Einstein manifolds with symmetry.

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Goal: To discuss recent developments & open questions about Einstein Riemannian manifolds on which a Lie group acts by isometries, including (but not restricting ourselves to) homogeneous Einstein spaces.

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PLAN

Lecture 1. Isometric group actions. Left-invariant metrics. Symmetric spaces. Ricci curvature & Einstein manifolds.

Lecture 2 Homogeneous Einstein manifolds: state of the art. Ricci curvature of left-invariant metrics and a link with moment maps and GIT. The topology of homogeneous spaces. The Alekseevskii conjecture: proof strategy.

Lecture 3 Non-transitive symmetries, principal bundles. Ricci curvature of a Riem. submersion. Einstein's equations with symmetry. Sketch of proof of the Alek. conj. Open questions: rigidity; Ricci flow; existence.

Lecture 1

(M^n, g) Riemannian manifold

$\hookrightarrow$ metric: $p \in M \mapsto g_p(\cdot,\cdot)$ inner product on T_pM (positive-definite)

In local coords (x^i) $(U \subseteq M)$, $g \leftrightsquigarrow (g_{ij})_{1\le i,j\le n}$

$g_{ij}: U \to \mathbb{R}$, $g_{ij}(p) = g_p(\partial_i, \partial_j)$ $\qquad \partial_i := \partial/\partial x^i$.

A (smooth) Lie group G acts (on the left) on M^n $(G \curvearrowright M^n)$ if $\exists$ (smooth) map

$$G \times M \longrightarrow M, \qquad (h,p) \longmapsto h\cdot p,$$

such that: $\quad e\cdot p = p, \quad \forall p \in M$

$$h\cdot(k\cdot p) = (hk)\cdot p, \qquad \forall h,k \in G,\ \forall p \in M.$$

Notice: $g \in G \longmapsto \tau_g \in \mathrm{Diff}(M)$, $\quad \tau_g(p) = g\cdot p$

is a group homomorphism. If $\tau: G \to \mathrm{Diff}(M)$ injective we call the action effective.

G acts on (M^n, g) by isometries if

$$\tau: G \longrightarrow \mathrm{Isom}(M^n, g) := \{f \in \mathrm{Diff}(M) : f^*g = g\}.$$

The action is proper if $\tau(G) \subseteq \mathrm{Isom}(M^n,g)$ is a closed subgroup.

Typically, we'll assume all these properties w/o. explicitly saying so.

Given $p \in M$, we call $G\cdot p := \{g\cdot p : g \in G\} \subseteq M$ the orbit through p, and $G_p := \{g \in G : g\cdot p = p\} \le G$ the stabiliser (or isotropy) subgroup at p.

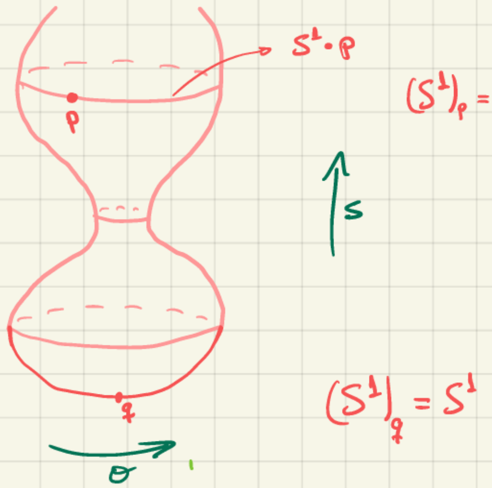
For proper actions, $G\cdot p$ is an embedded submanifold in M.

If all the stabilisers are trivial, we call the action free.

Ex. 1 Surfaces of revolution.

$(\Sigma^2, g = ds^2 + \underbrace{\varphi(s)^2}_{\text{indep. of } \theta \,!} \cdot d\theta^2)$

The group $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ acts by isometries. $\overset{\shortparallel}{U(1)} \simeq SO(2)$



Ex. 2. Linear actions (representations).

