QUANTUM RIEMANNIAN GEOMETRY AND APPLICATIONS

SHAHN MAJID (QMUL)

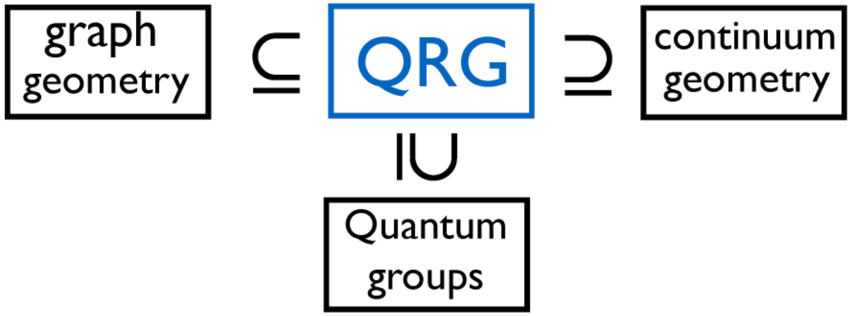
SRNI 24

Part I Formalism and applications to quantum gravity

- Formalism of quantum Riemannian geometry

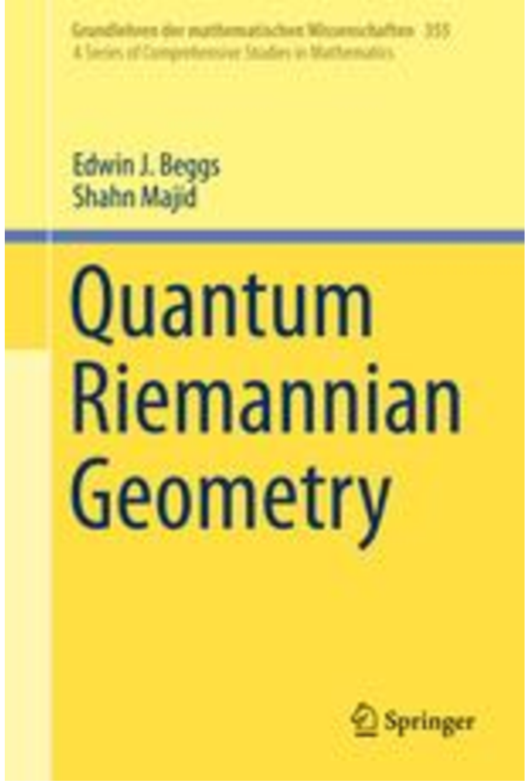
$$g\in\Omega^1\otimes_A\Omega^1,\quad \nabla:\Omega^1\to\Omega^1\otimes_A\Omega^1$$

Leibniz rule ➡ $\sigma:\Omega^1\otimes_A\Omega^1\to\Omega^1\otimes_A\Omega^1$



graph geometry $\subseteq$ QRG $\supseteq$ continuum geometry

QRG $\supseteq$ Quantum groups



Grundlehren der mathematischen Wissenschaften 355
A Series of Comprehensive Studies in Mathematics

Edwin J. Beggs
Shahn Majid

Quantum
Riemannian
Geometry

Springer

CQG 2019 (x2)

- Baby quantum gravity models

w/Argota-Quiroz CQG 2020
w/ Lira-Torres LMP 2021

- 5D black holes and FLRW models

w/ Argota-Quiroz CQG 2021

- Quantum gravity origin of the Kaluza-Klein ansatz

w/ Liu JHEP 2023, arXiv 2023

Differential calculus

Classically, $C^\infty(M)=\Omega^0(M)\subset\Omega(M)=\oplus_i\Omega^i(M)$

Ω^1 space of 1-forms, e.g. `differentials' $\mathrm{d}f=\sum_i\frac{\partial f}{\partial x^i}\mathrm{d}x^i$

$$f\mathrm{d}g=(\mathrm{d}g)f\in\Omega^1$$

$$\wedge:\Omega\otimes_A\Omega\to\Omega,\quad \mathrm{d}(\omega\wedge\eta)=(\mathrm{d}\omega)\wedge\eta+(-1)^{|\omega|}\omega\wedge\mathrm{d}\eta$$

$$\omega\wedge\eta=(-1)^{|\omega||\eta|}\eta\wedge\omega,\quad \mathrm{d}^2=0$$

graded Leibniz rule

- Algebra *A* over k; drop the (graded) commutativity but keep:

Ω^1 *a*((d*b*)*c*)=(*a*(d*b*))*c* bimodule

$\mathrm{d}:A\to\Omega^1$ d(*ab*)=(d*a*)*b*+*a*(d*b*) Leibniz rule

$\{\sum a\mathrm{d}b\}=\Omega^1$ surjectivity

- Extend to DGA of differential forms (generated by *A*, d*A*) $\Omega=T_A\Omega^1/\mathcal{I}=\oplus_n\Omega^n,\quad \mathrm{d}^2=0$

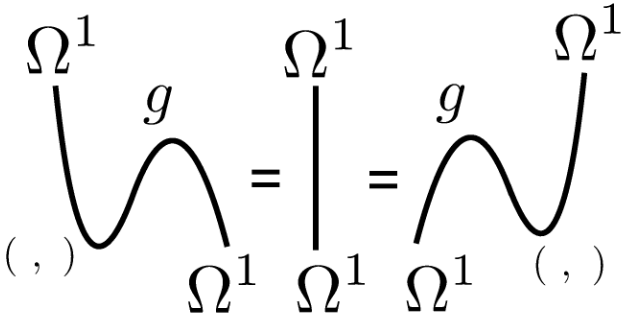
Propn. Every Ω^1 has a maximal prolongation Ω_{max}

metrics and connections

- **Metric** $g \in \Omega^1 \underset{A}{\otimes} \Omega^1$ $\quad ds^2 = \sum_{\mu,\nu} g_{\mu\nu} dx^\mu dx^\nu$

(i) bimodule map inverse $\quad (\ , \ ) : \Omega^1 \underset{A}{\otimes} \Omega^1 \to A$

(ii) some form of quantum symmetry e.g. $\wedge(g) = 0$



Lemma. a quantum metric is necessarily central

Proof: $(\omega, ag^1)g^2 = (\omega a, g^1)g^2 = \omega a = (\omega, g^1)g^2 a$

- **Connection/covariant derivative**

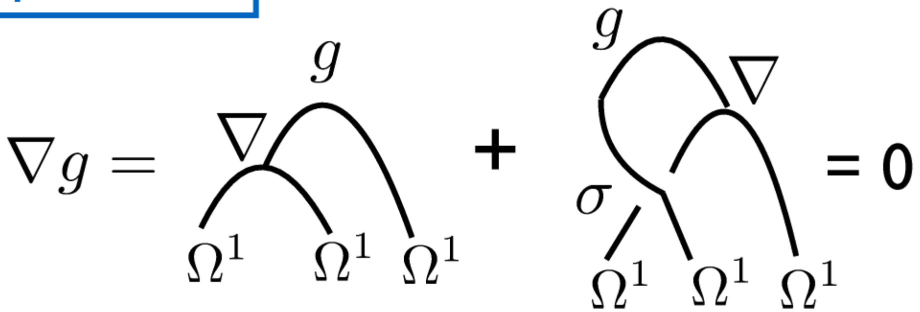
$$\nabla : \Omega^1 \to \Omega^1 \underset{A}{\otimes} \Omega^1 \qquad \sigma : \Omega^1 \underset{A}{\otimes} \Omega^1 \to \Omega^1 \underset{A}{\otimes} \Omega^1$$

$$\nabla(f\omega) = df \otimes \omega + f\nabla\omega \qquad \nabla(\omega f) = \sigma(\omega \otimes df) + (\nabla\omega)f$$

Lemma: bimodule connections extend to tensor products

$$\nabla(\omega \otimes \eta) = \nabla\omega \otimes \eta + (\sigma \otimes \mathrm{id})(\omega \otimes \nabla\eta)$$

➡ metric compatibility

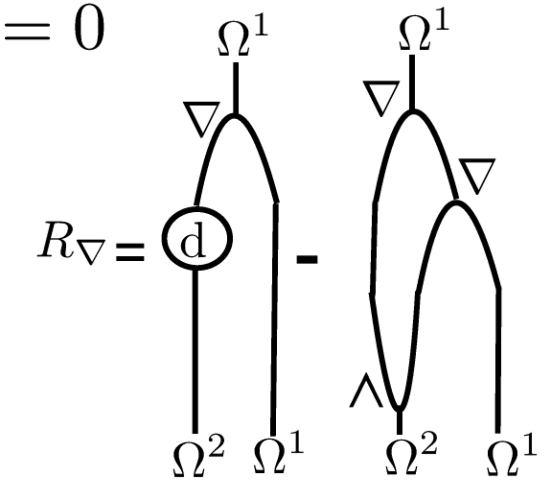


QLC and curvature

- Quantum Levi-Civita connection (QLC) $T_\nabla = \nabla g = 0$

$T_\nabla : \Omega^1 \to \Omega^2$ $T_\nabla = \wedge\nabla - \mathrm{d}$ torsion tensor

- Riemann curvature $R_\nabla : \Omega^1 \to \Omega^2 \underset{A}{\otimes} \Omega^1$



Propn: For a left connection $\wedge R_\nabla = \mathrm{d}T_\nabla - (\mathrm{id} \wedge T_\nabla)\nabla$ (Bianchi identity)

Lemma: T_∇, R_∇ are left-module maps

Proof e.g. $R_\nabla(a\omega) = (\mathrm{d} \otimes \mathrm{id} - \mathrm{id} \wedge \nabla)(\mathrm{d}a \otimes \omega + a\nabla\omega)$

$$= \mathrm{d}a \wedge \nabla\omega + a(\mathrm{d} \otimes \mathrm{id})\nabla\omega - \mathrm{d}a \wedge \nabla\omega - a(\mathrm{id} \wedge \nabla)\nabla\omega = aR_\nabla\omega$$

Propn:, T_∇, R_∇ resp. bimodule maps iff

$\wedge(\mathrm{id} + \sigma) = 0$ (torsion compatible)

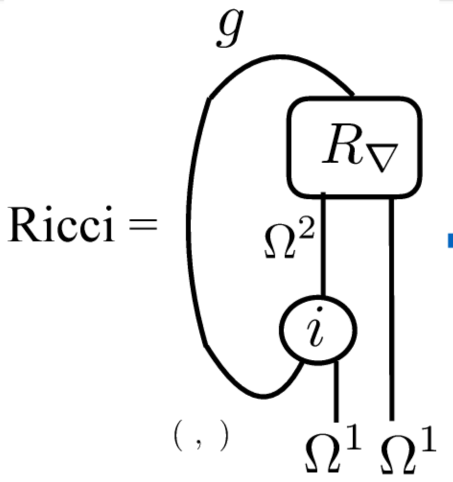
$(\mathrm{d} \otimes \mathrm{id} - \mathrm{id} \wedge \nabla)\sigma = (\mathrm{id} \wedge \sigma)(\nabla \otimes \mathrm{id})$ on $\Omega^1 \otimes \mathrm{d}A$ (Riemann compatible)

