Introduction to h-principle

<u>Main question:</u> Let $\mathcal{R}$ be a class of geometric strs. Fix manifold M.

What is the weak homotopy type of
$$Sol_{\mathcal{R}}(M) = \left\{ \begin{array}{c} \text{geometric structures of} \\ \text{class } \mathcal{R} \text{ in } M \end{array} \right\} ?$$

Concretely:

- is this $Sol_{\mathcal{R}}(M)$?
- what is π_0?
- π_1 ...

<u>Goal:</u> understand patterns common to many $\mathcal{R}$.

I. The language

Idea: • geometric strs

$\quad\quad\quad$ |||

sections of some bundle $E \to M$

satisfying a *partial differential relation.*

• we define $J^r E \to M$

$\quad\quad\quad\quad\quad\quad$ |||

Taylor pols of sections of E

of order r.

a relation will be $R \subset J^r E$

• *Ex:* Say we look at $f: \mathbb{R}^n \to \mathbb{R}^m$

Then the *space of r-jets* is:

$$J^r(\mathbb{R}^n, \mathbb{R}^m) = \mathbb{R}^n \times \mathbb{R}^m \times \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m) \times \dots \times \mathrm{Sym}^r(\mathbb{R}^n, \mathbb{R}^m)$$

given $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$\rightsquigarrow$ *r-jet extension*

$$j^r f: \mathbb{R}^n \longrightarrow J^r(\mathbb{R}^n, \mathbb{R}^m)$$
$$x \longmapsto (x, f(x), D_x f \dots D^r_x f)$$

Obs: not all sections of $J^r(\mathbb{R}^n, \mathbb{R}^m)$

are of this form.

$$x \longmapsto (x, y(x), z_1(x), \dots z_r(x))$$

Def: Given $f, g : M \to E$ they are

- 0 - tangent at $x \in M$ $\equiv$ $f(x) = g(x)$
- 1 - tangent at x $\equiv$ $d_x f = d_x g$
- r - tangent at x $\equiv$ df and dg are $(r-1)$-tangent at x.

Equivalence classes $\equiv$ <u>r-jets</u>

<u>Exercise</u>: TFAE:

- f, g are r-tangent at x
- in coordinates, they have same Taylor pol of order r at x.

Def: • Given $E \to M$, we define $J^r E$: as set:

$$J^r_x E \equiv \{ f : U \to E|_U \mid U \ni x \text{ open} \} \Big/ \begin{array}{l} r\text{-tangency} \\ \text{at } x. \end{array}$$

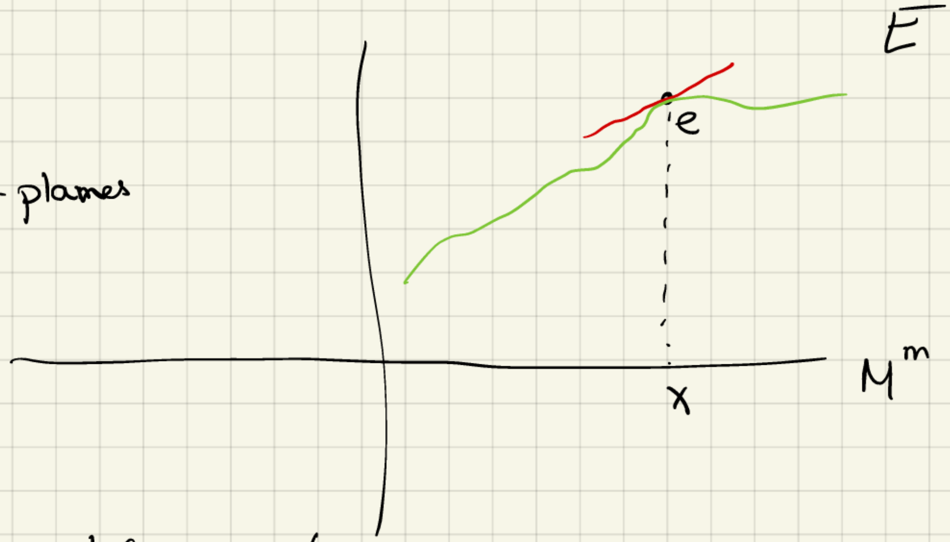
- Smooth str of $J^r E$ is given by requiring that