Any subgroup $G \leq Gl_n(\mathbb{R}) = \{n \times n \text{ invertible matrices}\}$ acts on $\mathbb{R}^n$ via $A \in G,\ x \in \mathbb{R}^n \longmapsto Ax \in \mathbb{R}^n$.

This action is by isometries iff $G \leq O(n) := \{A : A^T A = I\}$

- The round sphere $S^n = \{x \in \mathbb{R}^{n+1} : |x|^2_{Eucl} = 1\}$:
  $SO(n+1)$ acts (linearly) on $\mathbb{R}^{n+1}$, preserving S^n.
  $\rightsquigarrow$ it acts on S^n, by isometries and transitively
  (i.e. $\forall p, q \in S^n$, $\exists R \in SO(n+1) : R \cdot p = q$).

  If $p_N = (1, 0, \dots, 0) \in S^n \Rightarrow (SO(n+1))_{p_N} \simeq SO(n)$, thus
  $\underset{\text{diffeo}}{S^n \simeq} SO(n+1)/SO(n)$ as a homogeneous space
  (quotient of Lie group by closed subgroup).

Obs: S^2 is also a surface of revolution (Ex 1). This action is not transitive

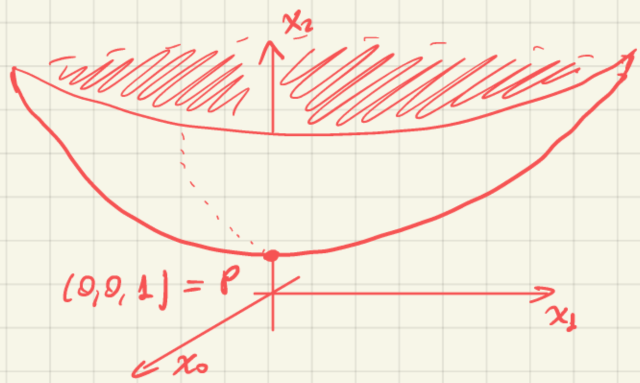
Ex 3 Hyperbolic space $H^n = \{x \in \mathbb{R}^{n,1} : \underbrace{x_0^2 + x_1^2 + \dots + x_{n-1}^2 - x_n^2}_{=:\langle x, x\rangle_M \,\longrightarrow\, \text{Minkowski norm}} = 1,\ x_n > 0\}$

$\langle \cdot, \cdot \rangle_M |_{T_x H^n}$ is positive-def. $\rightsquigarrow g_{Hyp}$ metric on H^n.

$SO(n,1) := \{A \in Gl_{n+1}(\mathbb{R}) : \langle Ax, Ay\rangle_M = \langle x, y\rangle_M,\ \forall x, y\}$.

$SO(n,1)_0 \curvearrowright (H^n, g_{Hyp})$ transitively and by isometries.

$SO(n,1)_p \simeq SO(n) \rightsquigarrow H^n \simeq SO(n,1)/SO(n)$



Obs: there are two other well-known models for H^n: the upper half-space & the Poincaré ball.

Ex 4 G Lie group. It acts transitively & freely on itself by left-multiplication: $g \cdot h = gh$. The maps $L_g : G \to G$, $L_g(h) = gh$ are diffeomorphisms called left-translations.

A tensor T on G is called left-invariant if

$$T \circ L_g = (L_g)_* T \quad , \quad \forall g \in G.$$

Left-invariant vector fields are closed under bracket, thus they form a Lie algebra

$$\mathfrak{g} = \{ X \in \mathfrak{X}(G) : X \text{ left-inv} \} =: Lie(G)$$

Evaluation at $e \in G$ yields a linear isomorphism

$$\mathfrak{g} \simeq T_e G.$$

For each choice of basis $(X^i)_{i=1}^n$ of $\mathfrak{g}$, if $(\theta_i)_{i=1}^n$ denotes the dual co-frame, then a left-invariant metric is given by.

$$g = \theta_1^2 + \cdots + \theta_n^2$$

For a different choice of basis we will get (in general) non-isometric metrics.

