Generalized Root Systems

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Plan of the Talk

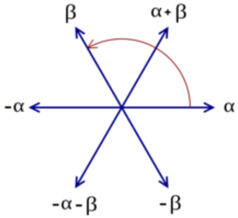
1. **Motivation:**
   A unifying conceptual point of view on root systems in Lie Theory (for semisimple Lie algebras, contragredient superalgebras and special types of hyperplane arrangements)
2. **Definition of Generalized Root Systems (GRS):**
   All "meaningful" root systems in Lie theory are examples.
3. **The Category of GRS:**
   In this category we can take subobjects and **quotients**
4. **Classification of rank 2 GRS**
5. **Virtual Reflections**
6. **Application to Reductive Supergroups**

Definition

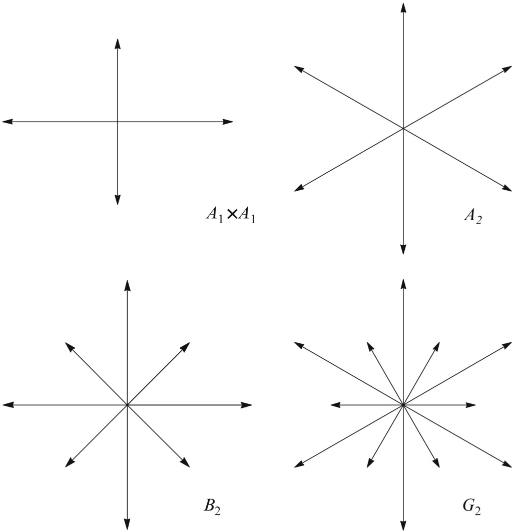
A **root system** Δ *in* E *euclidean space* $(\ ,\ )$ *satisfies:*

- Δ *is finite,* $0 \notin \Delta$
- Δ *spans* E*,*
- $\sigma_\alpha(\beta) := \beta - 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}\alpha \in \Delta$
- $\langle \beta, \alpha \rangle := 2\frac{(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$
- *The only multiple of a root* α *is* $\pm\alpha$ *(reduced root system)*

Example. $\Delta = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta)\}$



Root Systems of Rank 2



Notion of Root System in Lie Theory/1

Definition (Root Systems of semisimple Lie algebras)

A **root system** $\Delta \subset \mathfrak{h}^*$ *of a complex simple Lie algebra* $\mathfrak{g}$ *is obtained via the root space decomposition:*

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha, \qquad \mathfrak{h} \text{ Cartan subalgebra} \tag{1}$$

$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \,|\, [H, X] = \alpha(H)X, \forall H \in \mathfrak{h}\}$

Example: $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$

$$\mathfrak{h} = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \right\}, \qquad \mathfrak{g} = \left\{ \begin{pmatrix} a & X_\alpha & X_{\alpha+\beta} \\ X_{-\alpha} & b & X_\beta \\ X_{-\alpha-\beta} & X_{-\beta} & -a-b \end{pmatrix} \right\}$$

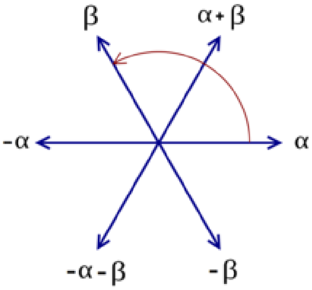
$$\alpha \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} = a - b, \qquad \beta \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} = b - c,$$

A_2 Root System of $\mathfrak{sl}_3(\mathbb{C})$

Example:

$\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$, $\Delta = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta)\}$

$$\mathfrak{h} = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \right\}, \quad \alpha \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} = a - b, \quad \beta \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} = b - c$$



Notion of Root System in Lie Theory/2

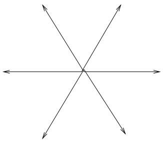
Definition (Kostant root system)

R is obtained via the root space decomposition of a semisimple Lie algebra $\mathfrak{g}$ with respect to a toral subalgebra $\mathfrak{t} \subset \mathfrak{h}$, $\pi : \mathfrak{h}^* \longrightarrow \mathfrak{t}^*$:

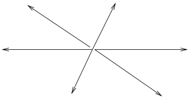
$$\mathfrak{g} = \mathfrak{c}(\mathfrak{t}) \oplus \sum_{\nu \in R} \mathfrak{g}_\nu, \quad \mathfrak{g}_\nu = \{x \in \mathfrak{g} \,|\, [h, x] = \nu(h)x, \ h \in \mathfrak{t}\}$$

Example $\Delta = \{\epsilon_i - \epsilon_j, \ i \neq j = 1 \dots n+1\}$ root system of $\mathfrak{sl}_{n+1}(\mathbb{C})$.
If we orthogonally project it, we modify the lengths of vectors!
Suppose we project on a two dimensional space.

