Rational Homotopy Theory, del Pezzo surfaces, and Mysterious Triality

Lecture II: Supergravity and Rational Homotopy Theory (Hypothesis H)

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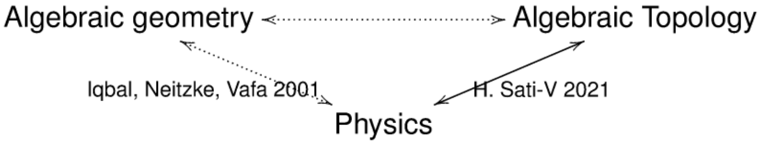
The Goal of These Lectures: Mysterious Triality



Main themes:

1. **Math Physics**: The RHT of iterated *cyclic loop spaces* $\mathcal{L}_c^k S^4$ is explicitly related to the M-theory story. (E.g., want equations of motion of M-theory wrapped on T^5, *i.e.*, 6d supergravity? — Read them off from the differential in the Sullivan minimal model of $\mathcal{L}_c^5 S^4$! (**Hypothesis H**));
2. **Mathematics**: S^4, $\mathcal{L}_c S^4$, $\mathcal{L}_c^2 S^4, \ldots$ is a new series of objects with hidden internal E_k symmetry. Such as 27 “lines” in $\mathcal{L}_c^6 S^4$, 28 “bitangents” in $\mathcal{L}_c^7 S^4 \ldots$

Rational homotopy theory (RHT)

Definition: $X \overset{\mathbb{Q}}{\sim} Y$ iff $X \to Y$ *rational (homotopy) equivalence* of path connected spaces, a continuous map inducing isomorphisms $H_\bullet(X;\mathbb{Q}) \xrightarrow{\sim} H_\bullet(Y;\mathbb{Q})$ on rational homology. The equivalence class of a space X is called the *rational homotopy type* of X.

Rational homotopy category: topological spaces with inverses of rational homotopy equivalences formally added.

Fact (Quillen, Sullivan, '60–70s): the rational homotopy category (of good enough spaces) is equivalent to the opposite category of DGCAs (or DGLAs, resp.) of a certain type:

$X \mapsto M(X)$, a DGCA, the *Sullivan minimal model* of X
$X \mapsto Q(X)$, a DGLA, the *Quillen minimal model* of X.

Fundamental Theorem of RHT: For good enough X and Y,
$X \overset{\mathbb{Q}}{\sim} Y \Leftrightarrow M(X) \cong M(Y) \Leftrightarrow Q(X) \cong Q(Y)$.

DGCA's

A *differential graded commutative algebra (DGCA)* is a graded commutative associative algebra

$$M = \bigoplus_{n \geq 0} M^n, \qquad M^i \cdot M^j \subset M^{i+j}, \quad ba = (-1)^{|a| \cdot |b|} ab,$$

$$d : M^n \to M^{n+1}, \quad d^2 = 0, \qquad d(ab) = (da)b + (-1)^{|a|} a(db).$$

A *minimal Sullivan DGCA* is a DGCA M which is free as a graded commutative algebra (i.e., a polynomial algebra in even and odd variables) with a *decomposable* differential $dM \subset M^+ \cdot M^+$ (and if $M^1 \neq 0$, satisfying a certain nilpotence condition). Here $M^+ := \bigoplus_{n>0} M^n$.

Example: S^4

$$M(S^4) = (\mathbb{Q}[g_4, g_7], d),$$
$$dg_4 = 0, \qquad dg_7 = -\tfrac{1}{2} g_4^2,$$
$$|g_4| = 4, \qquad |g_7| = 7.$$

$$M(S^4) \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\text{q-is}} \Omega^\bullet_{\mathrm{dR}}(S^4),$$
$$g_4 \mapsto \text{volume form on } S^4,$$
$$g_7 \mapsto 0.$$

Remark: We will have to use $\mathbb{R}$ in place of $\mathbb{Q}$ (*rational homotopy theory over the reals*), but we will ignore this.

M-theory and 11d supergravity

Fact: 11d supergravity is the low-energy (UV) limit of M-theory.

Equations of motion of 11d supergravity:

$$dG_4 = 0, \qquad dG_7 = -\tfrac{1}{2} G_4 \wedge G_4, \qquad *G_4 = G_7,$$

where $G_4 \in \Omega^4(X^{11})$ and $G_7 \in \Omega^7(X^{11})$ and $X^{11} =$ spacetime.