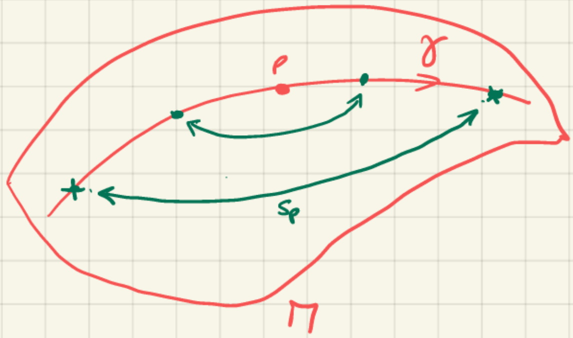
Obs: a left-invariant g is determined by a single inner product g_e on $T_e G \simeq \mathfrak{g}$.

Ex 5: A left-invariant metric which is also right-invariant is called biinvariant. These exist when G is compact.

Symmetric spaces

On a Riem. mfd (M^n, g), $p \in M$, the geodesic symmetry is a map $s_p : M \to M$ that, for each geodesic γ w/ $\gamma(0) = p$, maps

$$\gamma(t) \longmapsto \gamma(-t), \quad \forall t \in \mathbb{R}.$$



If s_p is well-defined and an isometry for all $p \in M$, we say M is a symmetric space.

Locally, this condition is equivalent to $\nabla Rm_g = 0$.

Levi-Civita

Riemann curvature tensor

Any symmetric space is the Riem. product of irreducible ones. These are classified (Cartan 1920's, see Helgason's book).

Ex:
- compact: S^n, $\mathbb{C}P^n$, $\mathbb{H}P^n$, $(G, g_{bi\text{-}inv})$, ...
- non-compact: H^n, $\mathbb{C}H^n$, $\mathbb{H}H^n$ ⟿ in general, G/K, (simple, non-cpt; maximal compact)

e.g. $Sl_n(\mathbb{R})/SO(n)$, $Sl_n(\mathbb{C})/U(n)$, etc...

Fact: The non-compact symmetric spaces also arise as left-invariant metrics on certain solvable Lie groups:

Iwasawa decomposition: $\underset{\text{non-compact semisimple}}{G} = \underset{\text{maximal compact}}{K} \cdot \underbrace{A \cdot N}_{\text{solvable subgroup}}$

E.g. $Sl_n(\mathbb{R}) = SO(n) \cdot \left\{ \begin{pmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix} : a_{ii} > 0 \right\} \cdot \left\{ \begin{pmatrix} 1 & * & * \\ & \ddots & * \\ 0 & & 1 \end{pmatrix} \right\}$

$A \cdot N \curvearrowright G/K$ transitively & freely $\therefore$ $G/K \simeq A \cdot N$, and the symmetric metric becomes a left-invariant metric on $A \cdot N$.

Einstein manifolds

(M^n, g) Einstein if

(E) $\quad Ric(g) = \lambda \cdot g, \quad \lambda \in \mathbb{R}$

cf. Einstein's field equations from General Relativity (but here $g>0$).

Albert Einstein, ca. 1915.

