Configuration space of a 3-link snake robot model

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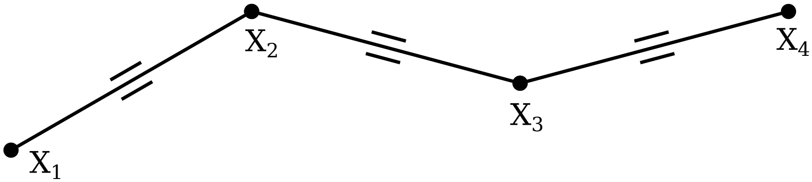
The model of a 3-link snake robot

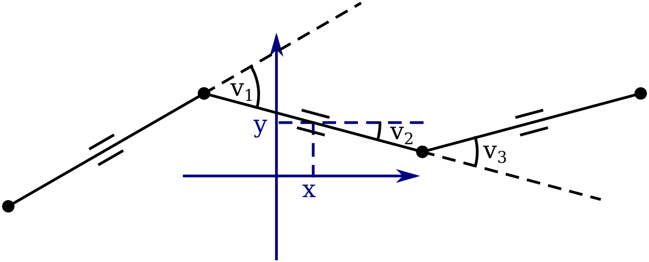


On a plane, i.e. $X_1, \ldots X_4 \in \mathbb{R}^2$

$$\frac{1}{2}(\dot{X}_1 + \dot{X}_2) = \alpha \cdot (X_2 - X_1)$$
$$\frac{1}{2}(\dot{X}_2 + \dot{X}_3) = \beta \cdot (X_3 - X_2)$$
$$\frac{1}{2}(\dot{X}_3 + \dot{X}_4) = \gamma \cdot (X_4 - X_3)$$
$$\|X_2 - X_1\| = const., \|X_3 - X_2\| = const., \|X_4 - X_3\| = const.$$

$\rightarrow$ foliation of subbundles of $\mathbb{R}^8$

The model, better coordinates



Choose $\|X_{i+1} - X_i\| = 1$.
New coordinates $(v_1, v_2, v_3, x, y) \in \mathcal{M} = S^1 \times S^1 \times S^1 \times \mathbb{R}^2$.

$$VF_1 = (1 + \cos(v_1))\,\partial_{v_1} - \partial_{v_2} + (1 + \cos(v_3))\,\partial_{v_3}$$
$$VF_2 = -2\sin(v_1)\,\partial_{v_1} + 2\sin(v_3)\,\partial_{v_3} + \cos(v_2)\,\partial_x + \sin(v_2)\,\partial_y$$

They represent steering of the central segment (move and rotation).

Distribution

- A distribution $\mathcal{D} = \langle VF_1, VF_2 \rangle$.
- $\mathcal{D}$ is bracket-generating

$$\mathcal{D}^{(1)} = \mathcal{D}, \quad \mathcal{D}^{(i+1)} = \mathcal{D} + [\mathcal{D}, \mathcal{D}^{(i)}],$$
$$\mathcal{D}^{(k)} = \mathcal{D}^{(k-1)} + [\mathcal{D}, \mathcal{D}^{(k)}] = T\mathcal{M}$$

- By Chow-Rashevskii thm, every position is achievable
- Differences of $\mathcal{D}^{(i)}$'s dimensions $\to$ growth vector
- In regular points, the growth vector is $(2, 1, 2)$

$$\begin{matrix} VF_1, \\ VF_2, \end{matrix} \qquad VF_3 = [VF_1, VF_2], \qquad \begin{matrix} VF_4 := [VF_1, VF_3], \\ VF_5 := [VF_2, VF_3] \end{matrix}$$

- In singular points (e.g. $v_1 = v_3 = 0$), $[VF_2, VF_3]$ isn't linearly independent, and the growth vector is $(2, 1, 1, 1)$.

Lie algebra

A vector field is $VF_6 := [VF_1, VF_4]$ is linearly independent only over real numbers. ($VF_6 = A \cdot VF_2 - VF_3 - A \cdot VF_4 - B \cdot VF_5$) One can show, $VF_1, \dots VF_6$ form a finite-dimensional solvable Lie algebra.

