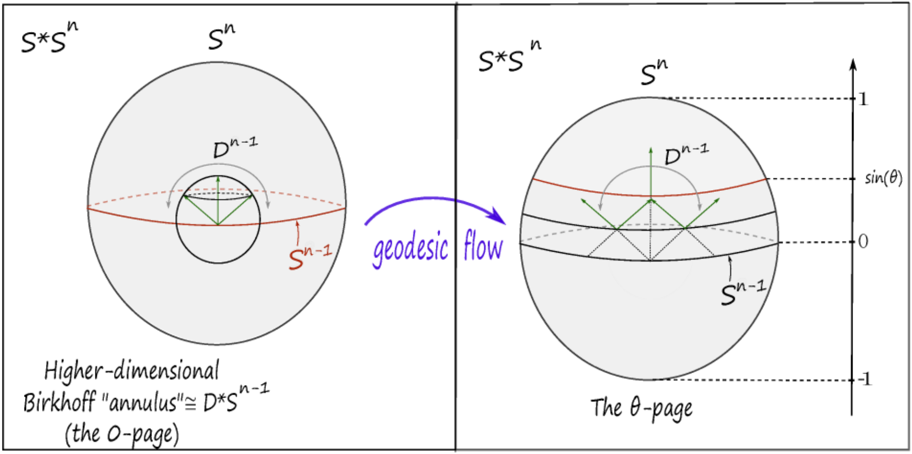
The fiber over $\pi/2$ corresponds to $q_3 = 0$, $p_3 > 0$, and the spatial orbits of S are transverse to the plane containing E, M away from collisions.

Polar orbits



Polar orbits prevent transversality on the collision locus.

The geodesic open book



The geodesic open book for S^*S^n.

Return map

Theorem (M.–van Koert)

For every $\mu \in (0,1]$, *c* $< H(L_1)$, *and page P, the return map f extends smoothly to the boundary* $B = \partial P$, *and in the interior it is an exact symplectomorphism*

$$f = f_{c,\mu} : (int(P), \omega) \to (int(P), \omega),$$

where $\omega = d\alpha|_P$, $\alpha = \alpha_{\mu,c}$ *contact form. Moreover, f is Hamiltonian in the interior, and the Hamiltonian isotopy extends smoothly to the boundary.*

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where $\omega = d\alpha|_P$, $\alpha = \alpha_{\mu,c}$ contact form. Moreover, f is Hamiltonian in the interior, and the Hamiltonian isotopy extends smoothly to the boundary.

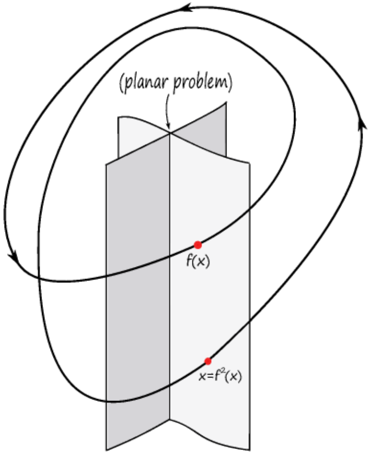
Here, ω degenerates at B, but after a continuous conjugation, it is actually symplectic and **deformation equivalent** to the standard symplectic form. The return map however extends only continuously after conjugation. The Hamiltonian is *not* rel boundary.

Remarks

- The fact that f is a symplectomorphism follows easily from Liouville's theorem.
- The fact that f extends to the boundary is non-trivial (relies on convexity in directions normal to the binding, cf. dynamical convexity by HWZ).
- The fact that f is Hamiltonian relies on: monodromy τ^2 is Hamiltonian, one can symplectically join f to the monodromy, and $H^1(\mathbb{D}^* S^2; \mathbb{R}) = 0$.

Step 2: Fixed-point theory of Hamiltonian twist maps

{**spatial** orbits} $\longleftrightarrow$ {**interior** periodic points}.



