Main question: what about M closed?

A. Microextension

Thm (Smale - Hirsch, 50's)

$$\mathrm{Imm}(M,N) \simeq \mathrm{Mon}(TM, TN)$$

if $\dim(M) < \dim(N)$.

Pf/

$$F \in \mathrm{Mon}(TM, TN)$$

$$\underbrace{F^* TN/_{TM}}_{\nu_F(M)} \to M \quad \text{" normal bundle "}$$

$\nu_F(M)$ open

$$\text{Fibre} \to \mathrm{Imm}(\nu_F(M), N) \stackrel{!}{\simeq} \mathrm{Imm}^F(\nu_F(M), N)$$
$$\downarrow$$
$$\mathrm{Imm}(M,N)$$

$\square$

Cor: sphere eversion (Smale)

Thm (McDuff):

h-principle holds for $\overbrace{\text{even - contact}}^{(M^{2n+2},\ \xi^{2n+1} = \ker(\alpha))\ \ d\alpha \wedge d\alpha^n \neq 0}$

$\underbrace{\text{odd - symplectic}}_{(M^{2n+1},\ \omega)\ \ d\omega = 0,\ \omega^n \neq 0.}$

Pf/ extend to contact / symplectic in $M \times \mathbb{R}$.

$\square$

⚠ quite special.

B. Convex integration

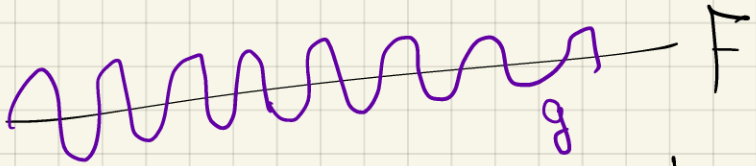
Baby problem Take $(S^1, d\theta^2) \xrightarrow{f} (\mathbb{R}^n, g_{std})$

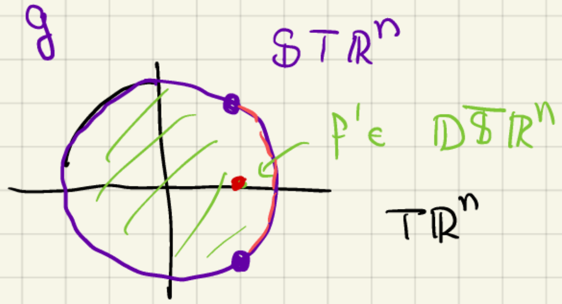
short $(f^* g_{std} < d\theta^2)$

$\Rightarrow \exists\ g : S^1 \to \mathbb{R}^n$ s.t

- $g^* g_{std} = d\theta^2$ isometry
- $g \underset{C^0}{\sim} f$

Pf





it works because

- Convex Hull $(S^{n-1}) = D^n$
- S^{n-1} is path-conn.

$\square$

Thm (1-dim convex int (Gromov)
Filipov's relaxation)

Fix $R \subset J^1 E \longrightarrow E \longrightarrow S^1$

If R is ample then h-principle!

$R_e \subset J^1_e E$ is path-connected

and Convex Hull $(R_e) = J^1_e E$

moreover, any formal sol can be C^0-approx by solution.

Thm: Let $R \subset J^r E \to M$ be open and ample.

Then R satisfies h-principle (Gromov).

Pf/ Assume $M = [0,1]^n$, $E = [0,1]^n \times \mathbb{R}^e$

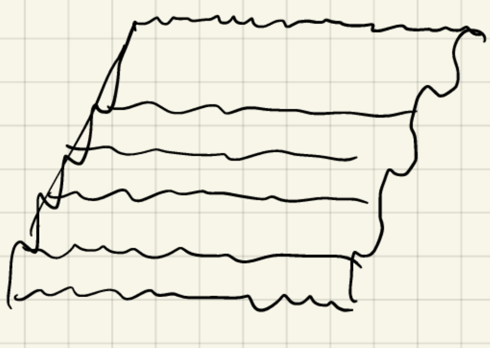
$$F : M \longrightarrow J^1 E$$
$$x \longmapsto (x, f(x), \underbrace{F_1(x)}_{\text{"}\partial_1 f(x)\text{"}} \dots F_n(x))$$

See $[0,1]^n$ as $[0,1]^{n-1}$-family of $[0,1]$

apply 1-dim case to F: ⊂ ampleness means ampleness in each 1-dim direction

$\rightsquigarrow \tilde{F} = (x, \tilde{f}(x), \partial_1 \tilde{f}(x), \tilde{F}_2(x), \dots \hat{F}_n(x))$

iterate being careful. □



Thm: (Nash, 50s)

Any embedding $(N, g_N) \overset{f}{\hookrightarrow} (\mathbb{R}^m, g_M)$

is homotopic to a C^1-isometric embedding.