$$\begin{tikzcd}
J^r(\mathbb{R}^n, \mathbb{R}^m) \arrow[r] \arrow[d] & J^r E \arrow[d] \\
\mathbb{R}^n \times \mathbb{R}^m \arrow[r] \arrow[d] & E \arrow[d] \\
\mathbb{R}^n \arrow[r] & M
\end{tikzcd}$$

smooth charts.

<u>Ex</u>: check these charts are smoothly compatible.

Ex: $J^1E \equiv$ {graphical m-planes in TE}



$J^1(\mathbb{R},\mathbb{R}) = \{(x, y, \underbrace{z}_{\text{slope}})\}$

Def: • A differential relation is a subset $R \subset J^rE$.

• A solution is $f: M \longrightarrow E$ s.t

$$j^r f : M \longrightarrow R \subset J^rE.$$

• A formal solution is $F: M \longrightarrow R$

! F is a smooth collection of pointwise solutions (formally up to order r) but need not patch to global solution.

Ex: A differential operator of order r is:

$$\begin{tikzcd} E \arrow[dr] & & \tilde{E} \arrow[dl] \\ & M & \end{tikzcd}$$

$\Gamma(E) \xrightarrow{P} \Gamma(\tilde{E})$ in such a way:

$P = \sigma(P) \circ j^r$ where

$\overbrace{\sigma(P)}^{\text{symbol}} : J^rE \longrightarrow \tilde{E}$ bundle map.

• j^r is universal diff operator

- Fix $g: M \to \tilde{E}$ then

$$\{ f: M \to E \mid P(f) = g \} \equiv \text{Solutions of relation } \{\sigma(P)^{-1}(g)\} \subset J^r E$$

$$= \{ \sigma(P)(j^r f) = g \}$$

<u>Ex</u>: • $f: M \to N$ is an <u>immersion</u> if $\operatorname{rank}(df) = \dim(M)$

$\rightsquigarrow$ relation $R_{imm} \subset J^1(M,N)$

$$R_{imm} = \left\{ j^1_x f \;\middle|\; d_x f \text{ has rank} = \dim(M) \right\}$$

- Formal solutions of R_{imm} are:

$$\left\{ \begin{array}{ccc} TM & \xrightarrow{F} & TN \\ \downarrow & & \downarrow \\ M & \xrightarrow[f]{} & N \end{array} \;\middle|\; \operatorname{rank}(F) = \dim(M) \right\}$$

$$= \operatorname{Mon}(TM, TN) \qquad \text{⚠ there is no geometry here}$$
$$\downarrow$$
$$\text{"}E\text{"} = M \times N \longrightarrow M$$

<u>Main goal</u>:

(<u>scanning map</u>)

$$\operatorname{Sols}_R(M) \xrightarrow{\;i\;} \operatorname{Sols}^f_R(M)$$

$$\{\text{solutions of } R \text{ on } M\} \qquad \{\text{formal sols}\} = \Gamma(R)$$