Ricci, *-structures over $\mathbb{C}$, integration

- Ricci curvature (working definition)

wrt bimodule a lifting map

$i : \Omega^2 \to \Omega^1 \otimes_A \Omega^1, \quad \wedge \circ i = \mathrm{id}$



→ Ricci scalar

$R = (\, , \,) \circ \mathrm{Ricci} \in A$

Lack deeper picture w/ conserved Einstein tensor, but do have examples of that

For A a *-algebra, require

- $*$ extends to (Ω, d) as an antilinear graded anti-involution commuting with d
- $\dagger(g) = g, \quad \dagger = \mathrm{flip}(* \otimes *) : \Omega^1 \otimes_A \Omega^1 \to \Omega^1 \otimes_A \Omega^1$ `real’
- $\dagger \nabla = \sigma^{-1} \nabla *$ → $\dagger \sigma = \sigma^{-1} \dagger$ *-preserving → *-compatible
- $\int : A \to \mathbb{C}$ positive linear functional and e.g. trace, $\int \delta = 0, \quad \delta = (\, , \,) \nabla : \Omega^1 \to A$
- $\Delta = (\, , \,) \nabla \mathrm{d}$ QRG Laplacian

Quantum differentials on finite sets

Propn: X a finite set $A = \mathbb{C}(X)$, Ω^1 ⟷ Directed graph with vertices X

$$\Omega^1 = \mathrm{span}_{\mathbb{C}}\{e_{x\to y}\} \quad f.e_{x\to y} = f(x)e_{x\to y}, \quad e_{x\to y}.f = e_{x\to y}f(y) \quad e^*_{x\to y} = -e_{y\to x}$$

if bidirected

$$\mathrm{d}f = \sum_{x\to y}(f(y)-f(x))e_{x\to y}$$

bilocal object → bimodule

$T_A\Omega^1$ = Path algebra, in degree *i* $\Omega^{1\otimes i} = \{e_{x_1\to x_2\to\cdots\to x_i}\}$

- $\Omega_{max} = T_A\Omega^1/$ relations $\displaystyle\sum_{\substack{y:p\to y\to q\\ p\neq q,\quad \text{not } p\to q}} e_{p\to y}\wedge e_{y\to q} = 0$
- metric $g = \displaystyle\sum_{x\to y} g_{x\to y}e_{x\to y}\otimes e_{y\to x}, \quad g_{x\to y}\in\mathbb{R}\backslash\{0\}$ if bidirected

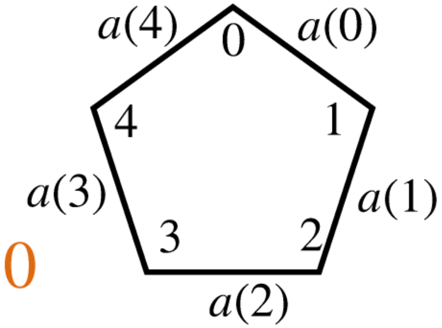
edge symmetric if $g_{x\to y} = g_{y\to x}$ → real `square-length' on each edge

QRG of $\mathbb{Z}, \mathbb{Z}_n \quad n > 2$

- Group structure/Cayley graph → left invariant basis $e^{\pm} = \sum_i e_{i \to i\pm 1}$

$$e^{\pm} f = R_{\pm} f e^{\pm}, \quad \mathrm{d}f = \sum_{\pm} (\partial_{\pm} f) e^{\pm}, \quad (R_{\pm} f)(i) = f(i \pm 1), \quad \partial_{\pm} = R_{\pm} - \mathrm{id}$$

$$(e^{\pm})^2 = 0, \quad e^{+} e^{-} + e^{-} e^{+} = 0, \quad \mathrm{d}e^{\pm} = 0 \qquad e^{+*} = -e^{-}$$



- $g = a e^{+} \otimes e^{-} + R_{\_}(a) e^{-} \otimes e^{+}$ ← lengths $a(i) \neq 0$

<u>Propn:</u> There is a QLC

$$\nabla e^{\pm} = (1 - \rho_{\pm}) e^{\pm} \otimes e^{\pm}, \quad \sigma(e^{\pm} \otimes e^{\pm}) = \rho_{\pm} e^{\pm} \otimes e^{\pm}, \quad \sigma(e^{\pm} \otimes e^{\mp}) = e^{\mp} \otimes e^{\pm}$$

unique for $n \neq 4$ $\qquad \rho = \rho_{+} = \dfrac{R_{+}(a)}{a}, \quad \rho_{\_} = R_{\_}(\dfrac{R_{\_}a}{a})$

→ $R_{\nabla} e^{\pm} = \partial_{\mp}(\rho_{\pm}) e^{\pm} \wedge e^{\mp} \otimes e^{\pm}$ vanishes iff $\{a(i)\}$ geometric sequence

<u>Ricci scalar curvature</u> $\quad R = \dfrac{1}{2a}\left(R_{\_}(\rho^{-1} \partial_{\_}(\rho) - \partial_{\_}(\rho^{-1})\right)$

Quantum gravity on $\mathbb{Z}, \mathbb{Z}_n$ looks like tropical scalar QFT

w/ Argota-Quiroz CQG (2020)

→ Einstein-Hilbert action

$$S[g] = \sum_i a_i R(i) = \frac{1}{2}\sum \rho\partial_+\rho = \frac{1}{4}\sum \rho\Delta_{\mathbb{Z}}\rho$$

discrete laplacian $\rho(i+1)+\rho(i-1)-2\rho(i)$

→ $\mathcal{Z} = \prod_i \int_0^\infty \mathrm{d}a_i\, e^{-\frac{1}{G}S[g]}$ Quantum gravity partition function

E.g. n=3

$$S[g] = \frac{1}{2}\left(\frac{a_0}{a_1}+\frac{a_1}{a_2}+\frac{a_2}{a_0}-\frac{a_0^2}{a_2^2}-\frac{a_2^2}{a_1^2}-\frac{a_1^2}{a_0^2}\right)$$

$L^{-m-3}\int_0^L \mathrm{d}a_0 \int_0^L \mathrm{d}a_1 \int_0^L \mathrm{d}a_2\; e^{\frac{1}{G}S[g]} a_{i_1}\cdots a_{i_m}$ finite → $\langle a_{i_1}\cdots a_{i_m}\rangle \sim L^m$

→ $$\frac{\langle a_i a_j\rangle}{\langle a_i\rangle\langle a_j\rangle} \to \begin{cases} \frac{4}{3} & i=j \\ 1 & i\neq j \end{cases}, \quad \frac{\Delta a_i}{\langle a_i\rangle} = \sqrt{\frac{\langle a_i^2\rangle - \langle a_i\rangle^2}{\langle a_i\rangle^2}} \to \frac{1}{\sqrt{3}} \qquad L\to\infty$$

similar results for ρ as primary field, constrained as $\prod \rho(i) = 1$

- Quantize fluctuations relative to the geom. mean $A = \left(\prod_i a_i\right)^{\frac{1}{n}}$

$$b_i = a_i/A$$

$$b_0 \cdots b_{n-1} = 1$$

$\xrightarrow{}$ *n*=3

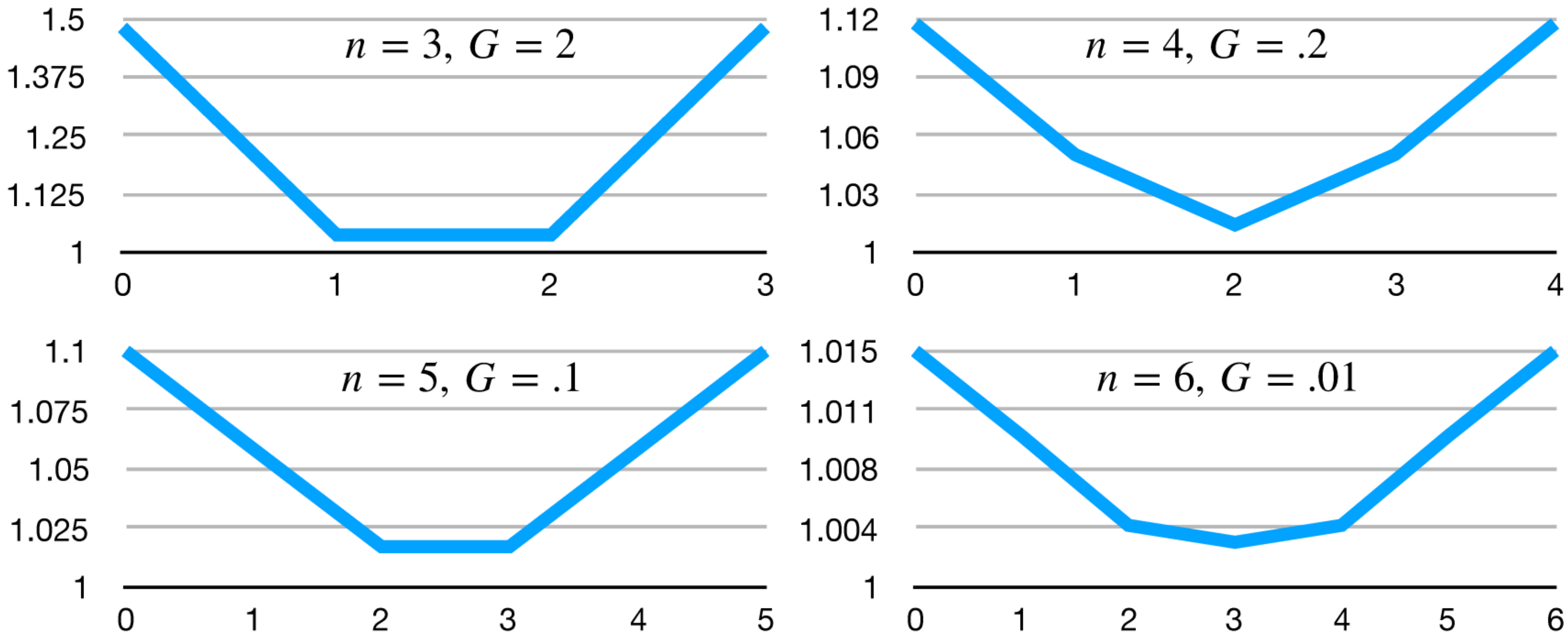
$$S[g] = \frac{1}{2}\left(\frac{b_0}{b_1} + \frac{b_1}{b_2} + \frac{b_0}{b_2} - \left(\frac{b_1}{b_0}\right)^2 - \left(\frac{b_2}{b_1}\right)^2 - \left(\frac{b_0}{b_2}\right)^2\right)$$

$$\mathrm{d}a_0\,\mathrm{d}a_1\,\mathrm{d}a_2 = \frac{3A^2}{b_0 b_1}\mathrm{d}b_0\,\mathrm{d}b_1\,\mathrm{d}A.$$

now omit the d*A* integration

$$Z = \int_0^\infty \mathrm{d}b_0 \int_0^\infty \mathrm{d}b_1 \frac{1}{b_0 b_1} e^{\frac{1}{2Gb_0^2 b_1^4}(-1+(1+b_0^3)b_1^3+(-1+b_0^3-b_0^6)b_1^6)}$$

