# On models of 2-nondegenerate CR hypersurfaces
# arXiv: 2310.18588

Jan Gregorovič (joint with D. Sykes)

TU Wien and Ostravska univerzita

# What are the models of CR geometries?

What are the maximally symmetric real hypesurfaces $M \subset \mathbb{C}^N$ realizing particular CR invariants?

# What are the models of CR geometries?

What are the maximally symmetric real hypesurfaces $M \subset \mathbb{C}^N$ realizing particular CR invariants?

## Example

Quadric models of Levi nondegenerate CR hypersurfaces

$$\Re(w) = \sum_{i=1}^{p} |z_i|^2 - \sum_{j=1}^{q} |z_j|^2.$$

# What are the models of CR geometries?

What are the maximally symmetric real hypesurfaces $M \subset \mathbb{C}^N$ realizing particular CR invariants?

## Example

Quadric models of Levi nondegenerate CR hypersurfaces

$$\Re(w) = \sum_{i=1}^{p} |z_i|^2 - \sum_{j=1}^{q} |z_j|^2.$$

## Example

Model of uniformly 2-nondegenerate CR hypersurfaces in $\mathbb{C}^3$: the tube over the light cone

$$\Re(w) = \frac{|z_1|^2 + \Re\left(\overline{\zeta_1} z_1^2\right)}{1 - |\zeta_1|^2}.$$

# How could the 2-nondegenerate models look like?

## Weighted homogeneity

The model is weighted homogeneous for weights

$$\mathrm{wt}(w) = 2, \mathrm{wt}(z_j) = 1, \mathrm{wt}(\zeta_\alpha) = 0,$$

where $z_1, \dots, z_s$ should correspond to rank $s$ of the Levi form.

# How could the 2-nondegenerate models look like?

## Weighted homogeneity

The model is weighted homogeneous for weights

$$\mathrm{wt}(w) = 2, \mathrm{wt}(z_j) = 1, \mathrm{wt}(\zeta_\alpha) = 0,$$

where $z_1, \ldots, z_s$ should correspond to rank $s$ of the Levi form.

## Rigidity

The model is rigid (the defining equation does not contain $\Im(w)$.)

# How could the 2-nondegenerate models look like?

## Weighted homogeneity

The model is weighted homogeneous for weights

$$\mathrm{wt}(w) = 2, \mathrm{wt}(z_j) = 1, \mathrm{wt}(\zeta_\alpha) = 0,$$

where $z_1, \dots, z_s$ should correspond to rank $s$ of the Levi form.

## Rigidity

The model is rigid (the defining equation does not contain $\Im(w)$.)

## General form of a model

$$\Re(w) = z^T H(\zeta, \bar{\zeta})\bar{z} + \Re(z^T S(\zeta, \bar{\zeta}) z),$$

where $z = (z_1, \dots, z_s)$, $H(\zeta, \bar{\zeta})$ and $S(\zeta, \bar{\zeta})$ are Hermitian matrix and symmetric matrix depending on $\zeta = (\zeta_1, \dots, \zeta_{n-s})$.

# Rank condition for the Levi form

## Levi form

Hypersurface $\Re(w) = z^T H(\zeta,\bar\zeta)\bar z + \Re(z^T S(\zeta,\bar\zeta)z)$ has Levi form

$$\mathcal{L}^1 = \begin{pmatrix} H(\zeta,\bar\zeta) & H_{\bar\zeta}(\zeta,\bar\zeta)\bar z + S_{\bar\zeta}(\zeta,\bar\zeta)z \\ z^T H_\zeta(\zeta,\bar\zeta) + \bar z^T \overline{S_{\bar\zeta}(\zeta,\bar\zeta)} & z^T H_{\zeta,\bar\zeta}(\zeta,\bar\zeta)\bar z + \Re(z^T S_{\zeta,\bar\zeta}(\zeta,\bar\zeta)z) \end{pmatrix}$$

