Rational Homotopy Theory, del Pezzo surfaces, and Mysterious Triality

Lecture III: The E_k symmetry of cyclic loop spaces

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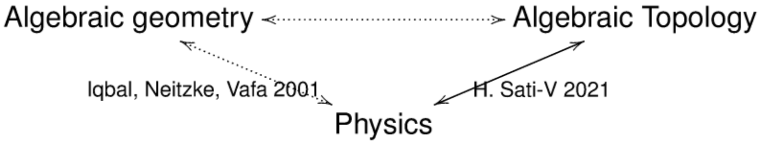
The Goal of These Lectures: Mysterious Triality



Main themes:

1. **Math Physics**: The RHT of iterated *cyclic loop spaces* $\mathcal{L}_c^k S^4$ is explicitly related to the M-theory story. (E.g., want equations of motion of M-theory wrapped on T^5, *i.e.*, 6d supergravity? — Read them off from the differential in the Sullivan minimal model of $\mathcal{L}_c^5 S^4$! (**Hypothesis H**));
2. **Mathematics**: $S^4, \mathcal{L}_c S^4, \mathcal{L}_c^2 S^4, \ldots$ is a new series of objects with hidden internal E_k symmetry, yielding such features as 27 "lines" in $\mathcal{L}_c^6 S^4$, 28 "bitangents" in $\mathcal{L}_c^7 S^4 \ldots$

Cyclic Loop Spaces $\mathcal{L}_c^k S^4$

The *free loop space* of a topological space Z:

$$\mathcal{L}Z = \mathrm{Map}(S^1, Z).$$

It admits a natural action of the group S^1 by rotating loops, and we define the *cyclic loop space* $\mathcal{L}_c Z$ to be the *homotopy quotient*

$$\mathcal{L}_c Z := \mathcal{L}Z /\!\!/ S^1 = \mathcal{L}Z \times_{S^1} ES^1,$$

the *Borel construction*. For $k \geq 0$, the *iterated cyclic loop space* (*cyclification*) $\mathcal{L}_c^k Z$ is the k-fold iteration of the cyclic loop space construction:

$$\begin{aligned} \mathcal{L}_c^0 Z &:= Z, \\ \mathcal{L}_c^k Z &:= \mathcal{L}_c(\mathcal{L}_c^{k-1} Z) \qquad \text{for } k \geq 1. \end{aligned}$$

We will be interested in the RHT of the iterated cyclic loop spaces $\mathcal{L}_c^k S^4$ of the 4-sphere S^4 for $0 \leq k \leq 8$ and beyond.

Recipe for Cyclic Looping

Theorem (Vigué-Poirrier, Burghelea 1985)

$$\textit{If} \quad M(Z) = (S(V), d), \quad \textit{then}$$

$$M(\mathcal{L}_c Z) = (S(V \oplus sV \oplus \mathbb{R}w), d_c) \quad \textit{with}$$

$$|w| = 2, \quad |sv| = |v| - 1,$$

$$\begin{aligned} d_c v &:= dv + sv \cdot w, \\ d_c sv &:= -sdv, \\ d_c w &:= 0. \end{aligned}$$

Remark: If $\mathcal{L}_c Z$ acquires connected components, drop all of them, expect for the component of the constant loop. Accordingly, if it happens that $|sv| = 0$, truncate: $sv = 0$, so as to stay within connected spaces and DGCAs.

Examples: Sullivan Minimal Models of S^4 and $\mathcal{L}_c S^4$

For example,

$$M(S^4) = (\mathbb{R}[g_4, g_7], d),$$
$$|g_4| = 4, \qquad |g_7| = 7,$$
$$dg_4 = 0, \qquad dg_7 = -\tfrac{1}{2} g_4^2,$$

i.e., $V = \mathbb{R}g_4 \oplus \mathbb{R}g_7$, whence $sV = \mathbb{R}sg_4 \oplus \mathbb{R}sg_7$ and

$$M(\mathcal{L}_c S^4) = (\mathbb{R}[g_4, g_7, sg_4, sg_7, w], d),$$
$$|w| = 2, \qquad |sg_4| = 3, \qquad |sg_7| = 6,$$
$$dg_4 = (sg_4) \cdot w, \qquad dg_7 = -\tfrac{1}{2} g_4^2 + (sg_7) \cdot w,$$
$$d(sg_4) = 0, \quad d(sg_7) = (sg_4) \cdot g_4, \quad dw = 0.$$