Plots of correlation functions $\langle b_0 b_i \rangle$ against *i*



QRG of the fuzzy sphere

w/ Lira-Torres LMP (2021)

- A $\quad [x_i, x_j] = 2\imath\lambda_p \epsilon_{ijk} x_k \qquad \sum_i x_i^2 = 1 - \lambda_p^2$

Ω^1 central basis s^i, $i = 1, 2, 3.$ $\qquad [s^i, x_j] = 0. \qquad \mathrm{d}x_i = \epsilon_{ijk} x_j s^k$

$$s^i \wedge s^j + s^j \wedge s^i = 0, \quad \mathrm{d}s^i = -\tfrac{1}{2}\epsilon_{ijk} s^j \wedge s^k \qquad \mathrm{d}f = (\partial_i f) s^i$$

- Metric $\quad g = g_{ij} s^i \otimes s^j \qquad$ real symmetric matrix

Propn. There is a natural QLC with $\quad \sigma(s^i \otimes s^j) = s^j \otimes s^i$

$$\nabla s^i = -\tfrac{1}{2}\Gamma^i{}_{jk} s^j \otimes s^k \qquad \Gamma^i{}_{jk} = g^{il}(2\epsilon_{lkm} g_{mj} + \mathrm{Tr}(g)\epsilon_{ljk})$$

$$\Rightarrow \quad R_\nabla(s^i) = \rho^i{}_{jk} \epsilon_{jmn} s^m \wedge s^n \otimes s^k$$

$$\rho^i{}_{jk} = \tfrac{1}{4}\Gamma^i{}_{jk} - \tfrac{1}{4}\epsilon_{jmn}\partial_m \Gamma^i{}_{nk} - \tfrac{1}{8}\epsilon_{jmn}\Gamma^i{}_{ml}\Gamma^l{}_{nk}$$

Ricci scalar curvature $\qquad R = \tfrac{1}{2}\left(\mathrm{Tr}(g^2) - \tfrac{1}{2}\mathrm{Tr}(g)^2\right)/\det(g).$

Quantum gravity on the fuzzy sphere

w/ Lira-Torres LMP (2021)

$g \in \mathcal{P}_3$ of 3×3 positive-definite symmetric matrices = $GL_3(\mathbb{R})/O_3(\mathbb{R})$

metric $\mathfrak{g}_{P_3}$ $\quad ds^2 = \mathrm{Tr}((g^{-1}dg)^2)$

➡ measure for field integration $\quad \sqrt{|\det(\mathfrak{g}_{\mathcal{P}_3})|} = |\det(g)|^{-2}$

➡
$$Z = \int \mathcal{D}g\, e^{-\frac{2}{G} S[g]}$$

$$\int 1 := \det(g)$$

$$= \int_{\mathcal{P}_3} \prod_{i\le j} dg_{ij}\ |\det(g)|^{-2}\, e^{-\frac{1}{G}(\mathrm{Tr}(g^2) - \frac{1}{2}\mathrm{Tr}(g)^2)}$$

Use Euler angles/spectral parametrisation

$$g = E(\theta,\phi,\psi)^t \mathrm{diag}(\lambda_1,\lambda_2,\lambda_3) E(\theta,\phi,\psi).$$

$$\prod_{i\le j} dg_{ij} = d\theta\, d\phi\, d\psi\, |\sin(\phi)| \prod_i d\lambda_i\ |(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)|$$

Suppose only look at spectral functions of g, then effectively:

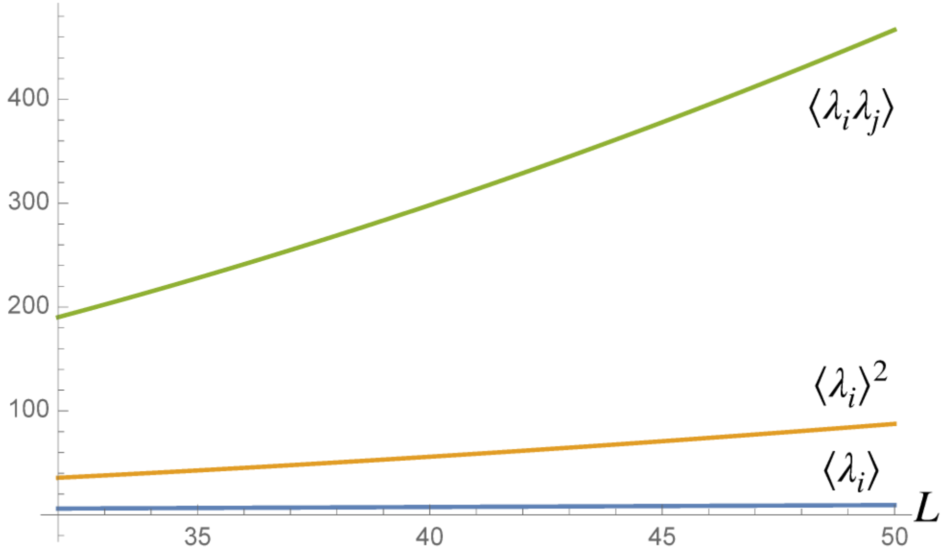
$$Z = \int_\epsilon^L \prod_i d\lambda_i\, \frac{|(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)|}{\lambda_1^2\lambda_2^2\lambda_3^2}\, e^{-\frac{1}{2G}(\lambda_1^2+\lambda_2^2+\lambda_3^2-2(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3))}$$

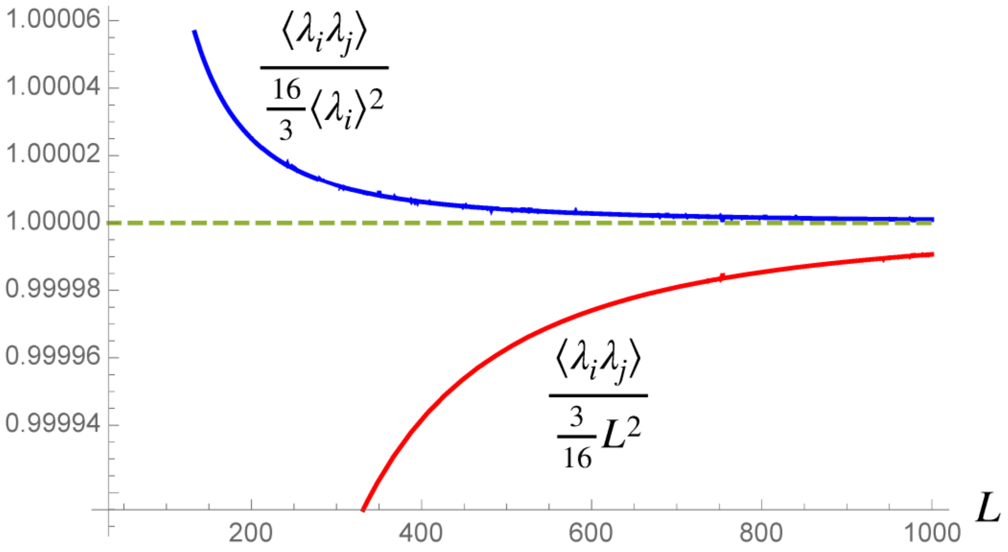
we have introduced cut-offs $L >> \epsilon > 0$ to regulate divergences at both ends

divergence as $\epsilon \to 0$ **does not show up in vevs** $\Rightarrow$ $\langle \lambda_{i_1} \cdots \lambda_{i_n} \rangle \sim \frac{3}{16} L^n$

$\Rightarrow$ $L \to \infty$ $$\frac{\langle \lambda_{i_1} \cdots \lambda_{i_n} \rangle}{\langle \lambda_i \rangle^n} = \left(\frac{16}{3}\right)^{n-1}$$

$$\frac{\Delta \lambda_i}{\langle \lambda_i \rangle} := \frac{\sqrt{\langle \lambda_i^2 \rangle - \langle \lambda_i \rangle^2}}{\langle \lambda_i \rangle} = \sqrt{\frac{13}{3}}$$





In both quantum gravity models above, and also for an earlier $\mathbb{Z}_2 \times \mathbb{Z}_2$ model, metric has a uniform uncertainty

Fuzzy sphere black hole

w/ Argota-Quiroz CQG (2021)

Fuzzy sphere at each *r,t*, which remain classical. Consider general static form

$$g=-\beta(r)\mathrm{d}t\otimes\mathrm{d}t+H(r)\mathrm{d}r\otimes\mathrm{d}r+r^2 g_{ij}s^i\otimes s^j$$

Solve for QRG with Ricci=0 ➡

$$g=-(1-\frac{r_H^2}{r^2})\mathrm{d}t\otimes\mathrm{d}t+(1-\frac{r_H^2}{r^2})^{-1}\mathrm{d}r\otimes\mathrm{d}r+r^2ks^i\otimes s^i$$

$$\nabla\mathrm{d}t=-\frac{r_H^2}{r(r^2-r_H^2)}\mathrm{d}r\otimes_s\mathrm{d}t,$$

$$k=\frac{1}{3}(\sqrt{7}-1)$$

$$\nabla\mathrm{d}r=\frac{r_H^2}{r(r^2-r_H^2)}\mathrm{d}r\otimes\mathrm{d}r-\frac{r_H^2}{r^3}\left(1-\frac{r_H^2}{r^2}\right)\mathrm{d}t\otimes\mathrm{d}t+rk\left(1-\frac{r_H^2}{r^2}\right)s^i\otimes s^i$$

$$\nabla s^i=-\frac{1}{2}\epsilon^i{}_{jk}s^j\otimes s^k-\frac{1}{r}\mathrm{d}r\otimes_s s^i,$$

