Rational Homotopy Theory, del Pezzo surfaces, and Mysterious Triality
Lecture I: Introduction to Mysterious Triality & del Pezzo surfaces

Sasha Voronov

Winter School on Geometry and Physics
Srní, Czechia

University of Minnesota, USA, and Kavli IPMU, Japan

January 14–19, 2024

Iqbal, Neitzke, and Vafa: Mysterious Duality

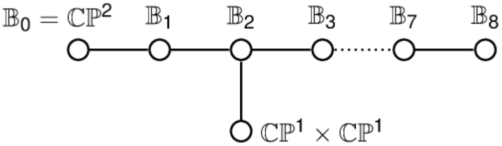
Once upon a time at Harvard... in 2001, A. Iqbal, A. Neitzke, and C. Vafa discovered a "mysterious duality:"

Physics: Toroidal compactifications of M-theory: X^{11}, $Y^{10} = X^{11}/S^1$ (type IIA), $X^{11}/(S^1)^2, \ldots, X^{11}/(S^1)^8$, and Z^{10} (type IIB)	have similar symmetry patterns, both governed by root system E_k, $0 \leq k \leq 8$.
Math: Del Pezzo surfaces $\mathbb{B}_0 = \mathbb{P}^2$, $\mathbb{B}_1 = \mathbb{P}^2 \# \overline{\mathbb{P}^2} = \mathrm{Bl}(\mathbb{P}^2, x_1), \ldots, \mathbb{B}_8$, and $\mathbb{P}^1 \times \mathbb{P}^1$	

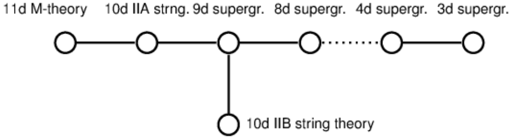
Del Pezzo's is classical algebraic geometry: cf. 27 lines on a cubic surface ($\mathbb{B}_6$) (A. Cayley and G. Salmon, 1849)

Math and Physics via the E_{10} Dynkin diagram

Del Pezzos of degrees $9 - k$:



Supergravity of dims $11 - k$: Dimensional reduction of supergravity on a k-torus $T^k = (S^1)^k$ a.k.a. toroidal compactifications of M-theory



Branes in toroidal compactifications of M-theory

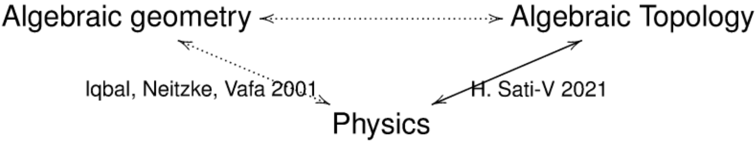
Two yogas: **homology classes** in $H_2(\mathbb{B}_1)$ and **tensions of branes** in type IIA string theory = M-theory$/S^1$ (the $k = 1$ case):

homology class	tension	type IIA meaning
E	$R^{-1} = l_s^{-1} g_s^{-1}$	D0-brane
$H - E$	$(2\pi)^2 R l_p^{-3} = (2\pi)^{-1} l_s^{-2}$	F-string
H	$(2\pi) l_p^{-3} = (2\pi)^{-2} l_s^{-3} g_s^{-1}$	D2-brane
$2H - E$	$(2\pi)^2 R l_p^{-6} = (2\pi)^{-4} l_s^{-5} g_s^{-1}$	D4-brane
$2H$	$(2\pi) l_p^{-6} = (2\pi)^{-5} l_s^{-6} g_s^{-2}$	NS5-brane
$3H - 2E$	$(2\pi)^3 R^2 l_p^{-9} = (2\pi)^{-6} l_s^{-7} g_s^{-1}$	D6-brane
$4H - 3E$	$(2\pi)^4 R^3 l_p^{-12} = (2\pi)^{-8} l_s^{-9} g_s^{-1}$	D8-brane

Table credit: Iqbal, Neitzke, and Vafa (2001)

Mystery: Physics and AG give rise to the E_k series, but no explicit connection between physics and del Pezzo surfaces.