Example: the Sullivan Minimal Model of $\mathcal{L}_c^2 S^4$

For the Sullivan minimal model $M(\mathcal{L}_c^2 S^4)$ of the double cyclic loop space of the sphere S^4, we have

$$M(\mathcal{L}_c^2 S^4) = (\mathbb{R}[g_4, g_7, s_1 g_4, s_1 g_7, w_1, s_2 g_4, s_2 g_7, s_2 s_1 g_4, s_2 s_1 g_7, s_2 w_1, w_2], d)\,,$$

$$dg_4 = s_1 g_4 \cdot w_1 + s_2 g_4 \cdot w_2, \qquad dg_7 = -\tfrac{1}{2} g_4^2 + s_1 g_7 \cdot w_1 + s_2 g_7 \cdot w_2,$$

$$ds_1 g_4 = s_2 s_1 g_4 \cdot w_2, \quad ds_1 g_7 = s_1 g_4 \cdot g_4 + s_2 s_1 g_7 \cdot w_2,$$

$$ds_2 g_4 = -s_2 s_1 g_4 \cdot w_1 + s_1 g_4 \cdot s_2 w_1,$$

$$ds_2 g_7 = s_2 g_4 \cdot g_4 - s_2 s_1 g_7 \cdot w_1 - s_1 g_7 \cdot s_2 w_1,$$

$$ds_2 s_1 g_4 = 0, \quad ds_2 s_1 g_7 = -s_2 s_1 g_4 \cdot g_4 + s_1 g_4 \cdot s_2 g_4,$$

$$dw_1 = s_2 w_1 \cdot w_2, \quad ds_2 w_1 = 0, \quad dw_2 = 0\,.$$

BTW, the physics form-field notation in 9d supergravity:

$$g_4 = F_4, \quad g_7 = H_7, \quad s_1 g_4 = H_3^{(1)}, \quad s_1 g_7 = F_6^{(1)}, \quad s_2 g_4 = H_3^{(2)}, \quad s_2 g_7 = F_6^{(2)},$$

$$s_2 s_1 g_4 = \mathcal{F}_2, \quad s_2 s_1 g_7 = \mathcal{F}_5, \quad w_1 = F_2^{(1)}, \quad w_2 = F_2^{(2)}, \quad s_2 w_1 = F_1^{(2)} \text{(axion field)}.$$

Toroidal Symmetries of $\mathcal{L}_c^k S^4$ and Its Rtnl Hmtpy Type

Theorem (Sati-V, *Comm. Math. Phys.* (2023), arXiv:2111.14810)

For each $k \geq 0$, the automorphism group $\operatorname{Aut} M$ of the Sullivan minimal model $M = M(\mathcal{L}_c^k S^4) \otimes_{\mathbb{Q}} \mathbb{R}$ is a real algebraic group which contains a canonically defined maximal $\mathbb{R}$-split torus

$$T \cong (\mathbb{R}^\times)^{k+1} \subseteq \operatorname{Aut} M,$$

where $\mathbb{R}^\times = \mathbb{R} \setminus \{0\} = \mathbb{G}_m(\mathbb{R})$.

The idea of proof. For any $q \neq 0 \in \mathbb{Z}$, define $\varphi_0(q) : S^4 \to S^4$ to be the degree-q folding map. It induces the homomorphism

$$\begin{aligned} \varphi_0(q)_* : \pi_4(S^4) &\longrightarrow \pi_4(S^4) \\ x &\longmapsto qx. \end{aligned}$$

Rationally, it is an isomorphism! Combine inverses of such with p-folding maps to get a natural geometric action of $p/q \in \mathbb{Q}^\times$ on S^4 and then on $\mathcal{L}_c^k S^4$. Extend by continuity to $\mathbb{R}^\times$.