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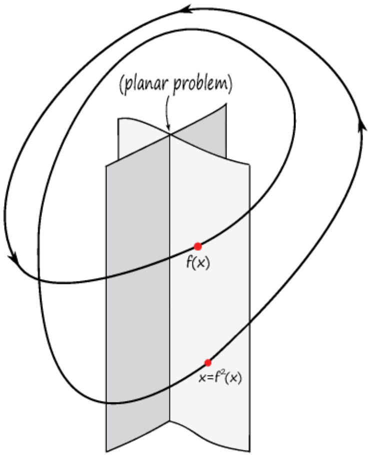

Goal: Find infinitely many *interior* periodic points.

Hamiltonian twist maps

$(W, \omega = d\lambda)$ Liouville domain, $\alpha = \lambda|_B$. Let $f : (W, \omega) \to (W, \omega)$ be a Hamiltonian symplectomorphism.

Definition

f is a **Hamiltonian twist map** if there exists a time-dependent Hamiltonian $H : \mathbb{R} \times W \to \mathbb{R}$ such that:

- H is *smooth* (or C^2);
- $f = \phi_H^1$;
- There exists a smooth function $h : \mathbb{R} \times B \to \mathbb{R}$ which is *positive* and

$$X_{H_t}|_B = h_t R_\alpha.$$

Index growth

We call a strict contact manifold $(Y, \xi = \ker \alpha)$ **strongly index-definite** if the contact structure $(\xi, d\alpha)$ admits a symplectic trivialization ϵ so that:

- There are constants $c > 0$ and $d \in \mathbb{R}$ such that for every Reeb chord $\gamma : [0, T] \to Y$ of Reeb action $T = \int_0^T \gamma^*\alpha$ we have

  $$|\mu_{RS}(\gamma; \epsilon)| \geqslant cT + d,$$

  where μ_{RS} is the Robbin–Salamon index.

Drop absolute value $\rightsquigarrow$ index-*positive*.

Examples

Lemma (Some examples)

- *If* $(Y, \alpha) \subset \mathbb{R}^4$ *is a strictly convex hypersurface, then it is strongly index-positive.*
- *If* $(Y, \ker \alpha) = (S^*Q, \xi_{std})$ *is symplectically trivial and* (Q, g) *has positive sectional curvature, then* (Y, α) *is strongly index-positive.*

Fixed-point theorem

Theorem (M.–van Koert, Generalized Poincaré–Birkhoff theorem)

Suppose that f is an exact symplectomorphism of a Liouville domain (W, λ), and let $\alpha = \lambda|_B$. Assume the following:

- ***(Twist condition)*** *f is a Hamiltonian twist map;*
- ***(index-definiteness)*** *If $\dim W \geqslant 4$, then assume $c_1(W)|_{\pi_2(W)} = 0$, and $(\partial W, \alpha)$ is strongly index-definite. In addition, assume all fixed points of f are isolated;*
- ***(Symplectic homology)*** *$SH_\bullet(W)$ is infinite dimensional.*

Then f has simple interior periodic points of arbitrarily large (integer) period.

Special case of fixed-point theorem

Theorem (M.–van Koert, special case)

Let $W \subset (T^*M, \lambda_{can})$ *be fiber-wise star-shaped, with* M *simply connected, orientable and closed. Let* $f : W \to W$ *be a Hamiltonian twist map. Assume:*

- *Reeb flow on* ∂W *is index-positive; and*
- *All fixed points of* f *are isolated.*

Then f *has simple interior periodic points of arbitrarily large period.*

Non-examples: Katok examples

There are examples of (non-reversible) Finsler metrics on S^n with only finitely many simple geodesics, which are perturbations of the round metric (and so close to the Kepler problem).

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The return maps are Hamiltonian and satisfy all conditions of the theorem, *except* the Hamiltonian twist condition (as a consequence of the above theorem).

Toy example: smoothness is relevant

$Q = S^n$ with round metric.

