Bounded Poinceré operators for twisted complexes and BGG complexes

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- This talk reports on joint work with K. Hu (Edinburgh) which rectenly appeared in J. Math. Pures Appl.
- This connects to our earlier work that constructs projective and conformal BGG sequences on bounded Lipschitz domains in $\mathbb{R}^n$ in a Sobolev setting.
- Building on existing results for the de Rham complex, we construct bounded Poincaré operators on these complexes, both in the case of star shaped domains and in the case of non-trivial topology.
- In the former case, we can use these operators to construct complexes of polynomial sections with trivial cohomoloogy, which are a basis for constructing finite elements.
- For non-trivial topology, more subtle analytic methods lead to results like closed range and uniform smooth representatives for cohomology.

Contents

The motivation for the notion of a Poincaré operator comes from the standard proof of the Poincaré lemma on star-shaped open subsets of $\mathbb{R}^n$: Let $(C^k, d^k : C^k \to C^{k+1})_{k\geq 0}$ be a complex and suppose one finds operators $P^k : C^k \to C^{k-1}$ for all $k \geq 1$ such that $d^0P^1d^0 = d^0$ and $d^{k-1}P^k + P^{k+1}d^k = \mathrm{id}$ for all $k \geq 1$.

This immediately implies that the cohomology of the complex is trivial in degrees > 0 and equals $\ker(d^0)$ in degree zero.

For smooth forms on a star shaped domain in $\mathbb{R}^n$, such an operator is obtained by defining $P\varphi(x)(\xi_2, \dots, \xi_k)$ to be $\int_0^1 \varphi(F_t(x))(\partial_t F_t(x), DF_t(x)(\xi_2(x)), \dots, DF_t(x)(\xi_k(x)))dt$, where $F_t(x) = tx + (1-t)x_0$. For purposes of analysis, one often has to work with differential forms of weaker regularity, e.g. Sobolev regularity H^s for some $s \in \mathbb{R}$. Unfortunately, the construction does not lead to bounded operators in this setting, mainly because of the distinguished role of the center x_0.

This problem was solved in 2010 in work of M. Costable and A. MacIntosh. They impose a slightly stronger assumption on the domain U, namely that it is star-shaped with respect to any point x_0 in a small ball $B \subset U$. Using a bump function supported in B as a weight, they then average over the integral operators obtained from the different centers. They prove:

- The operators satisfy the homotopy relations thus implying trivial cohomology.
- They are restrictions of pseudodifferential operators of order -1 on $\mathbb{R}^n$, so in particular define bounded operators form H^s-forms to H^{s+1}-forms.
- There is a simple characterization for families of forms with polynomial coefficients that are preserved by the operators.

The last condition allows one to construct complexes of polynomial forms with trivial cohomology on U, which is a starting point for finite element methods.

The simplified BGG construction

For $(G, G_0) = (SL(n+1,\mathbb{R}), SL(n,\mathbb{R}))$ or $(SO(n+1,1), O(n))$, representation theory provides a specific decomposition $\mathbb{V} = \oplus_{i=0}^N \mathbb{V}_i$ of any irreducible G-representation $\mathbb{V}$ into G_0-invariant subspaces $\mathbb{V}_i$ together with G_0-equivariant maps $\partial : \Lambda^k\mathbb{R}^{n*} \otimes \mathbb{V}_i \to \Lambda^{k+1}\mathbb{R}^{n*} \otimes \mathbb{V}_{i-1}$ for $k = 0, \dots, n$ and $i = 1, \dots, N$ such that $\partial \circ \partial = 0$. Given $U \subset \mathbb{R}^n$ open and $s_i \in \mathbb{R}$, we put $C^{k,i} := H^{s_i - k}(U, \Lambda^k\mathbb{R}^{n*} \otimes \mathbb{V}_i)$ and $C^k := \oplus_i C^{k,1}$.

Interpreting $C^{k,i}$ as $\mathbb{V}_i$-valued Sobolev forms, we obtain bounded operators $d : C^{k,i} \to C^{k+1,i}$. Passing to $\oplus_i$, the cohomology of (C^k, d) is the de Rham cohomolgy of U tensorized with $\mathbb{V}$. If $s_i \geq s_{i-1} - 1$, then we get $S : C^{k,i} \to C^{k+1,i-1}$ (bounded) via ∂.

It is easy to see that $dS = -Sd$ and hence $d - S$ defines a differential $C^k \to C^{k+1}$ for any $k \geq 0$ ("twisted complex"). This computes the same chomology as (C^*, d).

For star shaped domains, this can be nicely proved via Poincaré operators. Tensorizing the de Rham Poincaré operators with identity maps, we obtain operators $P^{k,i} : C^{k,i} \to C^{k-1,i}$ for any $k \geq 1$, $i \geq 0$ such that $dP + Pd = \mathrm{id}$. Using this identity and $dS = -Sd$, one immediately shows that $d(\mathrm{id} - PS) = (\mathrm{id} - PS)(d - S)$, so $\mathrm{id} - PS$ is a chain map.

By construction PS maps each $C^{k,i}$ to $C^{k,i-1}$, and hence is nilpotent. Thus $\mathrm{id} - PS$ is an isomorphism with inverse $\sum_{\ell=0}^{\infty} (PS)^{\ell}$ (finite sum). In particular, we can pull back the operators $P^{k,i}$ to bounded Poincaré operators on the twisted complex.