The Idea of Proof, Continued

Similarly, the degree-q folding map $\varphi_i(q) : S^1 \to S^1$, acting on the ith source S^1 in $\mathcal{L}_c^k S^4$, $0 \leq i \leq k$, induces a rational isomorphism

$$\mathcal{L}_c^k S^4 \xrightarrow{\sim} \mathcal{L}_c^k S^4,$$

which extends to an action

$$\mathbb{R}^\times \times \mathcal{L}_c^k S^4 \to \mathcal{L}_c^k S^4$$

in the rational homotopy category. This gives an action

$$(\mathbb{R}^\times)^{k+1} \times M(\mathcal{L}_c^k S^4) \to M(\mathcal{L}_c^k S^4)$$

on the Sullivan minimal model.
The maximality of the resulting split real torus $(\mathbb{R}^\times)^{k+1}$ is pure RHT. The action of an $\mathbb{R}$-split torus T on M is determined by its action on $P(M) := Z(M^+)/Z(M^+) \cap (M^+)^2$. This implies $\dim T \leq \dim P(M) = k + 1$. □

Extracting the E_k Data from $\mathcal{L}_c^k S^4$

Theorem (Sati-V, *Comm. Math. Phys.* (2023), arXiv:2111.14810)

The Lie algebra $\mathfrak{h}_k = \mathrm{Lie}(T)$ *of* $T \cong (\mathbb{R}^\times)^{k+1}) \subseteq \mathrm{Aut}\, M(\mathcal{L}_c^k S^4)$ *has a natural basis, giving a* ***lattice*** $\mathfrak{h}_k^{\mathbb{Z}} \subseteq \mathfrak{h}_k$*, an* ***integral inner product****, and a* ***distinguished element*** $-K_k \in \mathfrak{h}_k^{\mathbb{Z}}$*.*
The triple $(\mathfrak{h}_k^{\mathbb{Z}}, (-,-), -K_k)$ *associated to the cyclic loop spaces* $\mathcal{L}_c^k S^4$ *and their Sullivan minimal models* $M(\mathcal{L}_c^k S^4)$ *consists of*

- *a free abelian group* $\mathfrak{h}_k^{\mathbb{Z}}$ *with a basis* $h_0, h_1, \ldots, h_k$*;*
- *a symmetric bilinear form* $\mathfrak{h}_k^{\mathbb{Z}} \otimes \mathfrak{h}_k^{\mathbb{Z}} \to \mathbb{Z}$ *given by*

$$(h_0, h_0) = 1, \qquad (h_i, h_j) = -\delta_{ij}, \qquad i > 0, j \geq 0;$$

- *an element* $-K_k = 3h_0 - h_1 - \cdots - h_k$*.*

This algebraic structure produces the root system E_k*, as for del Pezzo surfaces, now in the context of cyclic loop spaces* $\mathcal{L}_c^k S^4$*.*

Cf. Extracting the E_k Data from del Pezzo $\mathbb{B}_k$

1. The rank-$(k+1)$ lattice $N_k = H_2(\mathbb{B}_k) \subset N_k \otimes \mathbb{R}$;
2. The Lorentzian inner product $(-,-) : N_k \otimes N_k \to \mathbb{Z}$;
3. The anticanonical class $-K_k = 3H - E_1 - \cdots - E_k$.

Theorem (Y. I. Manin, *Cubic Forms*, 1972)

The triple $(N_k, (-,-), -K_k)$ gives rise to the wealth of combinatorial data associated with the root system of type E_k:

- *The very root system $R_k = \{\beta \in N_k \mid (\beta, -K_k) = 0, (\beta, \beta) = -2\}$ sitting in the Euclidean space $(-K_k)^\perp := \{\beta \in N_k \otimes \mathbb{R} \mid (\beta, -K_k) = 0\}$;*
- *The Weyl group $W(E_k)$;*
- *The system of “lines” $l_k = \{\beta \in N_k \mid (\beta, -K_k) = 1, (\beta, \beta) = -1\}$;*
- *Etc.*

The Idea of Proof

Since the action of the folding maps on the iterated cyclic loop space is by **precomposition** in the k sources S^1 and **postcomposition** in the single target S^4, one gets from this that the weights for the infinitesimal generators of the folding actions are -1 for the infinitesimal source foldings and $+1$ for the target folding, thus giving a natural Lorentzian metric.

The basis is given by these infinitesimal generators of the torus action.