➡ $R_\nabla\to\infty$ at $r=0$

$$\Delta=-\left(1-\frac{r_H^2}{r^2}\right)^{-1}\partial_t^2+\left(\frac{3}{r}-\frac{r_H^2}{r^3}\right)\partial_r+\left(1-\frac{r_H^2}{r^2}\right)\partial_r^2+\frac{1}{kr^2}\sum_i\partial_i^2.$$

would be Δ_{S^3} for Tangherilini 5D classical bh

Fuzzy FLRW cosmology

w/ Argota-Quiroz CQG (2021)

Expanding round fuzzy sphere at each *t*

$$g = -\mathrm{d}t \otimes \mathrm{d}t + R^2(t) s^i \otimes s^i,$$

- QLC $\quad \nabla \mathrm{d}t = -R\dot{R} s^i \otimes s^i; \quad \nabla s^i = -\frac{1}{2}\epsilon^i{}_{jk} s^j \otimes s^k - \frac{\dot{R}}{R} s^i \otimes_s \mathrm{d}t,$

➡ $\quad R_\nabla \mathrm{d}t = -R\ddot{R}\mathrm{d}t \wedge s^i \otimes s^i$

$$R_\nabla s^i = \left(\frac{1}{4}\epsilon^{pi}{}_n \epsilon_{pkm} - \dot{R}^2 \delta^i{}_m \delta_{nk}\right) s^m \wedge s^n \otimes s^k + \frac{\ddot{R}}{R}\mathrm{d}t \wedge s^i \otimes \mathrm{d}t,$$

$$\mathrm{Ricci} = -(\dot{R}^2 + \frac{1}{2}R\ddot{R} + \frac{1}{4}) s^i \otimes s^i + \frac{3}{2}\frac{\ddot{R}}{R}\mathrm{d}t \otimes \mathrm{d}t, \quad S = -3\left(\frac{\dot{R}^2}{R^2} + \frac{\ddot{R}}{R} + \frac{1}{4R^2}\right)$$

Ricci scalar

natural Einstein tensor

$$\mathrm{Eins} = \mathrm{Ricci} - \frac{S}{2} g = \left(\ddot{R} + \frac{1}{2}\dot{R}^2 + \frac{1}{8}\right) s^i \otimes s^i - \frac{3}{2}\left(\frac{1}{4R^2} + \frac{\dot{R}^2}{R^2}\right) \mathrm{d}t \otimes \mathrm{d}t$$

obeying $\quad \nabla \cdot \mathrm{Eins} := ((\ ,\ ) \otimes \mathrm{id}) \nabla(\mathrm{Eins})$

- Fluid Stress tensor $$T = f\mathrm{d}t \otimes \mathrm{d}t + pR^2 s^i \otimes s^i$$

density *f* pressure *p*

continuity equation $\nabla \cdot T = 0$ is $\dot{f} + 3(f+p)\dfrac{\dot{R}}{R} = 0$

- Einstein equation in our conventions $\mathrm{Eins} + 4\pi G T = 0$

➡ $$4\pi G f = \frac{3}{2}\left(\frac{\dot{R}^2}{R^2} + \frac{1}{4R^2}\right), \quad 4\pi G p = -\frac{\ddot{R}}{R} - \frac{1}{2}\frac{\dot{R}^2}{R^2} - \frac{1}{8R^2} = -\frac{\ddot{R}}{R} - \frac{4\pi G}{3} f$$

This is identical to classical 4D FLRW for closed universe with curvature constant $\kappa > 0$ and metric

$$-\mathrm{d}t \otimes \mathrm{d}t + R(t)^2\left(\frac{1}{r^2(1-\kappa r^2)}\mathrm{d}r \otimes \mathrm{d}r + g_{S^2}\right)$$

(Here *R*(*t*) includes *r* to match our conventions)

Same dimension jump phenomenon as for black hole

Application to elementary particle physics

Fuzzy sphere at each point of spacetime *M*

w/ Liu arXiv 2023

Lemma General form of a quantum metric on the product is

$$\mathfrak{g} = g_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu + A_{\mu i}(\mathrm{d}x^\mu \otimes s^i + s^i \otimes \mathrm{d}x^\mu) + h_{ij}s^i \otimes s^j$$

for fields $g_{\mu\nu}$, $A_{\mu i}$, h_{ij} on *M*

Proof. $\mathfrak{g}$ central and $\mathrm{d}x^\mu, s^i$ central requires the coeffs central. But the fuzzy sphere has trivial centre for $\lambda \neq 0$ hence the coeffs depend only on *M*.

Thm There is a unique QLC on the product → Ricci scalar on product

$$R = \tilde{R}_M + R_h + \frac{1}{8}h_{ij}\tilde{F}^i_{\mu\nu}\tilde{F}^{j\mu\nu} + \frac{1}{2}\tilde{g}^{\alpha\beta}\tilde{\nabla}_\alpha \mathrm{Tr}(\Phi_\beta) + \frac{1}{8}\tilde{g}^{\alpha\beta}\Big(\mathrm{Tr}(\Phi_\alpha\Phi_\beta) + \mathrm{Tr}(\Phi_\alpha)\mathrm{Tr}(\Phi_\beta)\Big)$$

- $\Phi = \ln(h)$, $\quad R_h = \dfrac{e^{-\mathrm{Tr}(\Phi)}}{2}\Big(\mathrm{Tr}(e^{2\Phi}) - \dfrac{1}{2}\mathrm{Tr}(e^{\Phi})^2\Big)$ Ricci scalar on fuzzy sphere
- $\tilde{g}_{\mu\nu} = g_{\mu\nu} - h^{ij}A_{i\mu}A_{j\nu}$ physical metric with Ricci $\tilde{R}_M$, $\quad \Phi^i_{\alpha j} := h^{ik}\tilde{\nabla}_{A\alpha}h_{kj}$
- $\tilde{A}_{\alpha i} := h^{ij}A_{\alpha j}$, $\quad \tilde{F}^i_{\alpha\beta} = \partial_{[\alpha}\tilde{A}_{\beta]i} - \tilde{A}_{\alpha j}\tilde{A}_{\beta k}\epsilon_{ijk}$, SU(2) Yang-Mills curvature

So gravity on the product = gravity + SU(2) Yang-Mills + Liouville-sigma field h_{ij}

Propn. QRG Laplacian on product is $\Delta = \tilde{\Delta}_A + \Delta_h + \frac{1}{2}\tilde{g}^{\alpha\beta}\mathrm{Tr}(\Phi_\alpha)\,\tilde{\nabla}_{\beta A}$

➡ for $h_{ij} = h\delta_{ij}$, massless scalar on product appears as tower of fields $\{\phi_l\}, \ l \in \mathbb{N} \cup \{0\}$ with mass $\Delta_h = l(l+1)/h$

Proof: decompose as eigenvalues $l(l+1)$ of Laplacian ∂_i^2 on fuzzy sphere

- If $\lambda = 1/(2j+1), \quad j \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ then spin j repn ρ_j of su(2) restricts to the fuzzy sphere with quotient the reduced fuzzy sphere matrix algebra

$$\begin{tikzcd}
\mathbb{C}_\lambda[S^2] \arrow[r, "\rho_j"] \arrow[d] & M_{2j+1}(\mathbb{C}) \\
c_\lambda[S^2] \arrow[ur, "\cong"'] &
\end{tikzcd}$$

➡ $\{\phi_l\}, l = 0, 1, \cdots, 2j$ a finite multiplet of fields of different masses

- For weak force coupling constants need $\sqrt{h} = 11$ Planck lengths

f.d. QRG fibre with trivial centre could explain the Standard Model

Part II Quantum geodesic flows

Classical geodesic flows

Beggs JGP 2020

w/ Beggs & SM LMP 2023, JMP 2024

- dust particles moving on geodesics → tangents define vector field X_s obeying *geodesic velocity equation*

$$\dot{X}_s + \nabla_{X_s} X_s = 0$$

we reverse usual concept and first solve for this X_s for flow to time s

density ρ obeys continuity *equation*

$$\dot{\rho} = -X_s(\mathrm{d}\rho) - \rho\,\mathrm{div}(X_s)$$

- Let $\rho = |\psi|^2$ for a wave function ψ obeying the amplitude flow equation

$$\dot{\psi} = -X_s(\mathrm{d}\psi) - \frac{1}{2}\psi\,\mathrm{div}(X_s)$$

→ Convective derivative $\frac{D}{Ds} := \frac{\partial}{\partial s} + X_s$ of the divergence is the Ricci tensor

$$\frac{D\mathrm{div}(X_s)}{Ds} = -X^j{}_{;i}X^i{}_{;j} - X^iX^j\mathrm{Ricci}_{ij}$$

- Wave function $\psi(x,t)$ on spacetime, s is external geodesic proper time

Quantum geodesic equations

$A,\ \Omega^1,\ \mathrm{d},\ g,\ \nabla:\Omega^1\to\Omega^1\otimes_A\Omega^1$ left conn, eg. QLC

→ *right* conn $\nabla_\chi:\chi\to\chi\otimes_A\Omega^1$ on $\chi:={}_A\mathrm{Hom}(\Omega^1,A)$

- $\int$ non-deg → $\mathrm{div}_\int$ defined by $\int(a\,\mathrm{div}_\int(X)+X(\mathrm{d}a))=0\quad\forall a\in A,$

$$\kappa_s=\frac{1}{2}\mathrm{div}_\int(X_s)$$ flow divergence

$$\dot X_s+[X_s,\kappa_s]+(\mathrm{id}\otimes X_s)\nabla_\chi(X_s)=0$$ velocity flow

$$\dot\psi_s=-\psi_s\kappa_s-X_s(\mathrm{d}\psi_s)$$ amplitude flow

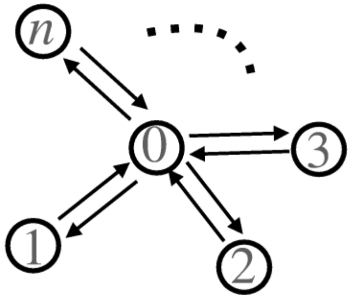
- Need $\int X_s(\omega^*)-X_s(\omega)^*=0\quad\forall\omega\in\Omega^1$ say X_s real with respect to $\int$

<u>Lemma</u> Then $\int\psi_s^*\psi_s$ constant in s → probabilistic picture

- Can add driving force *F* to velocity equation to ensure X_0 real → X_s real

QRG of the n-star graph

w/ Beggs arXiv 2023



$$\sum_{i=1}^{n} e_{0\to i\to 0} = 0,$$ Ω^2_{min} is $n-1$ dimensional $\Omega^{i>3}_{min}$=0

