Courant Algebroid Relations and T-duality

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Outline

Courant algebroids

The **generalised tangent bundle**, $TM \oplus T^*M$, can be equipped with a pairing and a bracket, $H \in \Omega^3_{\mathrm{cl}}(M)$:

$$\langle X + \xi, Y + \eta \rangle = \iota_X \eta + \iota_Y \xi,$$
$$[X + \xi, Y + \eta]_H = [X, Y] + \mathcal{L}_X \eta - \iota_Y d\xi + \iota_Y \iota_X H.$$

In general, an **exact Courant algebroid** (CA) is a vector bundle E over M with a pairing and a bracket of sections fitting the sequence

$$0 \to T^*M \to E \to TM \to 0. \qquad (E \cong TM \oplus T^*M)$$

CAs are classified by their *Ševera class* $[H] \in H^3_{cl}(M)$, with H given by a splitting $s\colon TM \to E$

$$H(X, Y, Z) = \langle [s(X), s(Y)]_E, s(Z) \rangle_E.$$

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Sigma-models in generalised geometry

Consider maps f from a worldsheet surface Σ to a target space M.

Sigma-Model	**Geometry**
Wess-Zumino term $S_H[f] = \int_V f^*H$	Courant algebroid E_H over M
$\delta S_H = 0$	$X + \xi$ such that $\iota_X H = d\xi$
Polyakov term $S_g[f] = \int_\Sigma \lVert df \rVert_g^2 d\mu_g$	Generalised metric $V_+ \subset E_H$ equiv. $\tau \in \mathrm{End}(E_H)$, $\tau^2 = 1$ equiv. $g \in \mathrm{Riem}(M)$
$\delta S_g = 0$	$\mathcal{L}_X g = 0$

The data of the full action $S = S_H + S_g$ is encoded in a CA equipped with a positive definite subbundle, (E, V_+).

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Courant algebroid relations

Isomorphism $\Phi\colon E_1 \to E_2$ of CAs covering a diffeomorphism $\varphi\colon M_1 \to M_2$ preserve all CA structure, hence the Sigma-models over M_1 and M_2 have the same equations of motion.

T-duality is an equivalence of Sigma-models on different target manifolds M_1 and M_2, though these are not necessarily diffeomorphic. A CA isomorphisms $\Phi\colon E_1 \to E_2$ must cover a diffeomorphism, so we seek to generalise the notion of isomorphism.

Consider the graph $\mathrm{gr}(\Phi) \subset E_1 \times E_2$. One sees that

$$[\Phi\cdot, \Phi\cdot]_2 = \Phi([\cdot,\cdot]_1) \iff \mathrm{gr}(\Phi) \text{ is involutive in } E_1 \times E_2$$

$$\langle \Phi\cdot, \Phi\cdot\rangle_2 = \langle \cdot,\cdot\rangle_1 \iff \mathrm{gr}(\Phi) \text{ is isotropic in } E_1 \times E_2$$

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Isometry and Composition

A **CA relation** $R\colon E_1 \dashrightarrow E_2$ is an isotropic, involutive subbundle of $E_1 \times E_2$, supported on a submanifold $C \subset M_1 \times M_2$.

If $R\colon (E_1, \tau_1) \dashrightarrow (E_2, \tau_2)$, then R is a **generalised isometry** if

$$(\tau_1 \times \tau_2)(R) = R.$$

Contingent on some smoothness conditions [Vysoký, 1], one can compose two relations $R\colon E_1 \dashrightarrow E_2$, $\tilde{R}\colon E_2 \to E_3$

$$\tilde{R} \circ R = \{(e_1, e_3) : (e_1, e_2) \in R \text{ and } (e_2, e_3) \in \tilde{R}\} \subset E_1 \times \overline{E}_3,$$