To pass to the BGG complex (for general U), we need the assumption that the S-operators have closed range, which amounts to $s_i = s_{i-1} - 1$. Then one considers the closed subspace $\Upsilon^k := \ker(S) \cap \mathrm{im}(S)^{\perp} \subset C^k$. The BGG machinery leads to operators $A : \Upsilon^k \to C^k$ (splitting operator) and $B : C^k \to \Upsilon^k$ such that $BA = \mathrm{id}$.

These operators can be used to “compress” $d - S$ to an operator $\mathcal{D} : \Upsilon^k \to \Upsilon^{k+1}$ such that $A \circ \mathcal{D} = (d - S) \circ A$ and $\mathcal{D} \circ B = B \circ (d - S)$. This readily implies $\mathcal{D}^2 = 0$, so $(\Upsilon^*, \mathcal{D})$ is a complex, the BGG complex and A and B define chain maps between the BGG complex and the twisted complex.

Theorem

The composition AB is chain homotopic to the identity, so A and B induce inverse isomorphisms in cohomolgy. In particular, the cohomology of the BGG complex is isomorphic to the de Rham cohomology of U tensorized with $\mathbb{V}$.

For star-shaped U, the statement on cohomology can be nicely prove using Poincaré operators: Denoting by P_V a Poincaré operator for the twisted complex, one proves that $\mathcal{P} = BP_VA$ defines a bounded Poincaré operator for the BGG complex, which thus has trivial chomology.

Complex property and polynomials

An important property of the de Rham Poincaré operators of Costable and MacIntosh is that $P^2 = 0$, so they also form a complex. This is preserved by the passage to the twisted complex but not by the passage to the BGG complex. However, one proves that $\tilde{\mathcal{P}} := \mathcal{P} - \mathcal{D}\mathcal{P}^2$ defines a Poincaré operator with $\tilde{\mathcal{P}}^2 = 0$.

Following Costable and MacIntosh, there are explicit conditions that one can impose on families $\Pi^{k,i} \subset C^{k,i}$ of polynomial maps for $U = \mathbb{R}^n$ which ensure that P_V maps $\Pi^k = \oplus_i \Pi^{k,i}$ to Π^{k-1} while $\mathcal{P}$ maps $\Pi^k \cap \Upsilon^k$ to $\Pi^{k-1} \cap \Upsilon^{k-1}$.

Using this, we can construct BGG sequences with spaces of polynomial forms which have trivial cohomology on star-shaped domains. Typically, these spaces are constructed using the Poincaré operators. They represent a first step towards finding finite elements for BGG complexes.

Here the homotopy relation has to be modified to the form $Pd + dP = \text{id} - L$ and the main question is what conditions on L to impose. The relation implies $dL = Ld$, so $(\text{im}(L), d)$ is a subcomplex, and the relation then implies that this computes the same cohomology.

Hence natural conditions to impose are $Ld = 0$ and $L^2 = L$ which means that L is a projection onto the cohomology. For the smooth de Rham complex on compact manifolds, this can be realized by Hodge theory. One puts $P = \delta G$, where δ is the codifferential and G is Green's operator. Observe that $\delta^2 = 0$ implies $P^2 = 0$.

The passage to the twisted complex and the BGG complex can be done as in the case of trivial cohomology and it is easy to see that the properties $Ld = 0$ and $L^2 = L$ are (independently) preserved in both steps. With a bit more effort, one proves that if one starts with P such that $P^2 = 0$ also $\tilde{\mathcal{P}}^2 = 0$ can be achieved on the BGG complex.

For Sobolev-de Rham complexes, I am not aware of a construction for Poincaré operators with these strong properties, but for the case of a bounded Lipschitz domain $U \subset \mathbb{R}^n$, Costable and MacIntosh find a surprising alternative solution:

- U can be covered by finitely many domains U_i which are star-shaped with respect to all points in a small ball
- Via a partition of unity $\{\chi_i\}$, the operators P_i on U_i as constructed before define P on U such that both P and L are restrictions of pseudodifferential operators of order -1.
- L has better properties around x if all χ_i are constant on some neighborhood x
- Using this and several coverings, the construction can be improved in such a way that the L-operators are restrictions of pseudodifferential operators of order $-\infty$.

Recall that pseudodifferential operators of order $-\infty$ are invertible modulo compact operators and have smooth values on all distributions. Using this, one proves analytical results like closed range of d, finite dimension of cohomologies, and existence of smooth representatives for the cohomology independent of the Sobolev regularity. We extend this to the BGG setting as follows.

We first run the construction of Costable and MacIntosh directly for the twisted complex to obtain P_V and L_V which are restrictions of pseudodifferential operators of order -1 and $-\infty$, respectively. These can then be carried over to the BGG complex as before, to obtain bounded operators $\mathcal{P}$ and $\mathcal{L}$, which are restrictions of pseudodifferential operators of order -1 and $-\infty$.

Hence all the analytical results mentioned above carry over to all BGG complexes obtained from projective or conformal geometry on bounded Lipschitz domains in $\mathbb{R}^n$.