The Goal of These Lectures: Mysterious Triality



Main themes:

1. **Math Physics**: The RHT of iterated *cyclic loop spaces* $\mathcal{L}_c^k S^4$ is explicitly related to the M-theory story. (E.g., want equations of motion of M-theory wrapped on T^5, *i.e.*, 6d supergravity? — Read them off from the differential in the Sullivan minimal model of $\mathcal{L}_c^5 S^4$! (**Hypothesis H**));
2. **Mathematics**: S^4, $\mathcal{L}_c S^4$, $\mathcal{L}_c^2 S^4, \ldots$ is a new series of objects with hidden internal E_k symmetry. Such as 27 "lines" in $\mathcal{L}_c^6 S^4$, 28 "bitangents" in $\mathcal{L}_c^7 S^4 \ldots$

Outline of Lectures

1. Introduction to Mysterious Triality & del Pezzo surfaces
2. Hypothesis H: Supergravity and Rational Homotopy Theory
3. The E_k symmetry of the cyclic loop spaces $\mathcal{L}_c^k S^4$

Del Pezzo surfaces

A *del Pezzo surface* is a complex compact smooth surface S whose anticanonical class $-K := -c_1(\Omega^1_S)$ is ample (sufficiently positive). Del Pezzo surfaces are classified topologically by belonging to one of the following types:

$$\mathbb{B}_0 := \mathbb{CP}^2, \mathbb{B}_1, \mathbb{B}_2, \ldots, \mathbb{B}_8$$

and

$$\mathbb{CP}^1 \times \mathbb{CP}^1.$$

Here

$$\mathbb{B}_k \quad = \quad \text{blowup of } \mathbb{CP}^2 \text{ at } k \text{ generic points} \quad = \quad \mathbb{CP}^2 \# k\overline{\mathbb{CP}^2}$$

Anticanonical "embedding:" $\mathbb{B}_k \hookrightarrow \mathbb{CP}^{9-k}$ associated with the *anticanonical line bundle* $(\wedge^2\Omega^1_S)^*$. $\mathbb{B}_k$ is a subvariety of degree $(-K, -K) = 9 - k$ in $\mathbb{CP}^{9-k}$.

Blowup; picture credit: R. Hartshorne

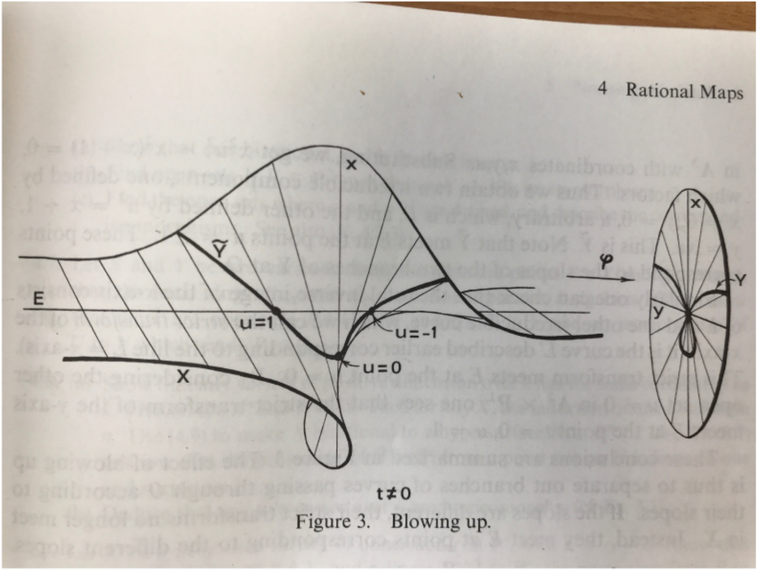


Figure 3. Blowing up.

Intersection theory on a del Pezzo surface

Surface: $\mathbb{B}_k$ = blowup of $\mathbb{CP}^2$ at k generic points $= \mathbb{CP}^2 \# k\overline{\mathbb{CP}^2}$

Homology: $H_2(\mathbb{B}_k) = \langle H, E_1, \ldots, E_k \rangle \cong \mathbb{Z}^{k+1}$
H = the class of a line in $\mathbb{P}^2$ (lifted to $\mathbb{B}_k$)
$E_1, \ldots, E_k$ = the exceptional divisors

Intersection form:

$$(H, H) = 1, \quad (H, E_i) = 0, \quad (E_i, E_j) = -\delta_{ij}, \qquad 1 \leq i, j \leq k$$

Combinatorial Extract

1. The rank-$(k+1)$ lattice $N_k = H_2(\mathbb{B}_k) \subset N_k \otimes \mathbb{R}$;
2. The Lorentzian inner product $(-,-) : N_k \otimes N_k \to \mathbb{Z}$;
3. The anticanonical class $-K_k = 3H - E_1 - \cdots - E_k$.