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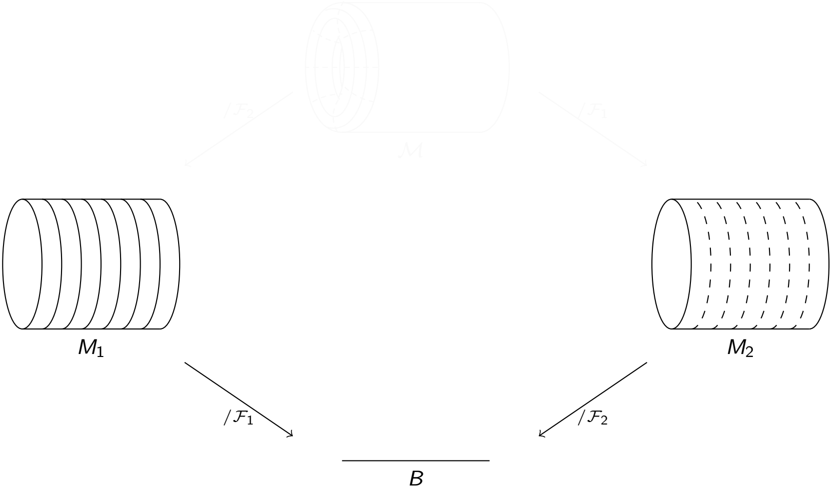
T-duality

T-dual spaces M_1, M_2 may be fibre bundles over a common base B



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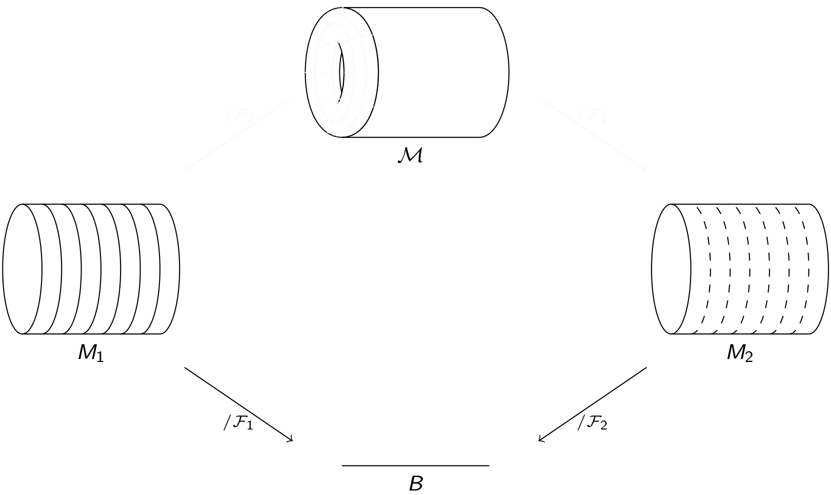

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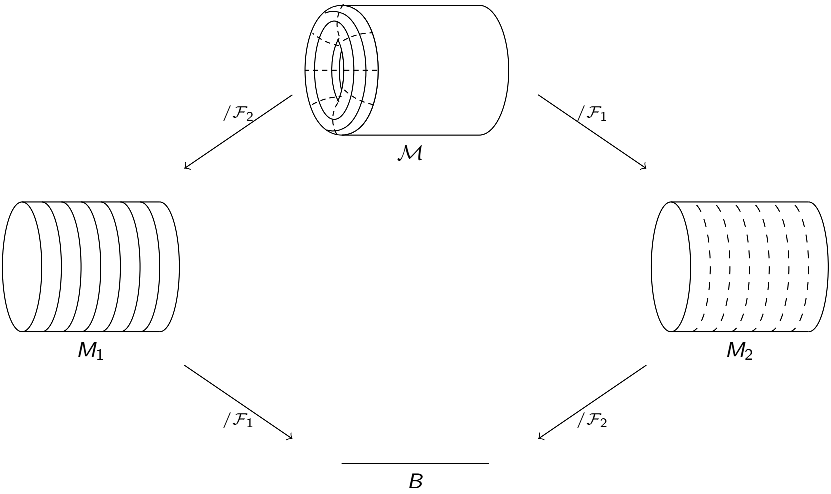

The T-duality relation I

Want to form a CA relation $R\colon E_1 \dashrightarrow E_2$ between CAs E_i over M_i. Need to know how to form CAs on a quotient manifold:

$\mathcal{M}$ is foliated by $\mathcal{F}_2$. If $\mathcal{E}$ is an exact CA over $\mathcal{M}$, then $\mathcal{E}/\mathcal{F}_2$ will not be an exact CA over $M_1 = \mathcal{M}/\mathcal{F}_2$.