Roots of $\mathfrak{sl}_3(\mathbb{C})$



Projected roots of $\mathfrak{sl}_{n+1}(\mathbb{C})$

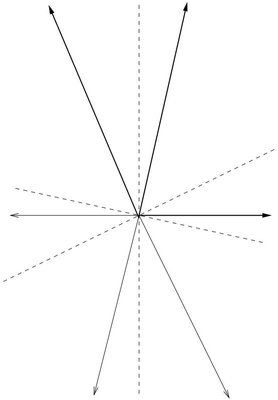


Kostant root systems

Observation

Kostant root systems are orthogonal projections of Lie algebra root systems, i.e. $R = \pi(\Delta)$. *They have* **less** *symmetries.*

Example



Notion of Root System in Lie Theory/3

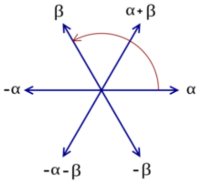
Definition (Root systems of basic classical Lie superalgebras)

Such root systems are obtained with a root space decomposition:

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha, \qquad \mathfrak{h} \text{ Cartan subalgebra}$$

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \,|\, [H, X] = \alpha(H)X, \forall H \in \mathfrak{h}\}$$

Example: $\mathfrak{g} = \mathfrak{sl}(2|1)$, $\mathfrak{h}$ diagonal matrices



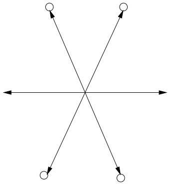
α even root, β odd root.

Lack of symmetries

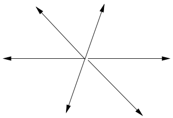
These root systems share many properties, but differ in some ways.

Key difference: lack of Weyl group.

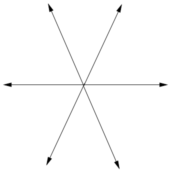
$\mathfrak{sl}(2|1)$



Kostant root system



$\mathfrak{sl}_3(\mathbb{C})$



How do we recover the "missing" symmetries?
Virtual Reflections (Serganova)

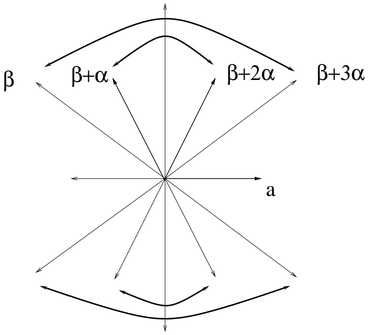
Virtual Reflections

Idea: A reflection of a simple root β with respect to a simple root α *reverses its α-string*

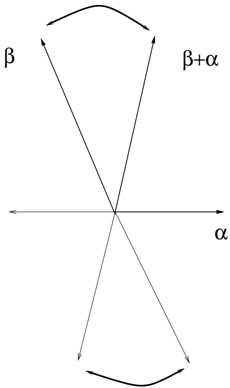
$$S_\alpha(\beta) = \begin{cases} -\beta & \beta = \alpha \\ \beta + \ell_{\beta,\alpha}\alpha & \beta \neq \alpha \end{cases}$$

$\ell_{\beta,\alpha}$ length of the α-string of β.

Example: The reflection s_α in the root system of the Lie algebra G_2



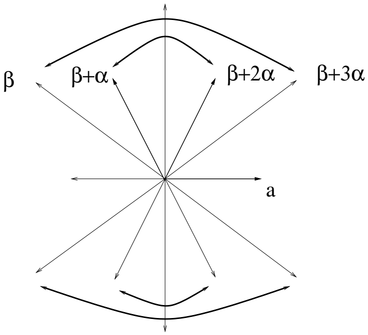
Recovering the Symmetries via virtual reflections



The virtual reflection is a linear map, but **not** an isometry.

Serganova gives this definition to recover the symmetries of root systems of Lie superalgebras. (**Weyl Groupoid**).