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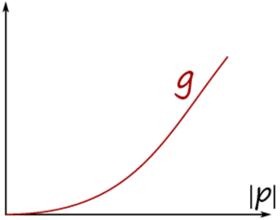
Let $K = 2\pi g$, with $g = g(|p|)$ smoothing of $|p|$ near $p = 0$. Then $\phi_K^1 = \phi_G^{2\pi g'(|p|)}$, where ϕ_G^t geodesic flow, is a Hamiltonian twist map. It has simple orbits of arbitrary period ($g'(|p|) = l/k$ coprime $\rightsquigarrow$ k-periodic orbit).



Idea of the proof

Extend a generating Hamiltonian to an ϵ-collar neighbourhood via a Taylor expansion, so it becomes admissible for SH. If $\hat{f}$ time-1 map, then twist condition implies

$$\lim_k HF_\bullet(\widehat{f}^k) = SH_\bullet(W)$$

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Contributions near the boundary escape any index window due to index–definiteness, and so fixed points are those of f. Iterating the same points is ruled out by grading considerations, using the linear growth of the mean index. Degeneracies are dealt with via local Floer homology.

A few remarks

- If $\dim W = 2$, $\dim SH_\bullet(W) = \infty$ iff $W \neq \mathbb{D}^2$.

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- A higher-dimensional generalization of the classical Poincaré-Birkhoff theorem, in the spirit of the Conley conjecture.
- We couldn't check the twist condition in the three-body problem. The boundary degeneracy of the symplectic form needs to be addressed.
- This opens up an completely unexplored line of research: *Hamiltonian dynamics on higher-dimensional Liouville domains.*

Hamiltonian dynamics on Liouville domains

Natural higher-dimensional analogue of dynamics on surfaces.

Hamiltonian dynamics on Liouville domains

Natural higher-dimensional analogue of dynamics on surfaces. Concrete starting question:

Q. If $f : W \to W$ Hamiltonian map on an *open* Liouville domain, does it have periodic points? How many? Are there obstructions of f and/or W?

Hamiltonian dynamics on Liouville domains

Natural higher-dimensional analogue of dynamics on surfaces. Concrete starting question:

Q. If $f : W \to W$ Hamiltonian map on an *open* Liouville domain, does it have periodic points? How many? Are there obstructions of f and/or W?

Analogue results in dimension 2: Brouwer translation theorem (open disk) and a theorem of Franks (open annulus).

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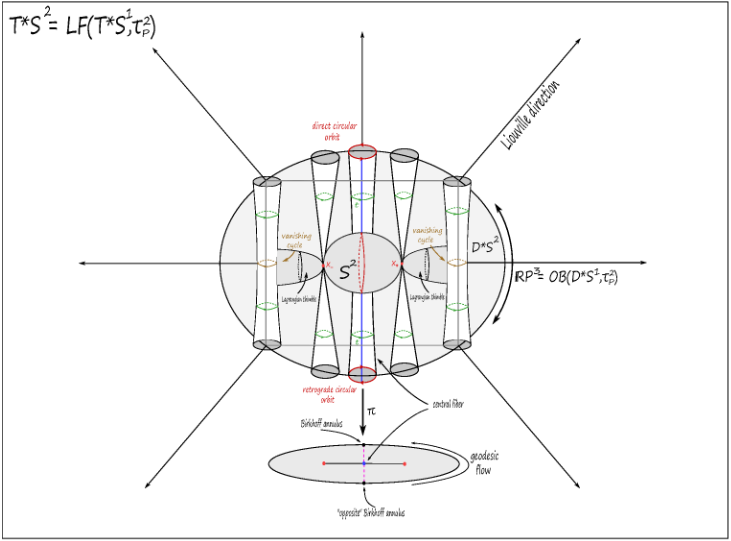
There is a fascinating interplay between interior and boundary phenomena.

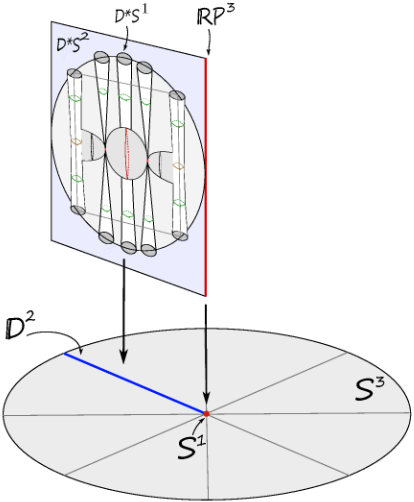
Pseudo-holomorphic foliations

Lefschetz fibration



Topological observation: The section $\mathbb{D}^* S^2$ admits a Lefschetz fibration with annuli fibers.

