A Holographic Approach to Submanifold Geometry

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Story

What **not** to do

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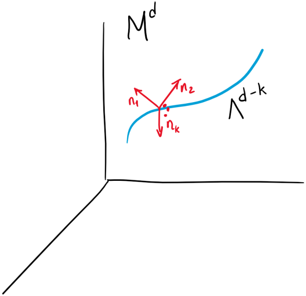
Q: Can we find any new invariants this way?

$\iota : \Lambda^{d-k} \hookrightarrow M^d$ smooth



(Always working locally, so assume triviality)

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$$\begin{array}{ccc} D : \Gamma(T\Lambda) \times \Gamma(N\Lambda) & \to & \Gamma(N\Lambda) \\ (v, n) & \mapsto & D_v n := \perp (\nabla_{\iota_* v} n) \, . \end{array}$$

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"Normal fundamental forms"

Curvature: $\mathcal{R}(u,v)n := D_u(D_v n) - D_v(D_u n) - D_{[u,v]} n$

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$\Rightarrow$ **"Gauge Fixing"**

Simplest case:

Theorem

If $\Lambda^{d-k} \hookrightarrow (M, g)$ *has* $\mathcal{R} = 0$, *then* $\{n_\alpha\}$ *is uniquely fixed (up to constant sections of* $O(k)$*) by fixing* $\beta = 0$.

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For $\mathcal{R} \neq 0$: **impossible?** $\Rightarrow$ Assume some $\{n_\alpha\}$ going forward.

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$$\Rightarrow \tilde{G}_{\alpha\beta} = \delta_{\alpha\beta} + F^{(2)}_{\alpha\beta\gamma_1\gamma_2} \tilde{s}_{\gamma_1} \tilde{s}_{\gamma_2} \,.$$

Problem

Let $\bar{f} \in C^\infty \Lambda$ *and* $\Lambda \hookrightarrow (M, g)$ *have defining map with* $G_{\alpha\beta} = \delta_{\alpha\beta} + \mathcal{O}(s^m)$. *Find a formal power series for* $f \in C^\infty M$ *solving*

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- $P(2, 1)$ always has a solution.
- $\mathcal{R} = 0 \Rightarrow P(2, 2)$ has a solution (in RMF).
- $\mathcal{R} = 0$ and (M, g) flat $\Rightarrow$ For $m \geq 3$, $P(m, m)$ has a solution (in RMF).

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Order ∞: If $\mathcal{R} = R = 0$, no obstructions, $\exists s_\alpha$ s.t. $G_{\alpha\beta} = \delta_{\alpha\beta} + \mathcal{O}(s^\infty)$.

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- For $g \in \boldsymbol{c}$, define "tractor bundle":
  $\mathcal{T}M \stackrel{g}{\cong} \mathcal{E}M[1] \oplus TM[-1] \oplus \mathcal{E}M[-1]$ with $g \mapsto \Omega^2 g$ transformation law.

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- Tractor connection:

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$$T^B \mapsto \nabla_a^{\mathcal{T}} T^B \stackrel{g}{=} \begin{pmatrix} \nabla_a \tau^+ - \tau_a \\ \nabla_a \tau^b + \boldsymbol{g}_a^b \tau^- + (P^g)_a^b \tau^+ \\ \nabla_a \tau^- - P^g_{ab} \tau^b \end{pmatrix}.$$

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$$T^B \mapsto \nabla^{\mathcal{T}}_a T^B \stackrel{g}{=} \begin{pmatrix} \nabla_a \tau^+ - \tau_a \\ \nabla_a \tau^b + \boldsymbol{g}^b_a \tau^- + (P^g)^b_a \tau^+ \\ \nabla_a \tau^- - P^g_{ab} \tau^b \end{pmatrix}.$$

- Thomas-D operator:

$$D_A : \Gamma(\mathcal{T}^\Phi M[w]) \to \Gamma(\mathcal{T}^* M \otimes \mathcal{T}^\Phi M[w-1])$$
$$T^\Phi \mapsto D_A T^\Phi \stackrel{g}{=} \begin{pmatrix} (d + 2w - 2) w T^\Phi \\ (d + 2w - 2) \nabla^{\mathcal{T}}_a T^\Phi \\ -(\Delta^{\mathcal{T}} + w J^g) T^\Phi \end{pmatrix},$$

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Frame-valued density

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Let $\bar{f} \in \Gamma(\mathcal{E}\Lambda[w])$ *and let* $\Lambda \hookrightarrow (M, \boldsymbol{c})$ *have defining densities satisfying* $\boldsymbol{G}_{\alpha\beta} = \delta_{\alpha\beta} + \mathcal{O}(\sigma^2)$*. Find a formal power series for* $f \in \Gamma(\mathcal{E}M[w])$ *solving* $N_\alpha \cdot \hat{D} f = 0$ *and* $f|_\Lambda = \bar{f}$*.*

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This spells the end....

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Unique invariant: $F^{(2)}_{\gamma[\alpha\beta]\gamma} = \bar{\nabla}^a \beta_{a\alpha\beta}$.

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Conformal Submanifolds

Order 3 and the Willmore Invariant

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$$F^{(3)}_{\alpha\gamma\gamma\rho\rho} \stackrel{\Lambda}{=} -\tfrac{d-2}{6} L_{ab} \mathring{\mathrm{I\!I}}^{ab}_{\alpha} - \tfrac{1}{3} \mathring{\mathrm{I\!I}}^{bc}_{\alpha} \bar{g}^{ad} W_{abcd} - \tfrac{d-2}{6} \mathrm{IV}_{\alpha}\,.$$

[$\mathrm{IV}_{\alpha} = C_{\alpha\beta\beta} + H_{\rho} W_{\alpha\beta\beta\rho} + \frac{1}{d-k-3} \bar{\nabla}^c W^{\top}_{c\beta\beta\alpha} \in \Gamma(\mathcal{E}\Lambda[-3])$]

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This differs from the Willmore invariant by a factor of 1/2.

Thank you

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