G_2 symmetries



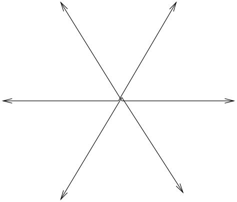
Generalized Root Systems

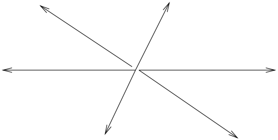
Definition

A **Generalized Root System** *is a pair* (E, R)*, where* E *is a finite-dimensional Euclidean space and* $R \subset E$ *is a finite set such that, for* $\alpha, \beta \in R$ *we have*

$$\begin{aligned} \langle\alpha,\beta\rangle > 0 &\implies \alpha - \beta \in R; \\ \langle\alpha,\beta\rangle < 0 &\implies \alpha + \beta \in R; \\ \langle\alpha,\beta\rangle = 0 &\implies \alpha + \beta \in R \text{ if and only if } \alpha - \beta \in R. \end{aligned}$$

Remark. $0 \in R$, $R = -R$.
Examples: "A type", $R = \pm\{\alpha, \beta, \alpha + \beta\}$.





Generalized Root Systems in the literature

- Root systems of semisimple Lie algebras (Cartan)
- Kostant root systems
  (Kostant "Borel-De Siebenthal Theorem")
- Root systems of contragredient Lie superalgebras
  (Kac "Lie Superalgebras")
- Crystallographic hyperplane arrangements
  (Refs in Cunz "Simplicial arrangements")
- Tits hyperplane arrangements
  (Wemyss private communication)
- Complex structures of flags (and superflags)
  (Alekseevski "Flag manifolds")

Examples of Generalized Root Systems: the reduced case

Definition

A GRS is **reduced** *if* $\alpha \in R$ *only* $\pm\alpha \in R$.

Proposition

The reduced irreducible rank 2 GRSs are:

"A_2": $R^+ = \{\alpha, \beta, \alpha + \beta\}$ *a two-parameter family.*

"B_2": $R^+ = \{\alpha, \beta, \beta + \alpha, \beta + 2\alpha\}$ *a one-parameter family* ($\beta \perp \beta + 2\alpha$).

G_2 *– this is a singleton: no "deformation" allowed, it is* **rigid**

Cartan Matrix

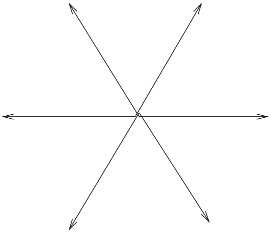
Let R be a (reduced) root system of rank, $S = \{\alpha, \beta\}$.
We can define the Cartan matrix $C = (c_{\alpha,\beta})$:

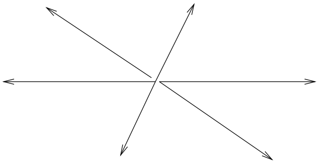
$$c_{\alpha,\beta} = \frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} \qquad \alpha, \beta \in S$$

Example: the two parameter family A_2

$$C = \begin{pmatrix} 2 & a \\ b & 2 \end{pmatrix} ,$$

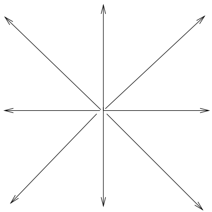
where $a := c_{\alpha,\beta} = \frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}$ and $b := c_{\beta,\alpha} = \frac{2\langle\beta,\alpha\rangle}{\langle\beta,\beta\rangle}$.

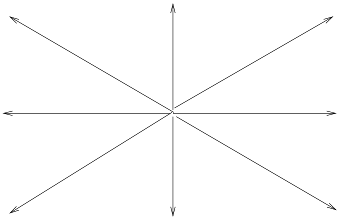




Examples:

"B type" $R = \pm\{\alpha, \beta, \beta + \alpha, \beta + 2\alpha\}$.



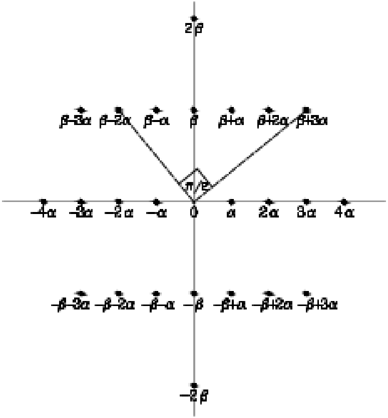


Exotic GRS: Quadruple root GRS

A more exotic rank 2 GRS – admits a root α for which 4α is also a root:

$$R^+ = \{k\alpha, 1 \le k \le 4, \beta + l\alpha, -3 \le l \le 3, 2\beta\} \ .$$

Here $\alpha \perp \beta$, $\|\beta\| = \sqrt{6}\,\|\alpha\|$.

