Algebraic structures and particles

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String Field Theory

Second quantization of a string:

$$|\Psi\rangle \propto \int \mathrm{d}k\, \Psi(k)\, |k\rangle$$

1. On-shell amplitudes: from punctured Riemann surfaces (expansion in Feynman diagrams). Moduli space:

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1. On-shell amplitudes: from punctured Riemann surfaces (expansion in Feynman diagrams). Moduli space:
   - it is a cochain complex $C_i \ni \nu_i$ with differential ∂
   - operation of joining two legs from different vertices and sewing two legs from the same vertex
   - MC equation

$$\partial \nu_4 + \frac{1}{2}\{\nu_3, \nu_3\} + \underbrace{(\Delta \nu_6)}_{\text{loops}} = 0$$

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     - closed SFT is based on L_∞ products.
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Normally in a QFT, 1. and 2. are reversed.

'Downgrade' to 1-dim

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Knowledge of the gauge-fixed $\mathcal{N} = 4$ **spinning** *particle allows to calculate the 1-loop* $\Gamma[g_{\mu\nu}]$ *of perturbative QG*

[Bastianelli et al.]

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$\mathcal{N} = 2$ BRST and quantization

- Needs a double cochain complex $\mathcal{C}^{p,q} := \Lambda^p \mathfrak{g}^* \otimes \Lambda^q \mathfrak{g} \otimes C^\infty(\mathcal{M})$

  inner derivations of $\mathfrak{g}$ are $\beta, \bar{\beta}, b$

  outer derivations of $\mathfrak{g}$ are $\bar{\gamma}, \gamma, c$ with $[\bar{\gamma}, \beta] = [\gamma, \bar{\beta}] = \{c, b\} = 1$

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- Pick vacuum $\bar{\gamma}\,|0\rangle = \bar{\beta}\,|0\rangle = b\,|0\rangle = 0$. States:

$$\underbrace{\omega_{k,m}(x, \psi)}_{\in \Gamma(E),\ E \xrightarrow{\pi} M \cong \mathcal{M}} \quad \gamma^k \wedge \beta^m \wedge c\,|0\rangle \ \in\ \hat{\mathcal{C}}^{p,q}$$

In $\hat{\mathcal{C}}^{p,q}$ one finds

$A_\mu(x)\psi^\mu |0\rangle\,,\, C(x)\beta |0\rangle\,,\, \varphi(x)c\beta |0\rangle$ $\qquad$ $A^*_\mu(x)\psi^\mu c |0\rangle\,,\, C^*(x)c\gamma |0\rangle\,,\, \varphi^*(x)\gamma |0\rangle$

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The BRST nilpotent operator $Q = e^i G_i + \mathrm{d}_{\text{CE}}$ is:

$$Q = \frac{1}{2}cP^2 + \gamma\bar{q} + \bar{\gamma}q + \gamma\bar{\gamma}b \tag{1}$$

Q acts on $\mathcal{C}^{p,q}$ (and $\hat{\mathcal{C}}^{p,q}$) by increasing $p-q$ by $+1$.

$\mathcal{N}=2$ BRST Cohomology

In the Hilbert space one can form an inner product (*involution*)

$$\int \mathrm{d}^4 x \, \langle \Phi | \, \Phi \rangle := \int \mathrm{d}^4 x \int \mathrm{d}^n\psi \mathrm{d}\gamma \mathrm{d}\beta \mathrm{d}c \; \bar{\Phi}_{p,q} \, \bar{\psi}^{\mu_1} \wedge \cdots \wedge \bar{\psi}^{\mu_k} \wedge \underbrace{\delta^p(\gamma)}_{\bar{\beta}^p} \wedge \underbrace{\delta^q(\beta)}_{\bar{\gamma}^q} \, |\Phi\rangle$$

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Hence it holds:

Vanishing Theorem **[Frenkel, Garland, Zuckerman]**

$$H^{\bullet}(\mathcal{C}^{p,q}, Q) \begin{cases} \neq 0 & \text{for } H^0(\mathcal{C}^{p,p}, Q) \\ = 0 & \text{otherwise} \end{cases}$$

The non-trivial cohomology is only in ghost degree zero.

An alternative way to find the dimension of the cohomology groups relies on

$$\mathbb{P}(t) = \sum_k t^k \dim \hat{\mathcal{C}}_k^{\bullet,\bullet} \tag{2}$$

because of the **Euler-Poincaré principle**:

$$\mathbb{P}(-1) = \sum_k (-1)^k \dim H_k = \chi.$$

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Observation: for $\mathcal{N} = 2$ spinning particle

$$\tfrac{1}{1-t}\mathbb{P}(t) = \dim H^\bullet \overset{\text{vanishing th.}}{\equiv} H^0(\hat{\mathcal{C}}^{p,p}, Q)$$

- We investigated also other **pictures**
- Topological sectors arise in the case of pseudoforms

BV on the fields

Data for Batalin-Vilkovisky: $T^*[-1]\mathcal{F}, \omega, \mathcal{Q}$.

- The states in $\hat{\mathcal{C}}^{p,q}$ are rearranged as $\mathcal{F} = \underbrace{\mathcal{F}_0}_{\ni A_\mu(x), \varphi(x)} \oplus \underbrace{\mathcal{F}_1}_{\ni C(x)}$

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Look for the Hamiltonian vector fields to retain the action functional

$$\mathcal{Q} = \{S_{\text{free}} + (\Phi^* Q \Phi)_{\text{free}}, -\} \tag{4}$$

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A solution:

The action turns out to be:

$$\int \mathrm{d}^4 x\, \frac{1}{2} \langle\beta|\, Q_{\text{int}} Q Q_{\text{int}}\, |\beta\rangle + \frac{1}{3} \langle\beta|\, Q_{\text{int}} Q_{\text{int}} Q_{\text{int}}\, |\beta\rangle \equiv S[A, A^*, \varphi, \varphi^*, C, C^*]_{\text{BV YM}} \tag{5}$$

Recap

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- A spinning particle on superworldline lends a BV structure to the space of fields in the target
- $\mathcal{N} = 2$ spinning particle $\Leftrightarrow$ open string
- the full action functional of the target Yang–Mills theory is retrieved

Thanks