# Homogeneous Einstein manifolds

a Lie group (Myers-Steenrod '39)

Recall: $(M^n, g)$ Riem. mfd is homogeneous if $\mathrm{Isom}(M^n, g)$ acts transitively. Choosing $G \leq \mathrm{Isom}(M, g)$ closed & transitive yields a presentation $(M^n, g) = (G/\underbrace{H}_{\text{compact}}, \underbrace{g}_{G\text{-invariant metric}})$.

Due to symmetries, $\mathrm{Ric}(g) = \lambda \cdot g$ (a nonlinear elliptic PDE) becomes a system of non-linear algebraic equations:
$M = G$, $g$ left-invariant, $\{E_i\}$ $g$-o.n.b. for $\mathfrak{g}$,
$[E_i, E_j] = c_{ij}^k E_k$ structure coefficients.

$$(E) \iff \sum_{k,l} -\frac{1}{2} c_{ik}^l c_{jk}^l + \frac{1}{4} c_{kl}^i c_{kl}^j - \frac{1}{2} c_{jk}^l c_{il}^k + \frac{1}{2} c_{kl}^l (c_{ki}^j + c_{kj}^i) = \lambda \cdot \delta_{ij}$$

$\lambda = 0$: (Alekseevskii-Kimelfeld '76) A homogeneous manifold with $\mathrm{Ric}_g = 0$ is flat, (thus a quotient of Euclidean space $(\mathbb{R}^n, g_{Eucl})$).

Exercise: Prove this using the Cheeger-Gromoll splitting theorem.

Even without symmetry assumptions!

$\lambda > 0$: $M$ is compact, and $\pi_1 M$ is finite (Bonnet-Myers)

Up to $\dim M \leq 11$, any compact homogeneous space $G/H$ admits a $G$-invariant Einstein metric. (Böhm-Kerr '06)

The space $M^{12} = SU(4)/SU(2)_{10}$ admits <u>no</u> invariant Einstein metric (Wang-Ziller '86).

Much progress has been made by exploiting a variational characterisation: on a compact $M$, Einstein metrics are precisely the critical points of the Einstein-Hilbert functional

$$g \longmapsto \int_M R(g)\, d\mu_g$$

scalar curvature $R(g) = g^{ij} \mathrm{Ric}_{ij}$

restricted to the space of unit-volume metrics.

However, the following fundamental problem remains open:

**Finiteness Conjecture** (Böhm, Wang, Ziller '04): Up to isometry and scaling, each compact homogeneous space $G/H$ admits only finitely many $G$-invariant Einstein metrics.

Rmk: BWZ only conjectured it for homog. spaces with $\chi(G/H) > 0$.

Euler characteristic

See: two great surveys by M. Wang:

- "Einstein metrics from symmetry & bundle constructions", 1999.
- " " " " : a sequel", 2012.

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From now on, and for the rest of these talks, we focus on $\lambda < 0$:

**Fact:** $M$ must be non-compact.

Indeed, (Bochner '49) $M$ cpt + $\mathrm{Ric}(g) < 0 \Rightarrow \dim \mathrm{Isom}(M) = 0$.

Pf/ $\mathrm{Lie}(\mathrm{Isom}(M)) = \{\text{Killing fields}\}$

$X : \mathcal{L}_X g = 0 \iff \nabla X$ skew-symmetric

If $X$ Killing, $f := \frac{1}{2}|X|^2 : M \to \mathbb{R}$ satisfies

$$\Delta f = |\nabla X|^2 - \underset{\mathrm{Ric}<0 \,+\, X\neq 0}{\mathrm{Ric}(X,X)} > 0 \quad \Rightarrow\Leftarrow \quad \text{(maximum principle)} \quad \square$$

Examples include non-compact symmetric spaces. More generally,
Einstein solvmanifolds :
Left-invariant Einstein metrics on (certain) simply-connected solvable Lie groups.

(Alekseevskii '75) First non-symmetric examples (quaternionic Kähler).

Lots of known examples (compare w/ $\lambda>0$ case):
Ex (Heber, '98): $\exists$ 46-dim family of pairwise non-isometric Einstein deformations of $M^{16} = \mathbb{H}H^4$.

Excellent survey: Lauret, "Einstein solvmanifolds & nilsolitons", 2009.

Obs: These are not only non-compact, but also diffeo. to $\mathbb{R}^n$.

Conjecture (Alekseevskii '75). Any connected homogeneous Einstein manifold with $\lambda<0$ is diffeomorphic to $\mathbb{R}^n$.

Our main result is:

Thm (Böhm, L. '21): Conjecture holds. Moreover, any such space is isometric to an Einstein solvmanifold.

"Non-compact Einstein manifolds with symmetry" JAMS 2023.

# Ricci curvature of left-invariant metrics.

$(G, g)$ Lie group w/ left-invariant metric.

$g \rightsquigarrow \langle \cdot, \cdot \rangle$ inner product on $\mathfrak{g} = \mathrm{Lie}(G)$.