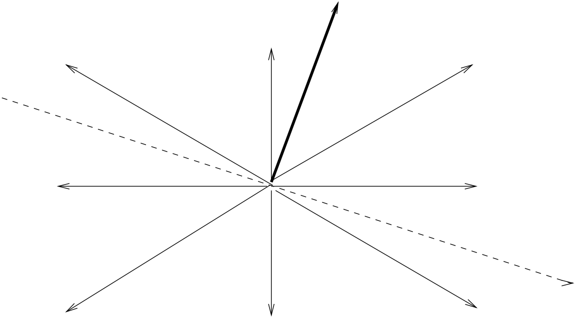


Positive systems of Generalized Root Systems

Definition

Given a Generalized Root System R we say that R^+ is a **positive systems** *if:*

- $R = R^+ \cup -R^+$.
- *If* $\alpha, \beta \in R^+$ *and* $\alpha + \beta \in R$, *then* $\alpha + \beta \in R^+$.



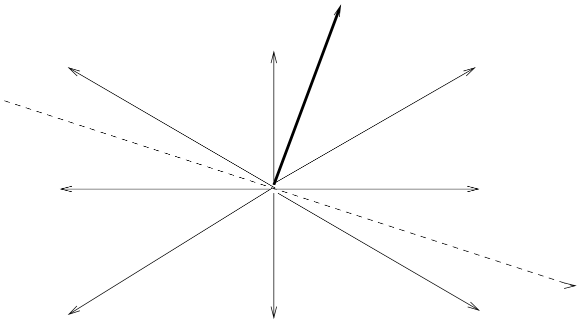
Some properties of Positive Systems

(i) Every positive system R^+ admits a base S.
Attention: Same R may look "different" if we choose different bases to represent it.

(ii) Every positive system has a highest root.

(iii) Every positive system is laying on one side of an hyperplane:

$$R^+ = \{\alpha \in R \,|\, \langle \alpha, \gamma \rangle > 0\}$$



Generalized Root Systems are Lattices

Let R be a GRS and S a base. We can define a partial order on R by the **height of a root**:

$$\mathrm{ht}(\alpha) = \sum_i a_i, \qquad \text{where} \quad \alpha = \sum_i a_i \alpha_i$$

Hence the pair (R, S) is a poset (partially order set).

Proposition

Let R be a generalized root systems and S a base. Then as a poset R is a lattices, i.e. we can define a $\vee$ join (least upper bound) and meet $\wedge$ (greatest lower bound). If $\alpha = \sum_i a_i \alpha_i$, $\beta = \sum_i b_i \alpha_i$, $S = \{\alpha_1, \dots, \alpha_n\}$ base:

$$\alpha \vee \beta = \sum_i \min(a_i, b_i) \alpha_i$$

Remark. In general $\alpha \wedge \beta \neq \sum_i \max(a_i, b_i)$.

Counterexample: in A_3 we have $\alpha_1 \wedge \alpha_3 \neq \alpha_1 + \alpha_3$ because this is not a root!

The Category of Generalized Root Systems

Let $R \subset V$ be a GRS, S a base of R.

(i) If $W \subset V$ is a subspace $R \cap W$ is a GRS, this is a *subsystem* of R.

(ii) If $I \subset S$, let $V_I = \mathrm{span}\{I\} \subset V$. We have $V = V_I \oplus V_I^{\perp}$, $\pi : V \longrightarrow V_I^{\perp}$. Define:

$$R_I := R \cap I, \qquad R/I := \pi_I(R) \subset V_I^{\perp}$$

Theorem. Let $I \subset S$ base of R.

a) Both R_I and R/I are GRS.

b) Let $I \subset J \subset S$. Then:

$$R/J \cong (R/I)/(J/I)$$

Proof. (a). R_I immediate, R/I harder to show. (b) easy.

Example of quotient

Take: $R = B_3 = \{e_i \pm e_j,\ i \neq j\}$

$S = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \alpha_3 = e_3\}$, $I = \alpha_2$.