Leaf space is S^3



The moduli space of fibers (i.e. the leaf space) is $S^3 = \mathbf{OB}(\mathbb{D}^2, \mathbb{1})$.

Pseudo-holomorphic foliations in the 3BC

Let $\alpha = \alpha_{\mu,c}$ contact form giving the 3BP. We say that (μ, c) lie in the convexity range if the *Levi–Civita regularization* of planar problem is a convex $S^3 \subset \mathbb{R}^4$.

Theorem (M.)

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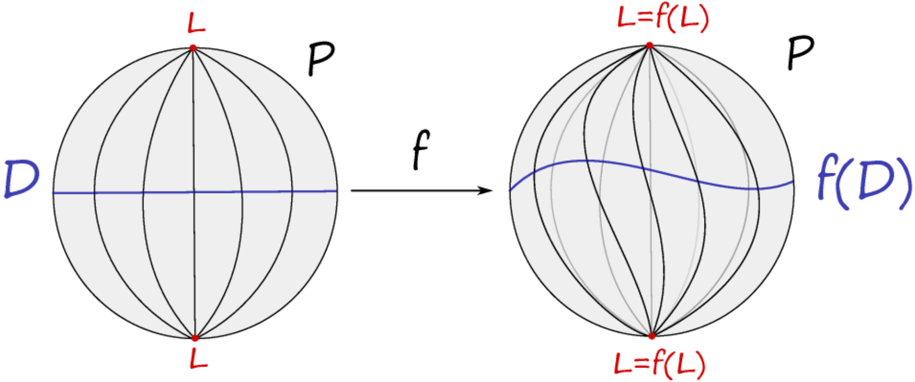
As the return map $f : \mathbb{D}^*S^2 \to \mathbb{D}^*S^2$ preserves ω, it sends a symplectic annulus to another symplectic annulus with the same boundary (direct/retrograde planar orbits), and same symplectic area (the sum of the period of these orbits). The adapted open book at the planar problem is given by Hrynewicz–Salomão–Wysocki.

Return map



The return map *f* in general does **not** preserve the foliation.

Contact structures and Reeb dynamics on moduli

$(M, \xi_M) = \mathbf{OB}(P, \phi)$ an *iterated planar* 5-fold, i.e. $P = \mathbf{LF}(F, \phi_F)$ has a 4D Lefschetz fibration with genus zero fibers.

$\mathbf{Reeb}(P, \phi) = \{\alpha$ adapted contact form: $\alpha|_B$ adapted to $B = \mathbf{OB}(F, \phi_F)\}$.

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Theorem (M., Contact structures and Reeb dynamics on moduli)

There is a moduli space $\mathcal{M}$ of holomorphic annuli foliating M, forming the fibers of a Lefschetz fibration on each page. It is a contact manifold $(\mathcal{M}, \xi_{\mathcal{M}}) \cong (S^3, \xi_{std}) = \mathbf{OB}(\mathbb{D}^2, \mathbb{1})$.

*Any $\alpha \in$ **Reeb**(P, ϕ) induces a contact form $\alpha_{\mathcal{M}} \in$ **Reeb**$(\mathbb{D}^2, \mathbb{1})$,* $\ker \alpha_{\mathcal{M}} = \xi_{\mathcal{M}}$.