Since $\mathrm{Ric}(g)$ is $G$-invariant, we view it as a symmetric bilinear form $\mathrm{Ric}(g) : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{R}$. We have:

$$\boxed{\mathrm{Ric}(g)(X,X) = -\frac{1}{2} \sum_{i,j}{}' \langle [X, E_i], E_j \rangle^2 + \frac{1}{4} \cdot \sum_{i,j}{}' \langle X, [E_i, E_j] \rangle^2 - \frac{1}{2} B_{\mathfrak{g}}(X,X) - \langle [H,X], X \rangle}$$

$$=: M(X,X) - \frac{1}{2} B(X,X) - \langle [H,X], X \rangle .$$

Here, $\{E_i\}_{i=1}^n$ is a $\langle \cdot, \cdot \rangle$-orthonormal basis for $\mathfrak{g}$,

$B_{\mathfrak{g}}(X,X) = \mathrm{tr}(\mathrm{ad} X)(\mathrm{ad} Y)$ is the Killing form of $\mathfrak{g}$

$(\mathrm{ad} X : \mathfrak{g} \to \mathfrak{g}, \quad Y \mapsto [X,Y])$,

$H$ is the so-called "mean curvature vector" of $(G, g)$, implicitly defined by

$$\langle H, X \rangle = \mathrm{tr}\, \mathrm{ad} X, \qquad \forall X \in \mathfrak{g}.$$

In particular, it depends on $g$, but $H = 0$ is independent of $g$.

When $H = 0$ we say $\mathfrak{g}$ is unimodular.

$$\boxed{\text{Question: what's the meaning of } M \text{ ?}}$$

Answer: it's a moment map (in the sense of real Geometric Invariant Theory) for the natural action of $Gl_n(\mathbb{R})$ on $\Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$.

Fix $\langle\cdot,\cdot\rangle_1$ on $\mathfrak{g}$. Formula for $Ric(g)$ only depends on $g$ via $\{E_i\}_{i=1}^n$, $g$-o.n.b. for $\mathfrak{g}$.

Changing metric $\Longleftrightarrow$ changing o.n.b. to $\{h^{-1}E_j\}_{j=1}^n$, $h \in Gl_n(\mathbb{R})$

$\langle\cdot,\cdot\rangle_h = \langle h\cdot, h\cdot\rangle_1$

$$Ric(\langle\cdot,\cdot\rangle_1)(X,X) = \overbrace{-\frac{1}{2}\sum_{i,j}\langle [X,E_i],E_j\rangle_1^2 + \frac{1}{4}\sum_{i,j}\langle X,[E_i,E_j]\rangle_1^2}^{M(X,X)} - \dots \quad \rightsquigarrow$$

$$\begin{aligned} Ric(\langle\cdot,\cdot\rangle_h)(h^{-1}X, h^{-1}X) &= -\frac{1}{2}\sum_{i,j}\langle [h^{-1}X, h^{-1}E_i], h^{-1}E_j\rangle_h^2 + \frac{1}{4}\sum_{i,j}\langle h^{-1}X, [h^{-1}E_i, h^{-1}E_j]\rangle_h^2 - \dots \\ &= -\frac{1}{2}\sum_{i,j}\langle h[h^{-1}X, h^{-1}E_i], E_j\rangle_1^2 + \frac{1}{4}\sum_{i,j}\langle X, h[h^{-1}E_i, h^{-1}E_j]\rangle_1^2 - \dots \\ &= \underbrace{-\frac{1}{2}\sum_{i,j}\langle \mu(X,E_i), E_j\rangle_1^2 + \frac{1}{4}\sum_{i,j}\langle X, \mu(E_i,E_j)\rangle_1^2}_{M_\mu(X,X)} - \dots \end{aligned}$$

$\mu(\cdot,\cdot) = h[h^{-1}\cdot, h^{-1}\cdot] =: h\bullet[\cdot,\cdot]$

$\rightsquigarrow Gl_n(\mathbb{R}) \curvearrowright \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n =: V_n$

$V_n$ has a natural inner product, compatible with the action:

$\underset{\text{Cartan decomp.}}{Gl_n(\mathbb{R}) = O(n)\cdot \exp(\underbrace{P}_{\text{positive definite}})}$ $\rightsquigarrow$ $O(n)$ acts "orthogonally" on $V_n$

$\exp(P)$ acts "self-adjointly" on $V_n$

induced representation of $Gl_n(\mathbb{C})$ in $V_n \otimes \mathbb{C}$ $\rightsquigarrow$

$Gl_n(\mathbb{C}) \curvearrowright \mathbb{P}(V_n\otimes\mathbb{C}) \simeq \mathbb{CP}^N$

compatibility $\Rightarrow$ $U(n) \leq Gl_n(\mathbb{C}) \curvearrowright \mathbb{P}(V_n\otimes\mathbb{C})$ is Hamiltonian

$\Rightarrow \exists$ moment map $m : \mathbb{P}(V_n\otimes\mathbb{C}) \to \mathfrak{u}(n)^* \simeq \mathfrak{u}(n)$

$\rightsquigarrow$ $m: V_n\setminus\{0\} \longrightarrow \underbrace{P}_{\text{symmetric matrices.}}$ is the "$\mathbb{R}$-moment map".

**Fact:** $M_\mu(X,X) = \frac{\|\mu\|^2}{4}\cdot\langle m(\mu)X, X\rangle_1$

Thus, $$\boxed{Ric = M_\mu - \frac{1}{2} B_g \underbrace{- \langle [H, \cdot], \cdot \rangle}_{\text{this term is easy to deal with}}}$$

To exploit this connection, $B_g$ should bother as little as possible:

$g$ semisimple $\Rightarrow$ $B_g$ non-degenerate $\Rightarrow$ $Ric \neq M_\mu$ ✗

$g$ nilpotent $\Rightarrow$ $B_g = 0$ $(+H=0)$ $\Rightarrow$ $Ric = M_\mu$. ✓

$g$ solvable $\Rightarrow$ $B_g$ has a big kernel $\Rightarrow$ $Ric \sim M_\mu$. ✓