Thm. There exists QLC iff $n \leq 4$ and $\dfrac{g_{i\to 0}}{g_{0\to i}} = \sqrt{n}$

$$\sigma(e_{0\to i} \otimes e_{i\to 0}) = \frac{q^{-1}}{\sqrt{n}} e_{0\to i} \otimes e_{i\to 0} + (1 + \frac{q^{-1}}{\sqrt{n}}) \sum_{j\neq i} e_{0\to j} \otimes e_{j\to 0},$$

$$\sigma(e_{i\to 0} \otimes e_{0\to i}) = q e_{i\to 0} \otimes e_{0\to i},$$

$$\sigma(e_{i\to 0} \otimes e_{0\to j}) = -\frac{g_{j\to 0}}{g_{i\to 0}} q^{-1} e_{i\to 0} \otimes e_{0\to j}$$

$$\nabla e_{0\to i} = \sum_{j} e_{j\to 0} \otimes e_{0\to i} - \sigma(e_{0\to i} \otimes e_{i\to 0})$$

$$\nabla e_{i\to 0} = e_{0\to i} \otimes e_{i\to 0} - \sum_{j} \sigma(e_{i\to 0} \otimes e_{0\to j})$$

$$q = \begin{cases} e^{\frac{3i\pi}{4}} & n = 2 \\ e^{\frac{5i\pi}{6}} & n = 3 \\ -1 & n = 4 \end{cases}$$

(extends to $U(1)$ moduli of QLCs for $n = 2$)

Similarly for A_n graph, QRG needs greater metric pointing into the bulk

Quantum geodesic flow on 4-star graph

$$\int f = \sum_X \mu(x) f(x) \quad \longrightarrow \quad (X^{y\leftarrow x})^* = -\frac{\mu_y}{\mu_x}\, X^{x\leftarrow y}$$

real w.r.t. $\int$

$$\mathrm{div}_{\int}(X)(x) = \sum_{y:x\to y} X^{y\leftarrow x} - \sum_{y:y\to x} \frac{\mu_y}{\mu_x}\, X^{x\leftarrow y} \quad \longrightarrow \quad \kappa_s = \frac{1}{2}\mathrm{div}_{\int}(X_s)$$

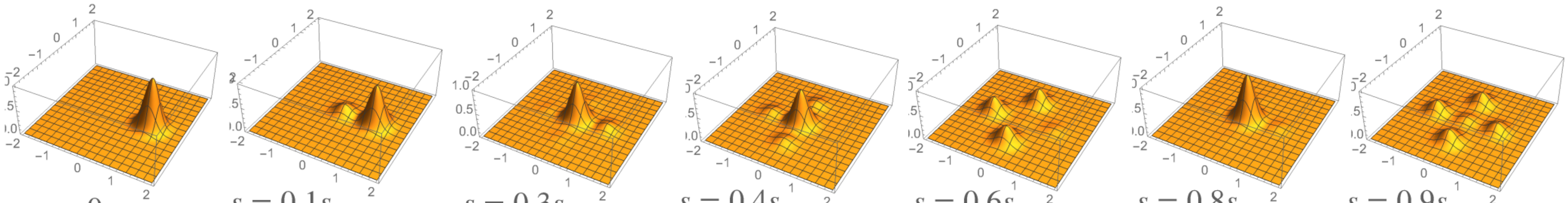
Propn. For *n*-star graph the geodesic velocity eqn with driving force is

$$-\dot{X}^{0\leftarrow y} = \frac{1}{2}\, X^{0\leftarrow y}\left(- X^{0\leftarrow y} + \sum_i \frac{\mu_i}{\mu_0}\, X^{0\leftarrow i} - \sum_i (X^{0\leftarrow i})^* \,\frac{\mu_i\, g_{i\to 0}}{\mu_0 g_{y\to 0}} + \left(2\frac{\mu_y}{\mu_0} - 1\right)(X^{0\leftarrow y})^* \right) + \frac{1}{4}\sum_i \frac{\mu_i}{\mu_y}\, |X^{0\leftarrow i}|^2$$

Then solve amplitude flow

$$\dot{\psi}_x = -\frac{1}{2}\psi_x \mathrm{div}_{\int}(X)_x - \sum_{p\leftarrow x} (\psi_p - \psi_x) X^{p\leftarrow x}$$

➡ See movie for $\mu_i, g_{i\to 0}$ constant and initial $X^{0\leftarrow i} = \psi_i = \delta_{i,1}$



$s = 0$ $s = 0.1 s_{max}$ $s = 0.3 s_{max}$ $s = 0.4 s_{max}$ $s = 0.6 s_{max}$ $s = 0.8 s_{max}$ $s = 0.9 s_{max}$

Strict quantum geodesic flow on fuzzy sphere

$\hat{\nabla} = \sigma_\chi^{-1}\nabla_\chi$ left connection → $\mathrm{div}_{\hat{\nabla}} = \mathrm{ev}\,\hat{\nabla}$

Propn. If $\mathrm{div}_{\int} = \mathrm{div}_{\hat{\nabla}}$ and $\int ab = \int \zeta(b)a$ where ζ extends to Ω^1 then

(a) $X^*(\xi) = \left(\mathrm{ev}\,\hat{\sigma}(X \otimes \xi^*)\right)^*$ defines * on vector fields

(b) X real w.r.t $\int$ ↔ $X^* = \varsigma \circ X \circ \varsigma^{-1}$

Fuzzy sphere case $\int$ = spin 0 component in orbital expansion, is a trace

→ $\{f_i\}$ dual basis to $\{s^i\}$ has $f_i^* = f_i$ → $X = f_i X^i$ real iff $X^{i*} = X^i$

$$\dot{X}^i = \tfrac{1}{2}[\partial_j X^j, X^i] - \Gamma^i{}_{jk} X^k X^j - (\partial_j X^i) X^j.$$

velocity flow eqn

$$\partial_j [X^i, X^j] = 2\epsilon_{ijk} X^j X^k$$

aux eqn (from conjugate velocity flow eqn)

$$\dot{\psi} = -X^i \partial_i \psi - \frac{\psi}{2} \partial_i X^i$$

amplitude flow eqn

We focus on $X^i \propto 1$

$$\dot{X}^i = -\Gamma^i{}_{jk} X^j X^k = g^{il} g_{mj} \epsilon_{lmk} X^j X^k$$

$$g = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3) \quad \rightarrow \quad \dot{X}^1 = \mu_1 X^2 X^3, \quad \dot{X}^2 = \mu_2 X^1 X^2, \quad \dot{X}^3 = \mu_3 X^1 X^2$$

$$\mu_1 = \frac{\lambda_2 - \lambda_3}{\lambda_1}, \quad \mu_2 = \frac{\lambda_3 - \lambda_1}{\lambda_2}, \quad \mu_3 = \frac{\lambda_1 - \lambda_2}{\lambda_3}$$

$\rightarrow$

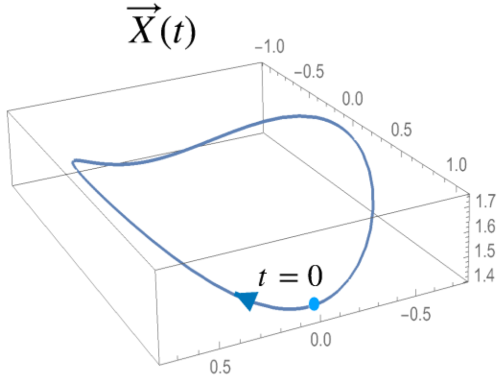
$$X^1(s) = i c_1 \sqrt{\mu_1} \mathrm{sn}\left(c_2 s | \mu\right), \quad X^2(s) = c_1 \sqrt{\mu_2} \mathrm{cn}\left(c_2 s | \mu\right), \quad X^3(s) = c_1 \sqrt{\frac{\mu_3}{\mu}} \sqrt{1 - \mu\, \mathrm{sn}^2\left(c_2 s | \mu\right)}$$

$$\mu = -\mu_1 \mu_2 \mu_3 \frac{c_1^2}{c_2^2}$$

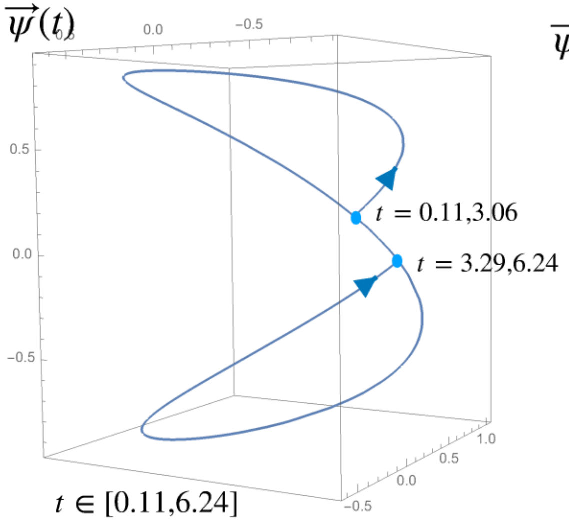
E.g. linear fields $\psi = \psi^i x_i$

$$\dot{\psi}^k = -\epsilon_{kij} X^i \psi^j$$

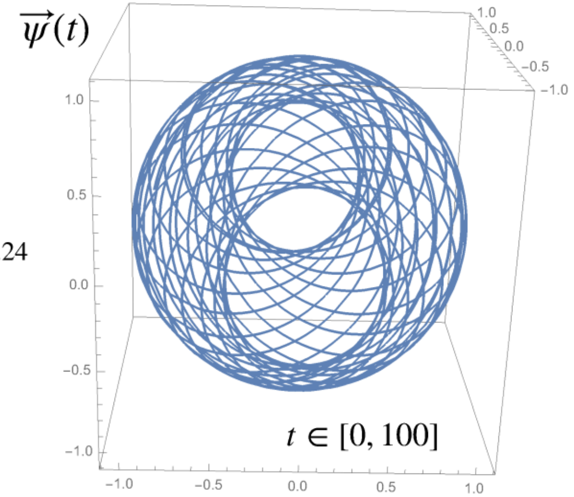
$g = \mathrm{diag}(4, 3, 1)$ initial $X = (0, 1, \sqrt{2}),\ \psi = (1, 0, 0)$



$t \in [0, 5.66]$



$t \in [0.11, 6.24]$



$t \in [0, 100]$

Quantum mechanics as quantum geodesic flow

Heisenberg algebra, fix Hamiltonian

w/ Beggs JMP 2024

$$[x^i, p_j] = \mathrm{i}\hbar\delta^i{}_j, \quad [x^i, x^j] = [p_i, p_j] = 0, \qquad h = \frac{p_1^2 + \cdots + p_n^2}{2m} + V(x^1, \cdots, x^n)$$