Take $K_2 = T\mathcal{F}_2 \subset \mathcal{E}$, then

$$E = \frac{K_2^\perp}{K_2} \Big/ \mathcal{F}_2$$

is an exact CA over M_1 [2,3].

Can form the CA relation $Q(K_2)$, supported on $\mathrm{gr}(q_2) \subset \mathcal{M} \times M_1$,

$$Q(K_2) = \{(e, \natural_2(e))\colon e \in K_2^\perp\} \subset \mathcal{E} \times E_1$$

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The T-duality relation $R\colon E \to E'$ is then the composition of CA relations:

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\Phi = \varphi_* \circ e^F} & \mathcal{E} \\ Q(K_2)^T \uparrow & & \downarrow Q(K_1) \\ E_1 & \xrightarrow[R]{} & E_2 \end{array}$$

Theorem (DF, Marotta, Szabo [4])

Let g_1 be a Riemannian metric on M_1. If $\mathcal{L}_X g = \mathcal{L}_X F = 0$, $\forall X \in \mathcal{F}_2$, then TFAE

1. $K_1^\perp \cap \Phi(K_2) \subseteq K_1$.
2. *There exists a unique Riemannian metric g_2 on M_2 such that R is a generalised isometry.*

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Application: T-dualities for Torus bundles

Cavalcanti-Gualitieri [5] formulate the correspondence space picture for T-dual torus bundles over a base B:

Endow M_i with H-flux H_i. Take $\mathcal{M} = M_1 \times_B \times M_2$ and $q_1 : \mathcal{M} \to M_2$ etc. Then M_1 is T-dual to M_2 if

$$q_2^* H_1 - q_1^* H_2 = dF,$$

with $F : \mathfrak{t}_1 \otimes \mathfrak{t}_2 \to \mathbb{R}$ invariant and non-degenerate.

The condition $\mathcal{L}_X F = 0$ gives requisite invariance, and item (1) gives non-degeneracy.

The resulting Riemannian metrics g_1, g_2 satisfy the Buscher rules.

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Application: Doubled Geometry

Instead of $\mathcal{M}$ being the fibred product space, we can consider it being the simultaneous doubled space of M_1 and M_2.

From a generalised metric (g_1, b_1) on E_1, one can form a Riemannian metric G_1 on $\mathcal{M}$.

Let $X_1, \ldots, X_N$ be coordinate vector fields on $\mathcal{M}$, and $X_1, \ldots, X_n$ generate the foliation $T\mathcal{F}_2$ (with $n = N/2$).

Now pick $k < n$ and a diffeomorphism φ, and define a foliation by $T\mathcal{F}_1$ generated by the following n vector fields:

$$\varphi_*(X_i) \text{ for } i = k+1, \ldots, n, \quad \varphi_*(X_p) \text{ for } p = n+1, \ldots, n+k.$$

Upon forming the T-duality relation $R\colon E_1 \to E_2$, one has the following results for the metrics:

$$\varphi^* G_2 = G_1.$$

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Outlook

1. Fit Poisson-Lie T-duality into this framework.
2. Add more structure: divergences (or CA connections), allowing to incorporate the dilaton, generalised Ricci tensors and Ricci flow; generalised complex structures, allowing incorporation of branes.

End

Thanks you for listening! Questions?

References: [1] Vysoký https://arxiv.org/abs/1910.05347
[2] Bursztyn, Cavalcanti, Gualtieri
https://arxiv.org/abs/math/0509640
[3] Zambon https://arxiv.org/abs/math/0701740
[4] DF, Marotta, Szabo https://arxiv.org/abs/2308.15147
[5] Cavalcanti, Gualitieri https://arxiv.org/abs/1106.1747