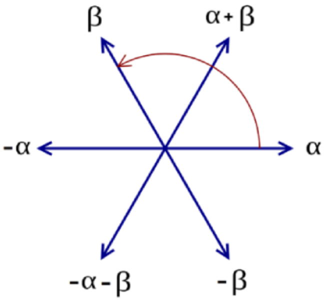
Theorem (Y. I. Manin, *Cubic Forms*, 1972)

The triple $(N_k, (-,-), -K_k)$ gives rise to the wealth of combinatorial data associated with the root system of type E_k:

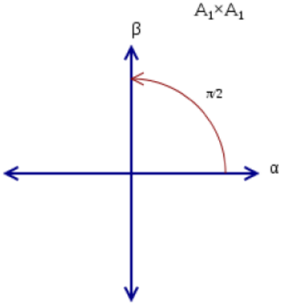
- *The very root system $R_k = \{\beta \in N_k \mid (\beta, -K_k) = 0, (\beta, \beta) = -2\}$ sitting in the Euclidean space $(-K_k)^\perp := \{\beta \in N_k \otimes \mathbb{R} \mid (\beta, -K_k) = 0\}$;*
- *The Weyl group $W(E_k)$;*
- *The system of "lines" $I_k = \{\beta \in N_k \mid (\beta, -K_k) = 1, (\beta, \beta) = -1\}$;*
- *Etc.*

Examples of Root Systems

The E_k root system arises in the orthogonal complement to K in the cohomology group $H^2(\mathbb{B}_k;\mathbb{Z})$ with the intersection form $H^2(\mathbb{B}_k;\mathbb{Z}) \otimes H^2(\mathbb{B}_k;\mathbb{Z}) \to \mathbb{Z}$. For example, the root system arising from $\mathbb{B}_2$ is A_1 and from $\mathbb{B}_3$ is $A_2 \times A_1$:



Root System A_2



Root System $A_1 \times A_1$

More on the E_k series

k	del Pezzo	Dynkin Diagram	Type of E_k	Lie Algebra
0	$\mathbb{P}^2$		A_{-1}	$\mathfrak{sl}_0 = \varnothing$
1	$\mathbb{B}_1$		A_0	$\mathfrak{sl}_1 = 0$
1	$\mathbb{P}^1 \times \mathbb{P}^1$		A_1	$\mathfrak{sl}_2$
2	$\mathbb{B}_2$		A_1	$\mathfrak{sl}_2$
3	$\mathbb{B}_3$		$A_2 \times A_1$	$\mathfrak{sl}_3 \oplus \mathfrak{sl}_2$
4	$\mathbb{B}_4$		A_4	$\mathfrak{sl}_5$
5	$\mathbb{B}_5$		D_5	$\mathfrak{so}_{10}$
6	$\mathbb{B}_6$		E_6	$\mathfrak{e}_6$
7	$\mathbb{B}_7$		E_7	$\mathfrak{e}_7$
8	$\mathbb{B}_8$		E_8	$\mathfrak{e}_8$
9	non-dP $\mathbb{B}_9$		$E_9 = \widehat{E}_8$	affine $\mathfrak{e}_9 = \widehat{\mathfrak{e}}_8$

Geometry & Root Systems: 27 Lines on a Cubic Surface

The del Pezzo surfaces of type $\mathbb{B}_6$ are exactly the smooth *cubic surfaces*, such as the *Fermat cubic* $x^3 + y^3 + z^3 = 1$, in $\mathbb{CP}^3$.

Theorem (A. Cayley and G. Salmon, 1849)

Every smooth complex cubic surface contains exactly 27 lines.

The 27 lines can be identified with the weights $I_k = \{\beta \in N_k \mid (\beta, -K_k) = 1, (\beta, \beta) = -1\}$ of a 27-dimensional fundamental representation of the Lie group E_6. See Manin's *Cubic Forms*.