Choose differential calculus with extra cotangent direction θ'

$$[\mathrm{d}p_i, p_j] = -\mathrm{i}\hbar \frac{\partial^2 V}{\partial x^i \partial x^j}\theta', \quad [\mathrm{d}p_i, x^j] = [\mathrm{d}x^i, p_j] = 0, \quad [\mathrm{d}x^i, x^j] = -\frac{\mathrm{i}\hbar}{m}\delta_{ij}\theta'$$

Thm: Then there exists a bimodule vector field $X : \Omega^1 \to A$ and ∇ obeying geodesic vel eqn with amplitude flow Schroedinger/(anti)Heisenberg eqns

$$X(\theta') = 1, \quad X(\mathrm{d}p_i) = -\frac{\partial V}{\partial x^i}, \quad X(\mathrm{d}x^i) = \frac{p_i}{m}$$

➡ $\dot{x}^i = -X(\mathrm{d}x^i) = -\dfrac{p_i}{m}$ etc

$$\nabla(\mathrm{d}x^i) = \frac{1}{m}\theta' \otimes \mathrm{d}p_i, \qquad \nabla(\mathrm{d}p_i) = -\frac{\partial^2 V}{\partial x^i \partial x^j}\theta' \otimes \mathrm{d}x^j + \frac{\mathrm{i}\hbar}{2m}\frac{\partial\partial^2 V}{\partial x^i}\theta' \otimes \theta' \qquad \nabla(\theta') = 0$$

$$\boldsymbol{\sigma}(\mathrm{d}x^i \otimes \mathrm{d}p_j) = \mathrm{d}p_j \otimes \mathrm{d}x^i + \frac{\mathrm{i}\hbar}{m}\frac{\partial^2 V}{\partial x^j \partial x^i}\theta' \otimes \theta', \qquad \boldsymbol{\sigma}(\mathrm{d}p_i \otimes \mathrm{d}x^j) = \mathrm{d}x^j \otimes \mathrm{d}p_i - \frac{\mathrm{i}\hbar}{m}\frac{\partial^2 V}{\partial x^i \partial x^j}\theta' \otimes \theta'$$

Propn. (a) ∇ is metric compatible for generalised metric

$$G = \mathrm{d}p_i \otimes \mathrm{d}x^i - \mathrm{d}x^i \otimes \mathrm{d}p_i + \frac{\partial V}{\partial x_i}(\theta' \otimes \mathrm{d}x^i - \mathrm{d}x^i \otimes \theta') + \frac{p_i}{m}(\theta' \otimes \mathrm{d}p_i - \mathrm{d}p_i \otimes \theta') + \frac{\mathrm{i}\hbar}{m}\partial^2 V \theta' \otimes \theta'$$

(b) $\omega_i = \mathrm{d}p_i + \partial_i V \theta', \quad \eta^i = \mathrm{d}x^i - \frac{p_i}{m}\theta'$ and G killed by X

- G quantises something antisymmetric and is degenerate.
- If we extend the algebra by central geodesic time variable s with $\theta' = \mathrm{d}s$

→ $\omega_i = \eta_i = 0$ would then reproduce the Hamilton-Jacobi eqns

- Ω_{max} modulo $\mathrm{d}\theta' = \theta'^2 = 0$ → $\wedge(G)$ closed and ∇ flat

Relativistic electromagnetic version

$$[x^a, p_b] = \mathrm{i}\hbar \delta^a{}_b, \quad [x^a, x^b] = 0, \quad [p_a, p_b] = \mathrm{i}\hbar q F_{ab}$$

electromagnetic
-Heisenberg alg

Ω^1

$$[\mathrm{d}x^a, x^b] = -\frac{\mathrm{i}\hbar}{m}\eta^{ab}\theta' \ , \quad [\mathrm{d}x^a, p_c] = \frac{\mathrm{i}\hbar q}{m}\eta^{ab}F_{bc}\theta' = [\mathrm{d}p_c, x^a],$$

$$[\mathrm{d}p_c, p_d] = -\mathrm{i}\hbar q\, F_{ac,d}\, \mathrm{d}x^a - \frac{\hbar q}{2m}\eta^{ab}(\hbar F_{bc,ad} + 2\mathrm{i}q F_{ac}F_{bd})\theta'$$

Thm: Then there exists a bimodule vector field $X:\Omega^1\to A$ and ∇ obeying geodesic vel eqn such that amplitude flow is $\frac{\partial\phi}{\partial s}-\frac{\mathrm{i}\hbar}{2m}\eta^{ab}D_aD_b\phi=0$

$$D_a=\frac{\partial}{\partial x^a}-\mathrm{i}\tfrac{q}{\hbar}A_a$$

$$X(\theta')=1,\qquad X(\mathrm{d}x^a)=\frac{1}{m}\eta^{ab}p_b\ ,\quad X(\mathrm{d}p_c)=\frac{q}{2m}\eta^{ab}(2F_{ca}p_b-\mathrm{i}\hbar F_{cb,a})$$

$$\nabla(\mathrm{d}x^d)=-\frac{q}{m}\eta^{cd}F_{ac}\theta'\otimes\mathrm{d}x^a+\theta'\otimes\frac{\mathrm{i}\hbar q}{2m^2}\eta^{ab}\eta^{cd}F_{bc,a}\theta'\ ,\qquad \nabla\theta'=0$$

$$\nabla(\mathrm{d}p_c)=-q\,F_{dc,e}\mathrm{d}x^d\otimes\mathrm{d}x^e-\xi_c\otimes\theta'-\theta'\otimes\eta_c+N_c\theta'\otimes\theta',\qquad N_c=\dots,\quad \xi_c=\dots$$

- Ω^1_{red} : $\mathrm{d}x^0=-\frac{p_0}{m}\theta'$, $\mathrm{d}p_0=qF_{0i}\mathrm{d}x^i-\frac{\mathrm{i}\hbar q}{2m}F_{0i,i}\theta'$ → $\theta'=\mathrm{d}s$ s proper time

- If A_μ is *t*-independent (this breaks Lorentz inv)

→ $\mathfrak{u}:=-p_0-qA_0$ commutes with $x^i, p_i, \mathrm{d}x^i, \mathrm{d}p_i$ has $\mathrm{d}\mathfrak{u}=0$

→ Ω^1_u generated by $x^i, p_i, \mathrm{d}x^i, \mathrm{d}p_i, \theta'$ deforms NR Heisenberg case

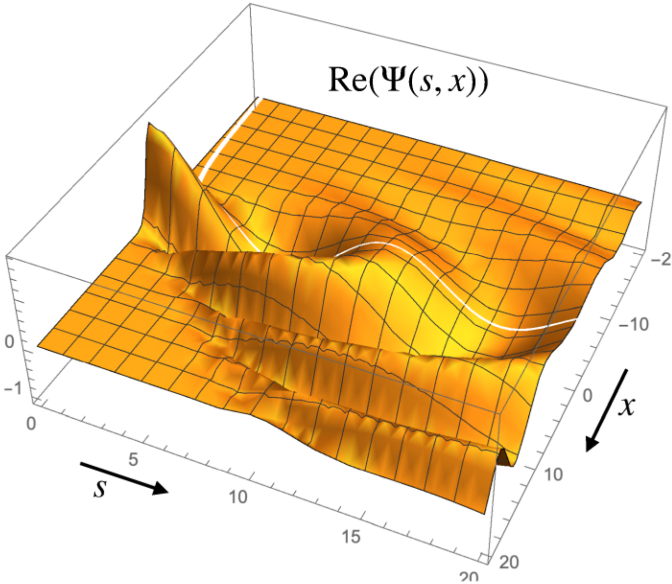
$\mathfrak{u}=u\in\mathbb{R}$ is now a `fixed energy' parameter

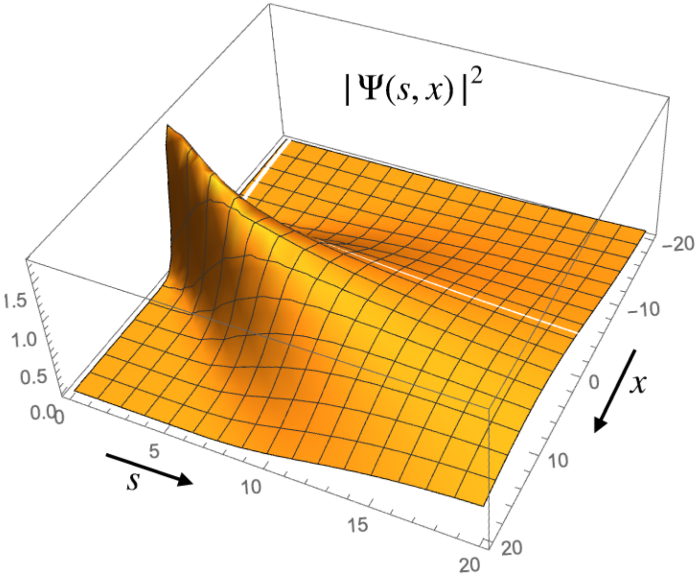
Propn (a) X, ∇ restrict to Ω^1_u and obey the velocity equation.

(b) amplitude flow is Schroedinger w.r.t. Hamiltonian $H = \frac{p_i p_i}{2m} - \frac{(u + qA_0)^2}{2m}$

Example free field on $\mathbb{R}^{1,1}$, $A_\mu = 0$ initial wave packet centred on on-shell value

$$\Psi(0,x) = \int \mathrm{d}k e^{-\frac{(ck-\sqrt{u^2-m^2c^4})^2}{\beta}} e^{\frac{\mathrm{i}kx}{\hbar}}, \quad \Psi(s,x) = \int \mathrm{d}k e^{\mathrm{i}s\frac{u^2-k^2c^2}{\hbar 2mc^2}} e^{-\frac{(ck-\sqrt{u^2-m^2c^4})^2}{\beta}} e^{\frac{\mathrm{i}kx}{\hbar}}$$





$$\langle p \rangle = \sqrt{u^2 - m^2}$$

$$\langle x \rangle = s\frac{\langle p \rangle}{m}$$

$$t := -\imath\hbar\partial_u \quad \rightarrow \quad \langle t \rangle = \frac{s}{\sqrt{1-v^2}}, \qquad v := \frac{\langle p \rangle}{u} = \frac{\langle x \rangle}{\langle t \rangle}$$

Poisson geometry of quantum geodesics

- M, ω symplectic, $h \in C^\infty(M)$, hamiltonian vector field $X_h^\mu = \omega^{\mu\nu}\partial_\nu h$

Lemma ∇ symplectic → $(\nabla_{X_h} X_h)^\mu = -\, g^{\mu\nu}\partial_\nu h, \quad g^{\mu\nu} = \omega^{\mu\alpha}\omega^{\nu\beta}\nabla_\alpha \partial_\beta h$