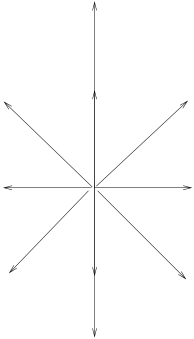
Take the quotient $R \longrightarrow R/I$

$\alpha_1 = e_1 - e_2 \mapsto \overline{e}_1 - \overline{e}_2$

$\alpha_2 = e_2 - e_3 \mapsto 0 \implies \overline{e}_2 = \overline{e}_3 \implies 2\overline{e}_2 \in R/I!$

$\alpha_3 = e_3 \mapsto \overline{e}_3$

We represent R/I (up to equivalence) as:



Morphisms of Generalized Root Systems

Definition

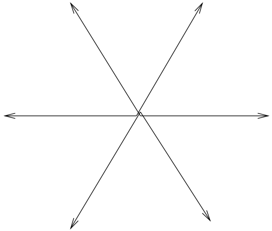
Two generalized root systems (R_1, V_1) and (R_2, V_2) are isomorphic if there is an isometry $\phi : V_1 \longrightarrow V_2$ and a bijection between R_1 and R_2

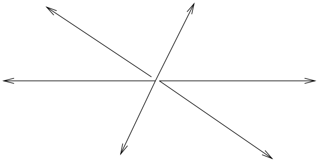
Definition

$(R1, V_1)$ and (R_2, V_2) are equivalent if there is a vector space isomorphism $\phi : V_1 \longrightarrow V_2$ and bijection between R_1 and R_2.

Clearly: isomorphic $\implies$ equivalent, but not vice-versa.

Example: the two root systems are equivalent, but not isomorphic.





Quotients of GRS

Theorem (Dimitrov-F. 2023)

Quotients of generalized root systems are generalized root systems.

Corollary (Dimitrov-F. 2023)

All Kostant root systems are GRS.

Note: They are quotients of Lie algebra root systems.

Theorem (Dimitrov-F. 2023)

All root systems of contragredient superalgebras are GRS.

Note: they are quotients of Lie algebra root systems (Dimitrov-Roth 2016, Dimitrov-F. 2021).

Lie algebras and quotients of Root systems

Theorem (Dimitrov-F. 2023, Dimitrov-Roth 2016)

Every root system of a simple Lie algebra is isomorphic to a quotient of one of the root system A_n, D_n, E_i, $i = 6, 7, 8$ (simply-laced root system). More precisely we have:

(i) $B_l \cong D_{l+2}^{1,2,\dots,l}$;

(ii) $C_l \cong D_{2l}^{2,4,\dots,2l}$;

(iii) $F_4 \cong E_7^{1,3,4,6} \cong E_8^{1,6,7,8}$;

(iv) $G_2 \cong E_6^{2,4} \cong E_7^{1,3} \cong E_8^{7,8}$.

Note: This is **not** obtained with technique of "diagram folding".

Note: For B_l and C_l we have listed just one of many ways these are quotients of D_N for an appropriate N, while for F_4 and G_2, we have listed all ways in which they are quotients of E_l.

Classification of GRS of rank 2

Theorem (Dimitrov-F. 2023)

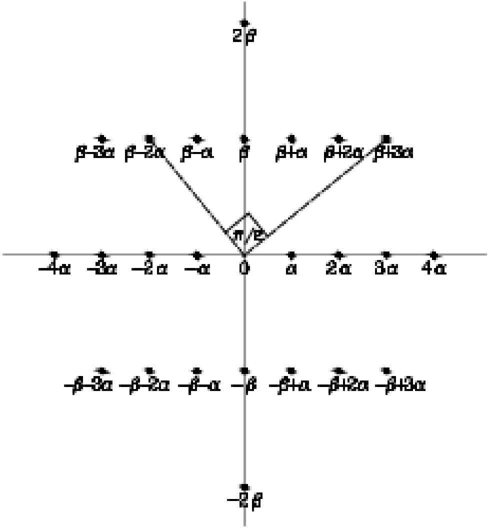
Up to equivalence, there are 16 irreducible GRSs of rank 2. Of these:

- *1 admits quadruple roots,*
- *4 admit a triple roots,*
- *8 admit double roots, and*
- *3 do not admit multiples of roots (reduced root systems A_2, B_2, G_2).*

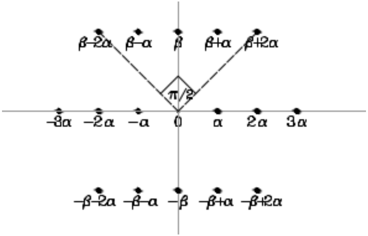
Note: This result agrees and recovers all root systems associated with rank 2 complex structures on flags G/P studied by Graev, 2015, JGP.