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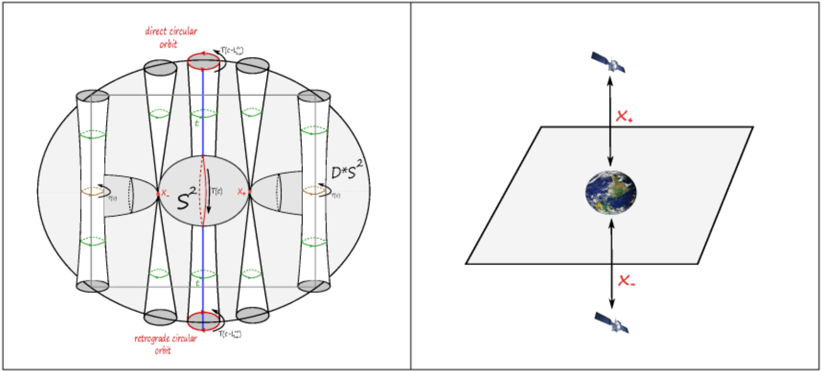
Fiberwise integration:

$$(\alpha_{\mathcal{M}})_u(v) = \int_u \alpha_z(v(z))dz,$$

with $dz = d\alpha|_u$, $\xi_{\mathcal{M}}$ corresponds to a symplectic connection on each page of M.

Integrable case $\mu = 0$.



If $\mu = 0 \rightsquigarrow$ f-invariant foliation, f is a **classical** twist map on the fibers with variable rotation angle $T(K) = \frac{\pi}{2(-K)^{3/2}}$ (Kepler's 3rd law), and the nodal Lefschetz singularities are fixed points (the polar orbits).

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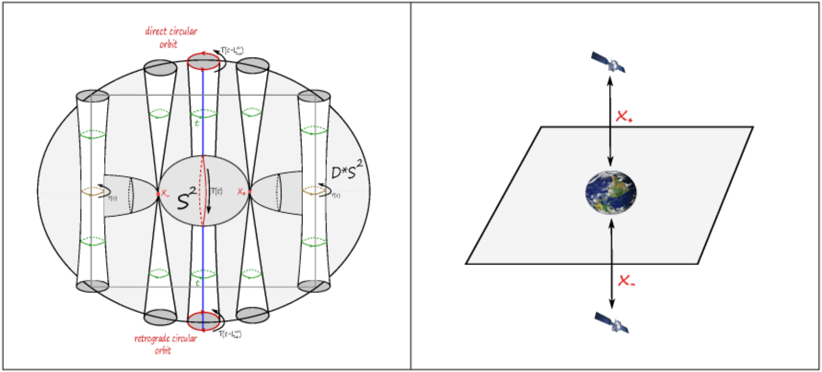

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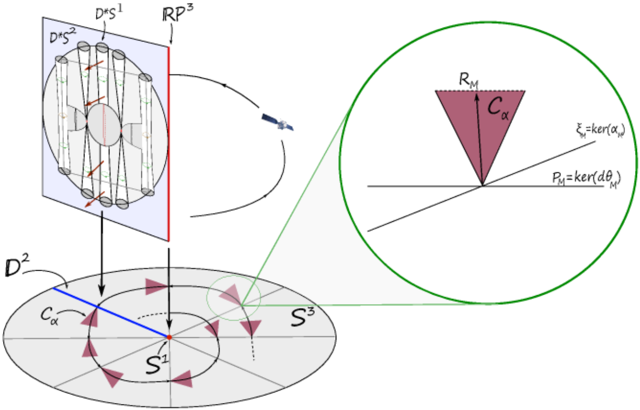
What happens when we perturb, i.e. $\mu \sim 0$? How does the dynamics interact with the foliation?

The shadowing cone



The shadowing cone is obtained by projecting the flow. Orbits of the flow project to orbits of the cone.