- Extend to $M \times \mathbb{R}$ by geodesic time s, $\theta' = \mathrm{d}s$

$$\tilde\omega = \omega - 2\mathrm{d}h \wedge \theta', \qquad \tilde\nabla \mathrm{d}x^\mu = \nabla \mathrm{d}x^\mu - g^{\mu\beta}\omega_{\beta\alpha}\mathrm{d}x^\alpha \otimes \theta', \qquad \tilde\nabla\theta' = 0$$

$$\tilde X_h = X_h + \frac{\partial}{\partial s} \quad \longrightarrow \quad \boxed{\tilde\nabla_{\tilde X_h}\tilde X_h = 0, \quad \tilde\nabla(\tilde\omega) = 0, \qquad i_{\tilde X_h}\tilde\omega = 0}$$

- Vanishing of $\eta^\mu := \mathrm{d}x^\mu - X_h^\mu \theta'$ would be Hamilton-Jacobi eqn

- $G = \omega_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu + \theta' \otimes \mathrm{d}h - \mathrm{d}h \otimes \theta' = \omega_{\mu\nu}\eta^\mu \otimes \eta^\nu$ lifts $\tilde\omega$

$$\tilde X_h(\eta) = \tilde\nabla(G) = (X_h \otimes \mathrm{id})(G) = 0$$

degenerate antisym. `metric’

$$R_{\tilde\nabla}(\mathrm{d}x^\mu) = \frac{1}{2}R^\mu{}_{\nu\alpha\beta}\eta^\nu \otimes \mathrm{d}x^\beta \wedge \mathrm{d}x^\alpha, \qquad T_{\tilde\nabla}(\mathrm{d}x^\mu) = g^{\mu\beta}\omega_{\beta\alpha}\mathrm{d}x^\alpha \wedge \theta'$$

Part III   Master equation $\boldsymbol{\nabla}(\sigma_E)=0$

Beggs & SM book

- Right *A*-*B*-Bimodule Connection

$$\nabla_E : E \to E\otimes_B \Omega^1_B \qquad \boxed{\sigma_E : \Omega^1\otimes_A E \to E\otimes_B\Omega^1_B}$$

$$\nabla_E(eb) = e\otimes \mathrm{d}b + (\nabla_E a)b\,, \qquad \nabla_E(ae) = \sigma_E(\mathrm{d}a\otimes e) + a\nabla_E e$$

- if $\phi : (E,\nabla_E)\to(F,\nabla_F)$, $\quad \boldsymbol{\nabla}(\phi) := \nabla_F\phi - (\phi\otimes\mathrm{id})\nabla_E : E\to F\otimes_B\Omega^1_B$

➡ 2-category of `differentiable' bimodules :

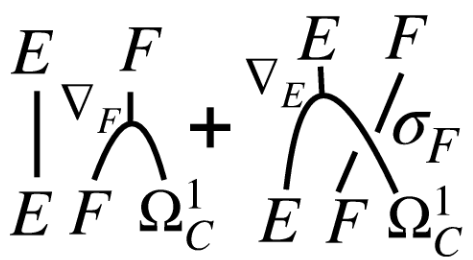
Objects: diff algebras (A, Ω^1,d)

Mor(*A*,*B*): category of *A*-*B*-bimodules w/ con (E,∇_E,σ_E) with 2-Mor bimodule maps ϕ with $\boldsymbol{\nabla}(\phi)=0$ and usual (vertical) composition

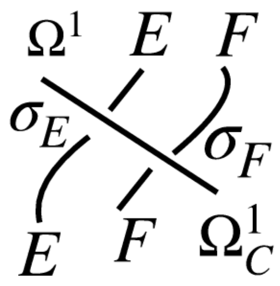
$$\otimes_B : \mathrm{Mor(B,C)}\times\mathrm{Mor(A,B)}\to\mathrm{Mor(A,C)}$$
$$F \qquad\quad E \quad\mapsto\quad E\otimes_B F$$

(horizontal) composition

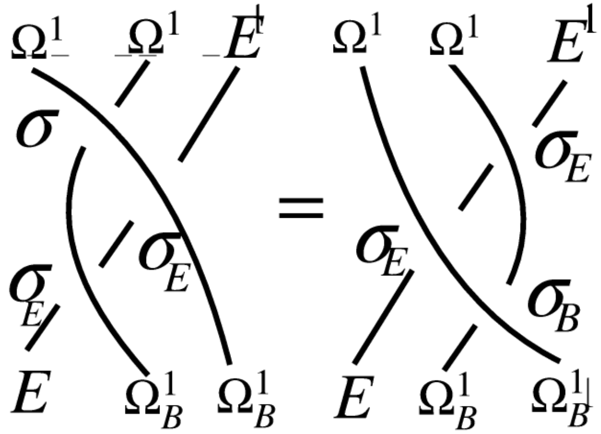
$\nabla_{E\otimes F} =$



$\sigma_{E\otimes F} =$



- If Ω^1 and Ω^1_B have right bimodule connections then both sides of σ_E have tensor product right A-B-bimodule connections.



Propn $\boldsymbol{\nabla}(\sigma_E)$ a bimodule map (eg 0) iff
Yang-Baxter or braid relations hold

- $B = C^\infty(\mathbb{R}), \Omega^1_B = B\mathrm{d}s,$ classical geodesic time s

  $\nabla_B\mathrm{d}s = 0, \quad \sigma_B(\mathrm{d}s \otimes \mathrm{d}s) = \mathrm{d}s \otimes \mathrm{d}s; \quad A, \Omega^1, \nabla, \sigma$ given eg QLC

- $E = A \otimes B \ni \psi$ → $\nabla_E\psi = (\dot{\psi} + \psi\kappa_s + X_s(\mathrm{d}\psi)) \otimes_B \mathrm{d}s$

  $\sigma_E(\mathrm{d}a \otimes_A \psi) = X_s((\mathrm{d}a) \,.\, \psi) \otimes_B \mathrm{d}s$ some $\kappa_s \in A, X_s \in \chi$

Thm. Beggs JGP 2020 $\boldsymbol{\nabla}(\sigma_E) = 0$ iff $\forall \omega \in \Omega^1$

$$\dot{X}_s(\omega) + [X_s, \kappa_s](\omega) + X_s(\mathrm{d}X_s(\omega)) - X_s(\mathrm{id} \otimes X_s)\nabla\omega = 0$$

geodesic velocity eqn

$X_s(\mathrm{id} \otimes X_s)(\sigma - \mathrm{id}) = 0$ braid relation (we replaced by X_s stays real)

- $E = C^\infty(\mathbb{R}, \mathscr{H}) \ni \psi,$ A acts on $\mathscr{H}$ e.g. Schroedinger repn, etc.

Quantum jet bundles

Classically, $J^k(M)$ has a prolongation map

SM & F. Simao LMP 2023

$$j^k : C^\infty(M) \hookrightarrow \Gamma(J^k(M)), \quad j^k(f)(x) = (f(x), \partial_i f(x), \partial_i\partial_j f(x)\cdots, \partial_{i_1}\cdots\partial_{i_k} f(x))$$

Given (A, Ω, d), $\quad \Omega^k_S := \cap_{i=1}^{k-1} \ker(\wedge_i) \subset \Omega^1 \otimes_A \Omega^1 \cdots \otimes_A \Omega^1$ (*k* times)

$\Omega^0_S := A$

adjacent wedge products

→ $\mathcal{J}^k_A := \oplus_{i=0}^k \Omega^i_S$ as an *A*-bimodule `Jet bundle'

cf Flood, Mantegazza, Winther arXiv 2022
Jet endofunctor

Want $j^k = \sum_{i=0}^k \nabla^i : A \to \mathcal{J}^k_A, \quad \nabla^i : A \to \Omega^i_S$

$\bullet_k$ bimodule structure such that j^k a bimodule map.

We consider $\quad \nabla^i = \nabla_{\Omega^{1\otimes(i-1)}} \nabla_{\Omega^{1\otimes(i-2)}} \cdots \nabla \mathrm{d}$

induced by (∇, σ)
with suitable properties

<u>Lemma</u>. $\mathcal{J}^1_A$ needs no ∇, $\; j^1(s) = s + \mathrm{d}s\;$ and $\forall s \in A,\ \omega \in \Omega^1 \subset \mathcal{J}^1_A$,

$$a \bullet_1 s = aj^1(s), \quad s \bullet_1 a = j^1(s)a, \quad a \bullet_1 \omega = a\omega, \quad \omega \bullet_1 a = \omega a$$

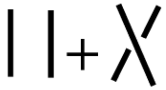
Propn. $\mathscr{J}^2_A$ needs ∇ torsion free but indept of it up to isom for fixed σ

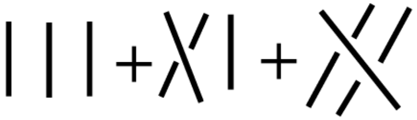
$j^2(s) = s + \mathrm{d}s + \nabla \mathrm{d}s$ and for all $s \in A, \eta \in \Omega^1, \eta^1 \otimes \eta^2 \in \Omega^2_S$

$$a \bullet_2 s = a \bullet_1 s + (\nabla \mathrm{d}a)s \qquad s \bullet_2 a = s \bullet_1 a + s\nabla \mathrm{d}a$$

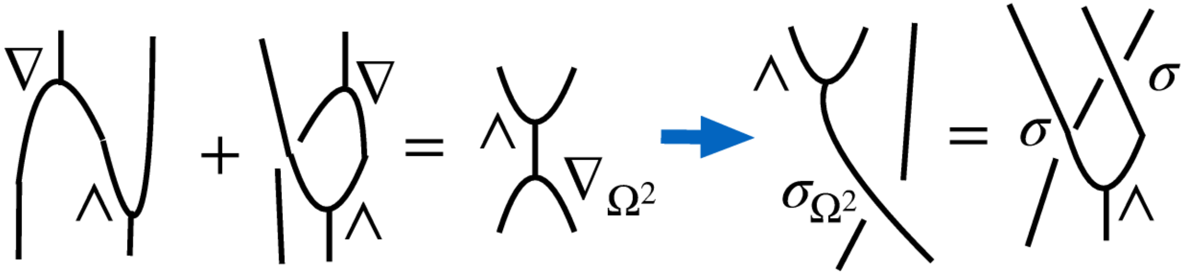
$$a \bullet_2 \eta = a \bullet_1 \eta + [2,\sigma](\mathrm{d}a \otimes \eta) \qquad \eta \bullet_2 a = \eta \bullet_1 a + [2,\sigma](\eta \otimes \mathrm{d}a)$$

$$a \bullet_2 (\eta^1 \otimes \eta^2) = a\eta^1 \otimes \eta^2 \qquad (\eta^1 \otimes \eta^2) \bullet_2 a = \eta^1 \otimes \eta^2 a$$