$[\cdot,\cdot]$	VF_1	VF_2	VF_3	VF_4	VF_5	VF_6
VF_1	0	VF_3	VF_4	VF_6	0	$-VF_4$
VF_2	$-VF_3$	0	VF_5	0	$4VF_3+4VF_6$	0
VF_3	$-VF_4$	$-VF_5$	0	0	0	0
VF_4	$-VF_6$	0	0	0	0	0
VF_5	0	$-4VF_3-4VF_6$	0	0	0	0
VF_6	VF_4	0	0	0	0	0

Extended model

$\mathcal{M}' = \mathcal{M} \times \mathbb{R}$, coordinates (v_1, v_2, v_3, x, y, t), $\mathcal{D}' = \langle VF'_1, VF'_2 \rangle$

$$VF'_1 = (1 + \cos(v_1))\, \partial_{v_1} - \partial_{v_2} + (1 + \cos(v_3))\, \partial_{v_3} + \partial_t$$
$$VF'_2 = -2\sin(v_1)\, \partial_{v_1} + 2\sin(v_3)\, \partial_{v_3} + \cos(v_2)\, \partial_x + \sin(v_2)\, \partial_y$$

- $VF'_1, \ldots VF'_6$ are now linearly independents over functions, thanks to the added coordinate t
- $\mathcal{D}'$ has a growth-vector $(2, 1, 2, 1)$
- $VF'_1, \ldots VF'_6$ form a basis of a Lie algebra, so they give us locally a structure of a Lie group on $\mathcal{M}'$

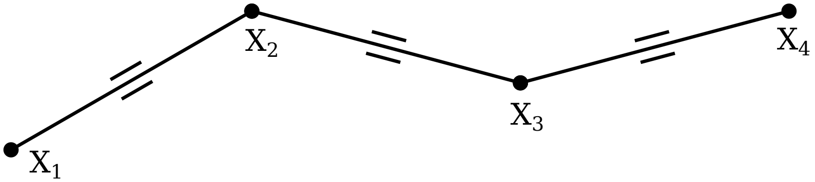
Symmetries on $\mathcal{M}'$

- Infinitesimal symmetries of a distribution $\mathcal{D}$ are vector fields v on $\mathcal{M}$ such that $[v, \mathcal{D}] \in \mathcal{D}$.
- $\mathcal{M}'$ is locally a Lie group, there are 6 right-invariant vector fields which commute with $VF'_1, \dots VF'_6$

$$\partial_x,\ \partial_y,\ \partial_{v_2} - y\,\partial_x + x\,\partial_y$$
$$(1+\cos v_3)e^{2v_2+2t}\partial_{v_3}, (1+\cos v_1)e^{-2v_2-2t}\partial_{v_1}, \partial_t$$

- distributions $(2, 1, 2, 1)$ can have maximally 8-dim symmetry (Anderson, Kruglikov 2011)
- it can by shown, $(\mathcal{M}', \mathcal{D}')$ has only these 6 symmetries
- $(\mathcal{M}, \mathcal{D})$ has only 3 symmetries (take quotient of $\mathcal{M}'$ by an automorphism generated by ∂_t)

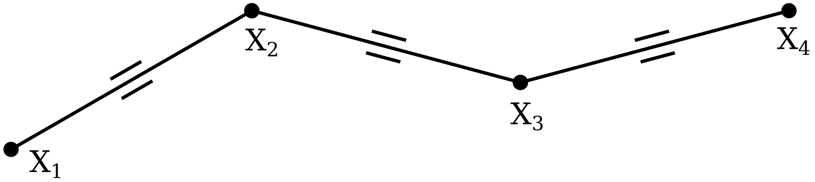
A modulation of the model



- For three different lengths $\ell_1, \ell_2, \ell_3 \in \mathbb{R}^+$, the distribution $\mathcal{D}$ is still $(2, 1, 2)$, but now it contains a 9-dim Lie algebra with a growth-vector $(2, 1, 2, 1, 2, 1)$

$$\|X_2 - X_1\| = \ell_1, \|X_3 - X_2\| = \ell_2, \|X_4 - X_3\| = \ell_3$$

Another modulation of the model



- If we move the rubber wheels, i.e. choose some $a, b, c \in (0, 1)$, the distribution is still $(2, 1, 2)$, now the Lie algebra inside is $(2, 1, 2, 2, 2)$.

$$a\dot{X}_1 + (1-a)\dot{X}_2 = \alpha \cdot (X_2 - X_1)$$
$$b\dot{X}_2 + (1-b)\dot{X}_3 = \beta \cdot (X_3 - X_2)$$
$$c\dot{X}_3 + (1-c)\dot{X}_4 = \gamma \cdot (X_4 - X_3)$$

Conclusions and open questions

- If a distribution contains generators of a finite-dimensional Lie algebra, it can be used to estimate the number of symmetries.
- It is unclear how to find such generators.
- It isn't easy to add new coordinates in a right way and let the dimension of the Lie algebra unchanged. Is it necessary to add these coordinates?
- further research needed...

Thank you for your attention!