Holomorphic shadow

The *holomorphic shadow map* is obtained by taking the shadow:

$$\textbf{HS} : \textbf{Reeb}(\mathbb{D}^* S^2, \tau^2) \to \textbf{Reeb}(\mathbb{D}^2, \mathbb{1})$$

$$\alpha \mapsto \alpha_{\mathcal{M}}.$$

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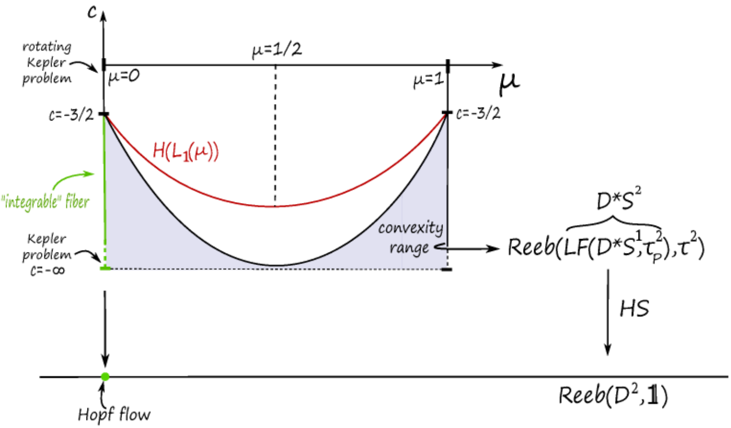
New program: Try to "lift" knowledge from dynamics on S^3.

Case of three-body problem

If (μ, c) in convexity range, combining our adapted open book with [HSW] on $B = \mathbb{R}P^3 \rightsquigarrow \alpha_{\mu,c} \in \textbf{Reeb}(\mathbb{D}^* S^2, \tau^2)$.



Dynamical applications

Definition

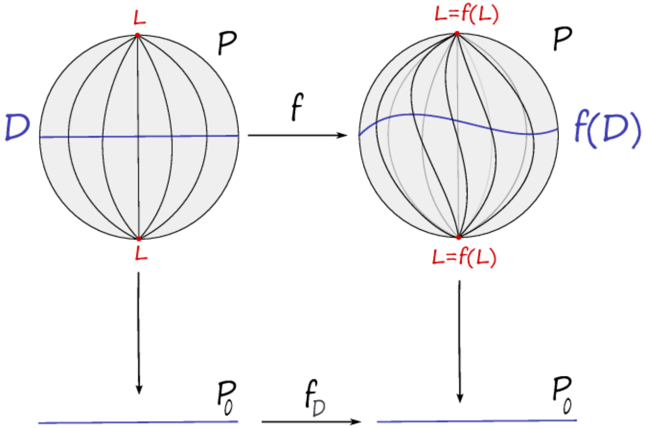
Let P be a page, and $f : \text{int}(P) \to \text{int}(P)$ a return map. A *fiber-wise* k-recurrent point is $x \in \text{int}(P)$ such that $f^k(\mathcal{M}_x) \cap \mathcal{M}_x \neq \varnothing$.

This is a "symplectic version" of a leaf-wise intersection.

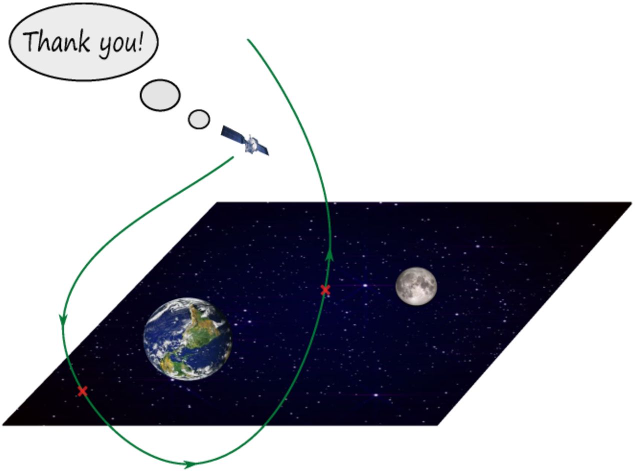
Theorem (M.)

In the SCR3BP, for every k, one can find sufficently small perturbations of the integrable cases which admit infinitely many fiber-wise k-recurrent points.

Idea of proof: symplectic tomographies



We induce maps $f_\mathcal{D} : \text{int}(D^2) \rightarrow \text{int}(D^2)$ for every symplectic disk section of the LF. These are the identity for the integrable case. These preserve area for near integrable cases, and hence Brouwer applies.


Thank you!

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