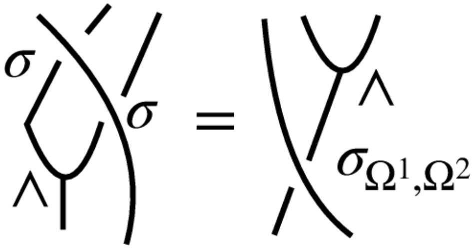
$[2,\sigma] = \mathrm{id} + \sigma =$  cf. $[k]_q = 1 + q + \cdots + q^{k-1}$

Similarly $[3,\sigma] =$  $[3,\sigma]' =$ 

- **Defn** ∇ is ∧-compatible if $\nabla_{\Omega^1 \otimes \Omega^1}$ descends to Ω^2 ⟷ restricts to Ω^2_S



- **Defn** ∇ on Ω^1 is extendable if well-defined map $\sigma_{\Omega^1,\Omega^2}$



Lemma $\boldsymbol{\nabla}(\sigma) = 0$ `Leibniz-compatible' iff

$$\nabla^3(ab) = (\nabla^3 a)b + [3,\sigma]'(\nabla^2 \otimes \mathrm{d}b) + [3,\sigma](\mathrm{d}a \otimes \nabla^2 b) + a\nabla^3 b$$

- Same as for quantum geodesics → braid relations

Propn. $\mathcal{J}^3_A$ needs ∇ torsion free, flat, extendable, $\wedge$-compatible and Leibniz-compatible.

$$j^3(s) = s + \mathrm{d}s + \nabla \mathrm{d}s + \nabla_{\Omega^1\otimes\Omega^1}\nabla \mathrm{d}s$$

$$a \bullet_3 s = a \bullet_2 s + (\nabla_{\Omega^1\otimes\Omega^1}\nabla \mathrm{d}a)s \qquad s \bullet_3 a = s \bullet_2 a + s(\nabla_{\Omega^1\otimes\Omega^1}\nabla \mathrm{d}a)$$

$$a \bullet_3 \eta = a \bullet_2 \eta + [3,\sigma]'(\nabla \mathrm{d}a \otimes \eta) \qquad \eta \bullet_3 a = \eta \bullet_2 a + [3,\sigma](\eta \otimes \nabla \mathrm{d}a)$$

$$a \bullet_3 (\eta^1 \otimes \eta^2) = a \bullet_2 (\eta^1 \otimes \eta^2) + [3,\sigma](\mathrm{d}a \otimes \eta^1 \otimes \eta^2) \qquad (\eta^1 \otimes \eta^2) \bullet_3 a = (\eta^1 \otimes \eta^2) \bullet_2 a + [3,\sigma]'(\eta^1 \otimes \eta^2 \otimes \mathrm{d}a)$$

$$a \bullet_3 (\zeta^1 \otimes \zeta^2 \otimes \zeta^3) = a\zeta^1 \otimes \zeta^2 \otimes \zeta^3 \qquad (\zeta^1 \otimes \zeta^2 \otimes \zeta^3) \bullet_3 a = \zeta^1 \otimes \zeta^2 \otimes \zeta^3 a$$

Thm. $\mathcal{J}^k_A$ works equally well for all *k* with no more restrictions on ∇

- We also have bimodule maps where quotient out top degree

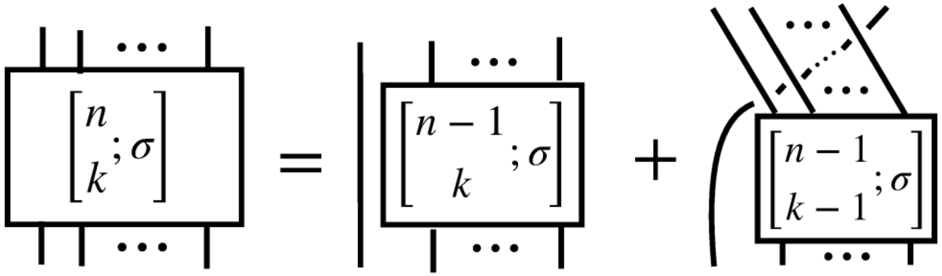
$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{J}^k_A \arrow[r] & \mathcal{J}^{k-1}_A \arrow[r] & \cdots \mathcal{J}^1_A \arrow[r] & A \arrow[r] \arrow[l, bend left, "j^1"] \arrow[ll, bend left, "j^{k-1}"] \arrow[lll, bend left, "j^k"] & 0
\end{tikzcd}$$

$$\longrightarrow \quad \mathcal{J}^{\infty}_A$$

Proof of the theorem

Defn *(cf. SM, 1993)* $\sigma : \Omega^1 \otimes_A \Omega^1 \to \Omega^1 \otimes_A \Omega^1$ a bimodule map. *binomial maps* defined recursively by

$$\left[\begin{matrix} n \\ 0 \end{matrix};\sigma\right] = \mathrm{id}$$



➡ $\left[\begin{matrix} n \\ 1 \end{matrix};\sigma\right] = [n,\sigma], \quad \left[\begin{matrix} n \\ n-1 \end{matrix};\sigma\right] = [n,\sigma]'$

➡ shuffle product on $T_A\Omega^1$ $\quad \circ = \left[\begin{matrix} i+j \\ j \end{matrix};\sigma\right] : \Omega^{1\otimes i} \otimes_A \Omega^{1\otimes j} \to \Omega^{1\otimes(i+j)}$

Propn $\circ$ restricts to shuffle product algebra $(\Omega_S, \circ\,)$ and under our assumptions,

$$\nabla^j(ab) = \sum_{i=0}^{j} \nabla^{j-i}a \circ \nabla^i b$$

- $\Omega_S \subset T_A^{sh}\Omega^1$ as Hopf algebras in braided subcategory generated by σ within A-bimodules
- Classically $\Omega_S = C^\infty_{poly}(TM)$, coproduct encodes pointwise addition on fibre

Quantum jet bundle $\mathcal{J}^k_E$ of bimodule E

$$j^k_E = \sum_{i=0}^{k} \nabla^i_E : E \to \mathcal{J}^k_E = \mathcal{J}^k_A \otimes_A E$$

We consider special case $\quad \nabla^i_E = \nabla_{\Omega^{1\otimes(i-1)}\otimes E} \cdots \nabla_{\Omega^1 \otimes E} \nabla_E$

$$a \bullet_k (\omega_i \otimes s) = j^{k-i}(a) \circ \omega_i \otimes s\,, \quad (\omega_i \otimes s) \bullet_k a = \omega_i \circ \sigma_E(s \otimes j^{k-i}(a))$$

$\forall s \in E, \omega_i \in \Omega^i_S$, needs additional data ∇_E flat, extendable and $\nabla(\sigma_E)$=0

- $\mathcal{J}^1_E = E \oplus \Omega^1_A \otimes_A E$ just needs bimodule map $\sigma_E : E \otimes_A \Omega^1 \to \Omega^1 \otimes_A E$

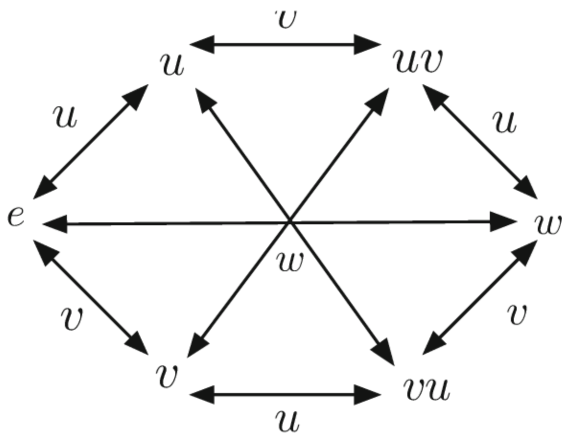
$$a \bullet_1 (s + \omega \otimes t) = as + \mathrm{da} \otimes s + a\omega \otimes t\ , \quad (s + \omega \otimes t) \bullet_1 a = sa + \sigma_E(s \otimes \mathrm{da}) + \omega \otimes ta$$

$0 \to \Omega^1 \otimes_A E \to \mathcal{J}^1_E \to E \to 0$ exact sequence of $\bullet_1$- bimodules

splitting map j^1_E ↔ a bimodule conn with braiding σ_E, $\quad j^1_E(s) = s + \nabla_E s$

$(\mathcal{J}^1_E, j^1_E)$ indept of (∇_E, σ_E) up to isom → *Atiyah class* for pair (E, σ_E)

Quantum jet bundle on Cayley graph on S_3



$A = \mathbb{C}(S_3)$ $\quad u = (12), v = (23), w = uvu = vuv = (13)$

Ω^1 left inv. basis $\{e^u, e^v, e^w\}$ $\quad da = (\partial_i a)e^i$

$\Omega_{wor} = T_A\Omega^1/\langle\ker(\mathrm{id} - \Psi)\rangle$ $\quad \partial_i = R_i - \mathrm{id}$ etc

crossed module braiding Ψ

$$\nabla e_u = \frac{1}{q-1}\Big(qe_u \otimes e_u + qe_u \otimes (e_v + e_w) + q(e_v + e_w) \otimes e_u + e_v \otimes e_w + e_w \otimes e_v + q^{-1}e_v \otimes e_v + q^{-1}e_w \otimes e_w\Big)$$

$$q = e^{\pm\frac{2\pi\imath}{3}}$$

$$\mathcal{J}^\infty_A = \mathrm{span}_{\mathbb{C}(S_3)}\langle 1, e_u, e_v, e_w, e_u \otimes e_u, e_v \otimes e_v, e_w \otimes e_w, e_{uv}, e_{vu}, \cdots\rangle,$$

$$j^\infty(a) = a + \sum_{i=u,v,w}(\partial_i a)e_i + \sum_{i=u,v,w} D_i(a)e_i \otimes e_i + D_{uv}(a)e_{uv} + D_{vu}(a)e_{vu} + \cdots$$

$$e_{uv} := e_u \otimes e_v + e_v \otimes e_w + e_w \otimes e_u, \quad e_{vu} := e_v \otimes e_u + e_w \otimes e_v + e_u \otimes e_w,$$

$$D_i = \frac{1}{q-1}(q^2(R_u + R_v + R_w) + R_i + 4q), \quad D_{uv} = R_{uv} + \frac{1}{q-1}(R_u + R_v + R_w - 3q),$$ etc.

- $\Omega_S \cong \mathbb{C}(S_3) \rtimes U(\mathscr{L}) \twoheadrightarrow \mathbb{C}(S_3) \rtimes \mathbb{C}S_3$ as an algebra, $\mathscr{L} = \mathrm{span}_{\mathbb{C}}\{\partial_i\}$