Recall: The Ricci curvature tensor is a symmetric, $(0,2)$-tensor (like g, but not necess. definite), defined by

$$Ric_{ij} = g^{kl} Rm_{iklj}, \qquad \text{or}$$

$$Ric(X_p, X_p) = \sum_{i=1}^{n-1} K(X, E_i), \qquad \{X, E_1, \ldots, E_{n-1}\} \text{ o.n.b. } T_pM.$$

In harmonic coordinates,

$$Ric_{ij} = -\frac{1}{2} \Delta_g g_{ij} + Q_{ij}(g, \partial g)$$

In normal coordinates, the Riemannian volume element $d\mu_g = \sqrt{\det g_{ij}}\, dx^1 \ldots dx^n$ has the following Taylor expansion

$$d\mu_g = \left(1 - \frac{1}{6} Ric_{jk} x^j x^k + O(|x|^3)\right) \cdot dx^1 \ldots dx^n$$

Rmks.
- (E) Non-linear system of elliptic PDEs.
- For $n \leq 3$, (E) $\iff$ constant curvature
- The case $n=4$ is particularly relevant, but won't be discussed here see survey by (Anderson '09).
- Key reference: (Besse, "Einstein manifolds" 1987)

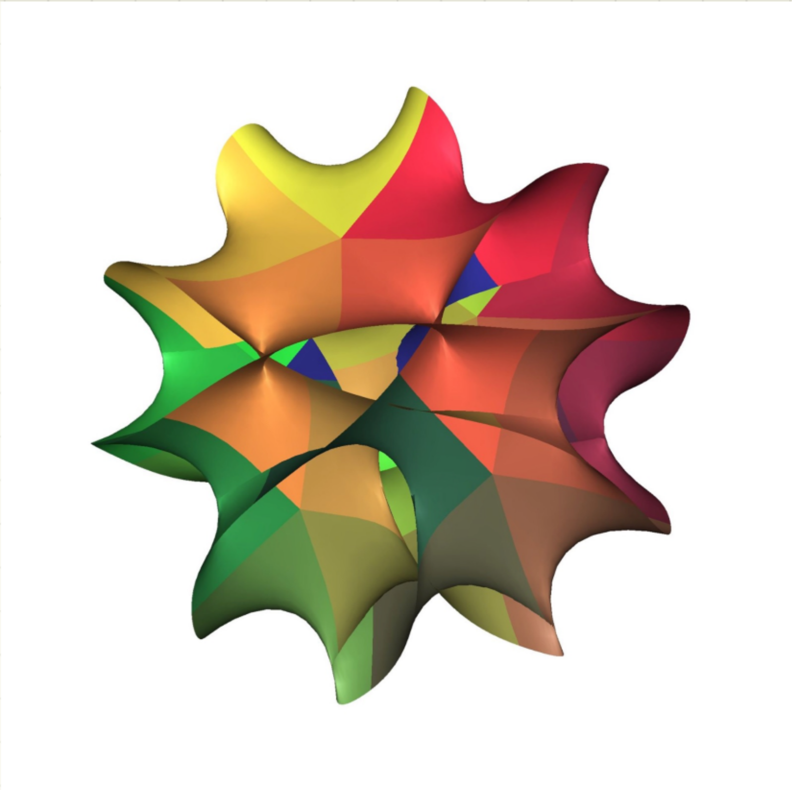
Examples: • Space forms ($\mathbb{R}^n$, S^n, H^n)

• Irreducible symmetric spaces. (Cartan 1929)

• Kähler-Einstein metrics (Aubin, Yau '76 + Yau '78)

(X, ω_0) compact Kähler mfd, $c_1(X) < 0$ (resp. $c_1(X) = 0$) $\Rightarrow$

$\exists! \, \omega \in [\omega_0]$ s.t. $Ric(\omega) = -\omega$ (resp. $\underbrace{Ric(\omega) = 0}_{\text{Calabi-Yau}}$).



img credit: Andrew J. Hanson, Wikipedia.

2-D projection of $X^6 = \{ z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 = 0 \} \subseteq \mathbb{CP}^4$.

These are obtained by solving a complex $\underbrace{\text{Monge-Ampere type}}_{\det(\mathrm{Hess}\,\varphi) = \cdots}$ equation on a single scalar function, by using the continuity method.

Fully non-linear!

as opposed to $\sim n^2$ variables g_{ij}.