Duality-Symmetric formulation (metric-free background):

$$dG_4 = 0, \qquad dG_7 = -\tfrac{1}{2} G_4 \wedge G_4.$$

In M-theory

$$G_4 \rightsquigarrow \text{M2-brane}$$
$$G_7 \rightsquigarrow \text{M5-brane}$$

Sati's "Hypothesis H" in 11d (2013)

We obviously (this is algebra, after all!) have a DGCA map

$$M(S^4) \to \Omega^\bullet_{\mathrm{dR}}(X^{11}),$$
$$g_4 \mapsto G_4,$$
$$g_7 \mapsto G_7.$$

According to the Fundamental Theorem of RHT, this gives a unique map in rational homotopy category

$$\varphi : X^{11} \to S^4.$$

Hisham Sati called this observation *Hypothesis H*.

Reduction to 10d: Type IIA string theory

Suppose X^{11} has a free action of S^1. Then $Y^{10} := X^{11}/S^1$ will be the spacetime reduced to 10 dimensions. The map φ will then induce a map φ_1:

$$\begin{array}{ccc} & X^{11} & \xrightarrow{\varphi} & S^4 \\ & \downarrow & & \\ Y^{10} = & X^{11}/S^1 & \xrightarrow{\varphi_1} & \mathcal{L}_c S^4 \end{array}$$

Actually, there is an adjunction (Fiorenza-Sati-Schreiber 2017)

$$[Y \times_{BS^1} ES^1, Z] \xrightarrow{\sim} [Y, \mathrm{Map}(S^1, Z) /\!\!/ S^1]_{/BS^1} \qquad (1)$$

between the homotopy categories of spaces and spaces over the classifying space $BS^1 = \mathbb{CP}^\infty$. Note that
$Y^{10} \times_{BS^1} ES^1 \sim X^{11}$ and $\mathrm{Map}(S^1, Z) /\!\!/ S^1 = \mathcal{L}_c Z$.

Type IIA string theory: Hypothesis H in 10d, as per Fiorenza-Sati-Schreiber 2017

$$M(\mathcal{L}_c S^4) = (\mathbb{R}[g_4, g_7, sg_4, sg_7, w], d),$$
$$|w| = 2, \qquad |sg_4| = 3, \qquad |sg_7| = 6,$$
$$dg_4 = (sg_4) \cdot w, \qquad dg_7 = -\tfrac{1}{2} g_4^2 + (sg_7) \cdot w,$$
$$d(sg_4) = 0, \quad d(sg_7) = (sg_4) \cdot g_4, \quad dw = 0.$$

This follows from a theorem of Vigué-Poirrier and Burghelea (1985), which tells you how to produce $M(\mathcal{L}_c Z)$ from $M(Z)$. Now we have

$$\varphi_1 : X^{11}/S^1 \to \mathcal{L}_c S^4 \qquad \text{and}$$

$$F_2 := \varphi_1^*(w), \quad H_3 := \varphi_1^*(sg_4), \quad F_4 := \varphi_1^*(g_4), \quad H_7 := \varphi_1^*(g_7).$$

Equations of motion (EOMs) of 10d type-IIA supergravity:

$$dF_4 = H_3 \wedge F_2, \qquad dH_7 = -\tfrac{1}{2} F_4 \wedge F_4 + F_6 \wedge F_2,$$
$$dH_3 = 0, \qquad dF_6 = H_3 \wedge F_4, \qquad dF_2 = 0.$$

Further Dimensional Reductions

This pattern continues for all $k \geq 0$ with $\mathcal{L}_c^k S^4$ serving as a universal target space for $(11-k)$-dim supergravity.

The equations of motion of $(11-k)$-dim supergravity are the equations for the differential in $M(\mathcal{L}_c^k S^4)$.

11d M-theory spacetime	$X^{11} \xrightarrow{\varphi} S^4$
Type IIA string in 10d	$X^{11}/S^1 \xrightarrow{\varphi_1} \mathcal{L}_c S^4$
9d supergravity spacetime	$(X^{11}/S^1)/S^1 \xrightarrow{\varphi_2} \mathcal{L}_c^2 S^4$

$$\begin{array}{ccc} X^{11} & \xrightarrow{\varphi} & S^4 \\ \downarrow & & \\ X^{11}/S^1 & \xrightarrow{\varphi_1} & \mathcal{L}_c S^4 \\ \downarrow & & \\ (X^{11}/S^1)/S^1 & \xrightarrow{\varphi_2} & \mathcal{L}_c^2 S^4 \\ \downarrow & & \\ \cdots & & \end{array}$$

Hypothesis H in all dimensions, as per Sati-V 2021

Hypothesis H

The dynamics of supergravity reduced to $11-k$ dimensions is governed by the rational homotopy theory (RHT) of $\mathcal{L}_c^k S^4$.

Principle H

Any feature of or statement about the Sullivan minimal model $M(\mathcal{L}_c^k S^4)$ of the iterated cyclic loop space $\mathcal{L}_c^k S^4$ (or the rational homotopy type thereof) may be translated into a feature of or statement about the reduction of M-theory to $11-k$ dimensions. Here $0 \leq k \leq 11$.