Pf/ Assume f is tiny thus short.

$$f \rightsquigarrow f_1 \text{ that is } 1/2\text{-short}$$
$$\rightsquigarrow f_2 \quad " \quad " \quad 1/2^2\text{-short}$$
$$\rightsquigarrow f_i \quad " \quad " \quad 1/2^i\text{-short}$$

$\xrightarrow{\text{limit}}$ g C^1 - isometric ☐

$\equiv$

!

<u>Open problem</u>: $C^{1,\alpha}$ - isometric embeddings

optimal α ?

<u>Thm</u> (de Lellis, Székelyhidi) . .)

Using convex int to construct low regularity sols of fluid equations.

(Onsager's conjecture)

<u>Thm</u>: (dP, Martínez Aguinaga, Zelenko, Jovanic)

There is h-principle for:

- $(3,5)$, $(3,6)$, bracket-gen in most cases.
- $(4,6)$ hyperbolic] convex integration w/ avoidance
- corank - 2, odd dim, maximally non-integrable.

<u>Open question</u>: what families of distributions satisfy h-principle ?

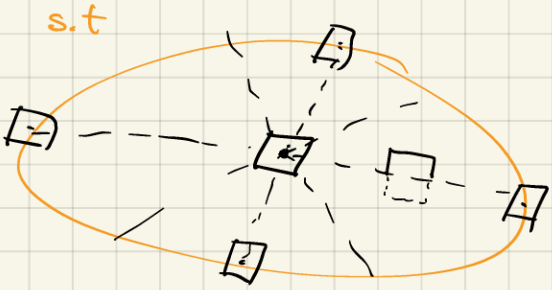
C. Removal of singularities

Obs: $\underbrace{\mathrm{Cont}}_{\text{contact str.}}(S^3) \not\simeq \mathrm{Cont}^f(S^3) = \text{Plane fields}(S^3)$

Bennequin.

∃ std and $\underbrace{\exists \text{ overtwisted}}_{\exists\, D^2 \hookrightarrow S^3}$ that are hom. as planes but not as contact

s.t

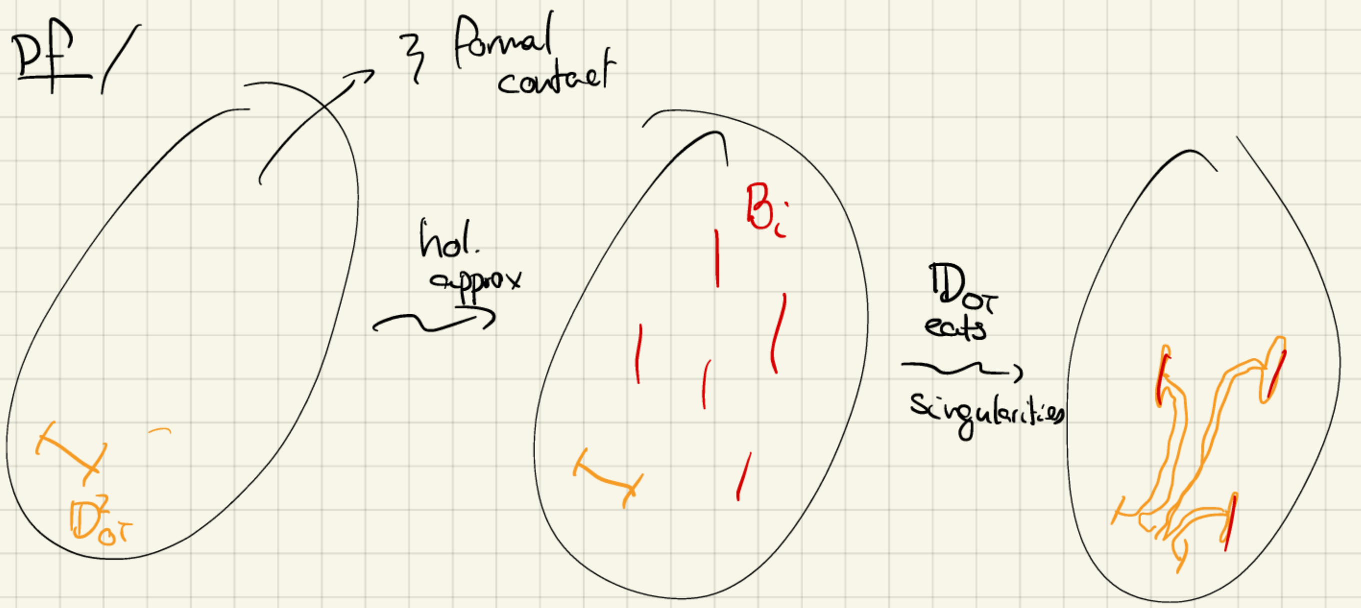


Thm (Eliashberg '89)

$$\mathrm{Cont}_{OT}(M, D^2) \xrightarrow[\simeq]{} \mathrm{Cont}^f_{OT}(M, D^2)$$

Q: what is live between h-principle and non-trivial invariants?

Pf/



□

Thm same for Engel.

Open question: how general is OT?