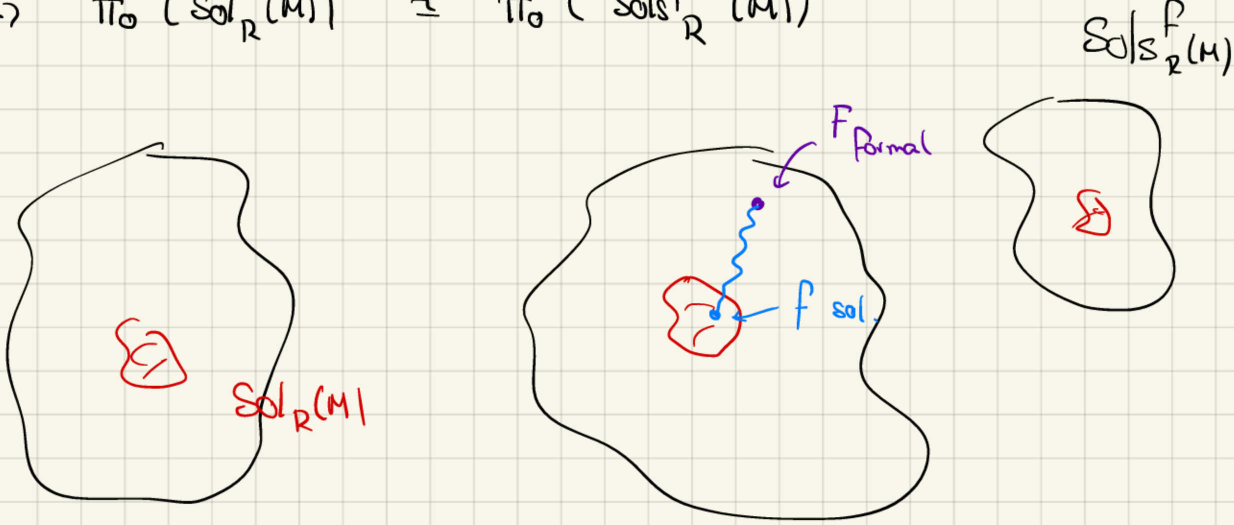
$$f \longmapsto j^r f$$

we will say that R satisfies the h-principle
if i is a weak hom. eq.

homotopy principle

- if we have weak eq.

$\Rightarrow \pi_0(Sol_R(M)) \cong \pi_0(Sols^f_R(M))$



- i being w. hom. eq. is local-global idea.

F formal $\xrightarrow{\text{homotopy}}$ f solution

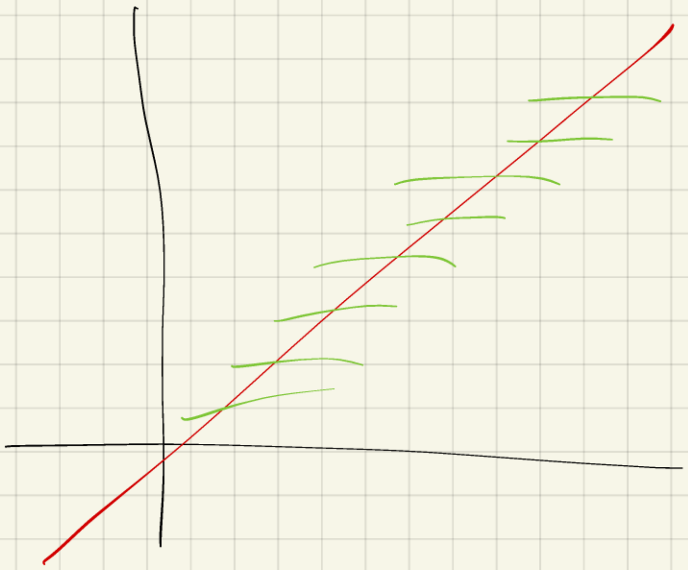
(pointwise infinitesimal sol.)

Def: A differential relation is Diff-invariant if
- the bundle $E \to M$ is natural

  ($Diff(M)$ lifts to E)
- since J^rE is natural, we want R to be invariant under $Diff(M)$.

Ex: (M, g_M) , (N, g_N) Riemannian

$\{$ isometric maps $M \to N \}$ not Diff-invariant



$$J^1(\mathbb{R},\mathbb{R}) \simeq \mathbb{R}^3 = \{(x,y,z)\}$$

slope

$$\cup$$

$$R = \{z=0\}$$