Einstein manifolds with symmetry
Lecture III

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- Very good understanding when F solvable. $\mathsf{F} \simeq_{\mathrm{Diff}} \mathbb{R}^n$
- Essentially nothing known when F semisimple. $\mathsf{F} \simeq_{\mathrm{Diff}} \mathsf{K} \times \mathbb{R}^p$.

Proof idea for Theorem A

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Key idea Forget homogeneity. Study $\mathrm{Ric}_g = -g$ using invariance under **non-transitive** group.

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Gain PDE on a compact manifold, involving algebraic data from Ric of G-orbits.

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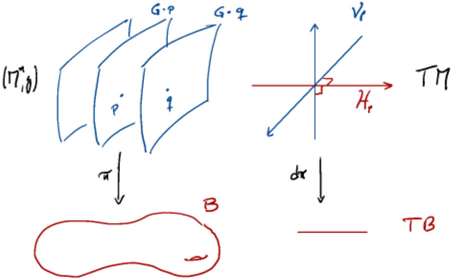

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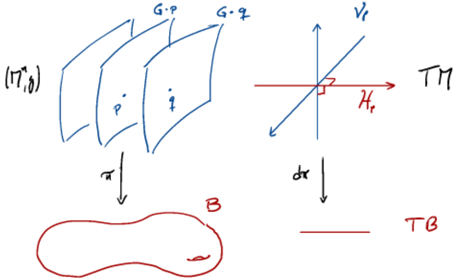

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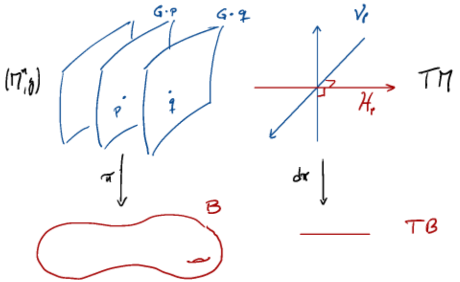

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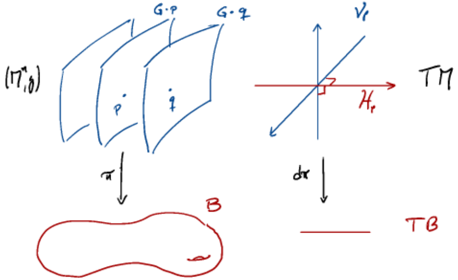

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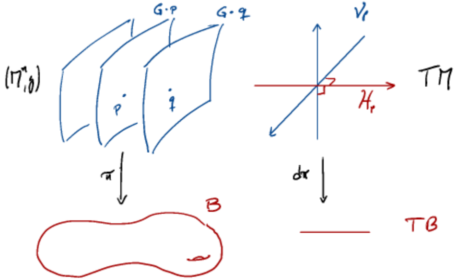

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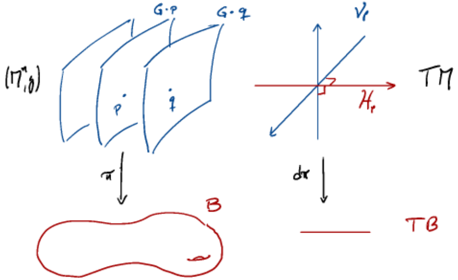

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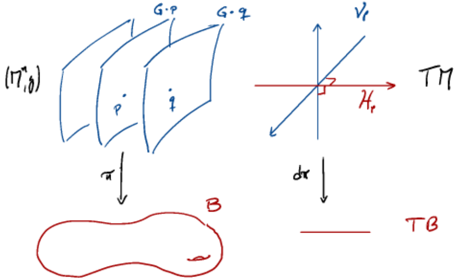

Rmk May view $h : B \to \mathrm{GL}(\mathfrak{g})/\mathrm{SO}(\mathfrak{g})$.

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Rmk If $M/\mathsf{G} = pt$, G solvable, **Thm B** (1) says Einstein solvmanifolds are *standard* (Lauret '10).

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Thm B implies Thm A: the Alekseevskii conjecture

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Remember homogeneity of M Using Killing fields from M, **Thm B** (1) and (2), plus some subtle new algebraic estimates, one shows $N_{\mathsf{F}}(\mathsf{G})$ acts transitively on $M = \mathsf{F}/\mathsf{H}$. **Thm A** then follows by (Jablonski, Petersen '17).

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Splitting conjecture (M^n, g), $\mathrm{Ric}_g = -g$ + cocompact symmetry $\implies$ M splits isometrically as a product of a compact Einstein manifold and an Einstein solvmanifold.

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Future directions

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Adam Thompson (UQ) was able to deal with the ODE case ($\dim B = 1$):
"Inhomogeneous deformations of Einstein solvmanifolds" [Thompson '23] arxiv:2305.05923.

Thank you!

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