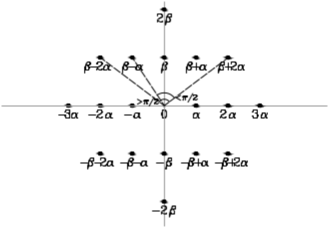
Important: We are not assuming these to be quotients of root systems of simple Lie algebras (Graev hypothesis).

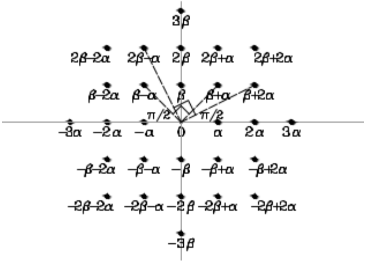
Generalized Root Systems with quadruple roots $(\alpha, 2\alpha, 3\alpha, 4\alpha)$

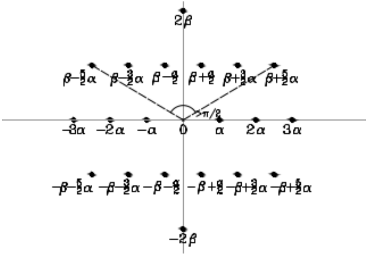


Root Systems with triple roots $(\alpha, 2\alpha, 3\alpha)$









Virtual Reflections

In ordinary root systems reflections play a key role (Weyl group).

Observation

Let R be a generalized root system.

(i) *The usual reflection about $\alpha \in R$* **may not leave** *R invariant.*

(ii) *For $\alpha \in R$, define virtual reflection $s_\alpha : R \to R$ as the bijection that reverses all α-strings.*

(iii) *s_α* **does not** *necessarily extend to an element of $GL(V)$.*

(iv) *The group generated by all (primitive) s_α* **does not seem** *to be a good analog of the Weyl group.*

(v) *Instead, we can define different groupoids in analogy with root systems of superalgebras.*

Bases and Virtual reflections

Virtual reflections "act transitively" on the set of bases of a generalized root system.
Important: there is no group here!

Theorem

Let S' and $S"$ be two bases of a generalized root system R. There exists a sequence of bases

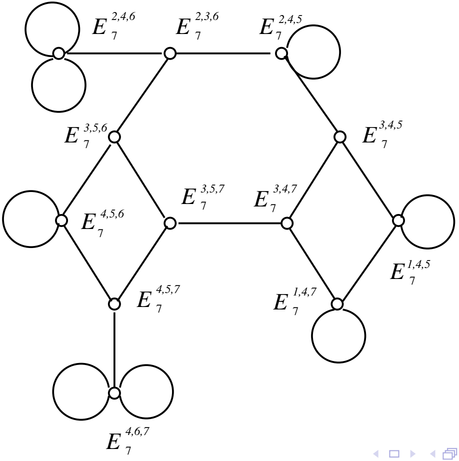
$$S' = S_0, S_1, \dots S_k = S"$$

and roots $\alpha_i \in S_i$, $i = 1 \dots k$ such that $S_{i+1} = \sigma_{\alpha_i}(S_i)$.

Virtual reflections in quotients of Lie algebras

Question. How do we establish whether two quotients of the same Lie algebra are **isomorphic**?

We have an algorithm to construct graphs whose connected components represent isomorphic quotients.



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Applications and Future work

Joint work with Bin Shu, ECNU, Shanghai
https://arxiv.org/abs/2303.18065

Quasi-reductive supergroup: algebraic supergroups $\mathbf{G}$ with reductive reduced group $G = \mathbf{G}_{ev}$.

Quasi-reductive pair (G, Y):

- G: connected reductive algebraic group,
- Y a G-superscheme of super-dimension $(0|N)$.

Quasi-reductive supergroups and GRS

Let T be the maximal torus of G, $X(T)$ character group of T,
$\epsilon : \mathcal{O}(Y) \longrightarrow k$.

Define

$$R = \Phi(T) \cup \Gamma(D)$$

Φ the root system of G.

Questions:

- When does a quasi-reductive pair correspond to quasi-reductive supergroup(s)?
- Can we classify quasi-reductive supergroups based on the properties of R?

Answers: (F.-Shu) A good class of quasi-reductive pairs defined via GRS corresponds to reductive supergroups that can be classified.