The distinguished vector $-K_k$, which is the unique element of $\mathfrak{h}_k$ which acts on the graded Lie algebra $Q(\mathcal{L}_c^k S^4)$ (the Quillen model) by the degree operator via the Sullivan model/Quillen model duality. □

27 “lines” on 6-Fold Cyclic Loop Space $\mathcal{L}_c^6 S^4$

Theorem (Sati-V, *Comm. Math. Phys.* (2023), arXiv:2111.14810)

The 27 exceptional vectors $\alpha \in (\mathfrak{h}_6^{\mathbb{Z}})^*$, $(\alpha, \alpha) = (\alpha, K_6^*) = -1$, *give rise to 27 canonically defined lines in the* $\mathbb{R}$*-vector space* $\pi_2^{\mathbb{R}}(\mathcal{L}_c^6 S^4) := \pi_2(\mathcal{L}_c^6 S^4) \otimes_{\mathbb{Q}} \mathbb{R}$*. Moreover, these lines freely generate* $\pi_2^{\mathbb{R}}(\mathcal{L}_c^6 S^4)$ *and thus* $\dim \pi_2^{\mathbb{R}}(\mathcal{L}_c^6 S^4) = 27$*.*

The idea of proof. The Sullivan minimal model $M(\mathcal{L}_c^6 S^4)$ is actually $S(\pi_\bullet^{\mathbb{R}}(\mathcal{L}_c^6 S^4)^*)$. Look at the weight spaces in $M(\mathcal{L}_c^6 S^4)$ corresponding to the above α's. They are 1-dimsnl and are generated by the elements

$$\begin{aligned} & w_1, \dots, w_6, & \\ & s_j s_i g_4, & 1 \le i < j \le 6, \\ & s_6 \dots \widehat{s_i} \dots s_1 g_7, & 1 \le i \le 6, \end{aligned} \tag{1}$$

On the other hand, $Q(\mathcal{L}_c^6 S^4) = s\pi_\bullet^{\mathbb{R}}(\mathcal{L}_c^6 S^4)$ (degree shift). □
The **lines**, $S^2 = \mathbb{CP}^1 \to \mathcal{L}_c^6 S^4$, in rational homotopy category.

The $k = 8$ Threshold

The distinguished element $K_k \in \mathfrak{h}_k^{\mathbb{Z}}$ with Minkowski inner product $(-,-)$ on $\mathfrak{h}_k^{\mathbb{Z}}$ gives $K_k^{\perp} = \{x \in \mathfrak{h}_k \mid (x, K_k) = 0\}$.

$$\dim \mathfrak{h}_k = k + 1, \qquad \dim K_k^{\perp} = k$$

- $\{\text{Roots of } E_k\} = \{x \in \mathfrak{h}_k^{\mathbb{Z}} \mid (x,x) = -2, (x, K_k) = 0\} \subset K_k^{\perp}$ form a finite set iff $k \leq 8$.
- $(-,-)|_{K_k^{\perp}}$ is negative definite iff $k \leq 8$.

Hypothesis H in All Dimensions, as per Sati-V 2021

Hypothesis H

The dynamics of supergravity reduced to $11 - k$ dimensions is governed by the rational homotopy theory (RHT) of $\mathcal{L}_c^k S^4$.

Principle H

Any feature of or statement about the Sullivan minimal model $M(\mathcal{L}_c^k S^4)$ of the iterated cyclic loop space $\mathcal{L}_c^k S^4$ (or the rational homotopy type thereof) may be translated into a feature of or statement about the reduction of M-theory to $11 - k$ dimensions. Here $0 \leq k \leq 11$.

Conjecture: duality between del Pezzo surfaces and loop spaces of S^4

$$\text{Algebraic geometry} \overset{?}{\longleftrightarrow} \text{Algebraic Topology}$$

Conjecture

There must be an explicit relation between the series of del Pezzo surfaces $\mathbb{B}_k$, $0 \leq k \leq 8$, and the series of iterated cyclic loop spaces $\mathcal{L}_c^k S^4$, $0 \leq k \leq 8$. This relation should match the E_k symmetry patterns occurring in both series, as well as relate other geometric data, such as the volumes of curves on del Pezzo surfaces, with some geometric data, such as the radii of S^4 and S^1s, for the iterated cylic loop spaces $\mathcal{L}_c^k S^4$.