# Rank condition for the Levi form

## Levi form

Hypersurface $\Re(w) = z^T H(\zeta,\bar{\zeta})\bar{z} + \Re(z^T S(\zeta,\bar{\zeta})z)$ has Levi form

$$\mathcal{L}^1 = \begin{pmatrix} H(\zeta,\bar{\zeta}) & H_{\bar{\zeta}}(\zeta,\bar{\zeta})\bar{z} + S_{\bar{\zeta}}(\zeta,\bar{\zeta})z \\ z^T H_{\zeta}(\zeta,\bar{\zeta}) + \bar{z}^T \overline{S_{\bar{\zeta}}(\zeta,\bar{\zeta})} & z^T H_{\zeta,\bar{\zeta}}(\zeta,\bar{\zeta})\bar{z} + \Re(z^T S_{\zeta,\bar{\zeta}}(\zeta,\bar{\zeta})z) \end{pmatrix}$$

## Rank $s$ condition

We assume that $H(\zeta,\bar{\zeta})$ is invertible. Then rank $s$ condition is equivalent to the following system of PDE's:

$$H_{\zeta_\alpha,\overline{\zeta_\beta}} = H_{\zeta_\alpha}(\zeta,\bar{\zeta})H(\zeta,\bar{\zeta})^{-1}H_{\overline{\zeta_\beta}}(\zeta,\bar{\zeta}) + S_{\overline{\zeta_\alpha}}(\zeta,\bar{\zeta})(H(\zeta,\bar{\zeta})^{-1})^T\overline{S_{\overline{\zeta_\beta}}(\zeta,\bar{\zeta})} +$$
$$+ H_{\zeta_\beta}(\zeta,\bar{\zeta})H(\zeta,\bar{\zeta})^{-1}H_{\overline{\zeta_\alpha}}(\zeta,\bar{\zeta}) + S_{\zeta_\beta}(\zeta,\bar{\zeta})(H(\zeta,\bar{\zeta})^{-1})^T\overline{S_{\zeta_\alpha}(\zeta,\bar{\zeta})}$$
$$S_{\zeta_\alpha,\overline{\zeta_\beta}} = H_{\zeta_\alpha}(\zeta,\bar{\zeta})H(\zeta,\bar{\zeta})^{-1}S_{\overline{\zeta_\beta}}(\zeta,\bar{\zeta}) + S_{\overline{\zeta_\beta}}(\zeta,\bar{\zeta})(H(\zeta,\bar{\zeta})^{-1})^T H_{\zeta_\alpha}(\zeta,\bar{\zeta})^T$$

# Admissible frame

## Admissible frame for $\Re(w) = z^T H(\zeta,\bar\zeta)\bar z + \Re(z^T S(\zeta,\bar\zeta)z)$

$$g := \frac{\partial}{\partial \Im(w)},\; f_a := \frac{\partial}{\partial z_a} - i(H_a(\zeta,\bar\zeta)\bar z + S_a(\zeta,\bar\zeta)z)\frac{\partial}{\partial \Im(w)},$$

$$e_\alpha := \frac{\partial}{\partial \zeta_\alpha} - i\left(z^T H_{\zeta_\alpha}(\zeta,\bar\zeta)\bar z + \Re(z^T S_{\zeta_\alpha}(\zeta,\bar\zeta)z)\right)\frac{\partial}{\partial \Im(w)}$$
$$- \sum_{\bar b,c}\left(z^T H_{\zeta_\alpha,\bar b}(\zeta,\bar\zeta) + \bar z^T \overline{S_{\overline{\zeta_\alpha},\bar b}(\zeta,\bar\zeta)}\right)(H(\zeta,\bar\zeta)^{-1})_{c,\bar b} f_c,$$

# Admissible frame

## Admissible frame for $\Re(w) = z^T H(\zeta,\bar\zeta)\bar z + \Re(z^T S(\zeta,\bar\zeta) z)$

$$g := \frac{\partial}{\partial \Im(w)},\ f_a := \frac{\partial}{\partial z_a} - i(H_a(\zeta,\bar\zeta)\bar z + S_a(\zeta,\bar\zeta)z)\frac{\partial}{\partial \Im(w)},$$

$$e_\alpha := \frac{\partial}{\partial \zeta_\alpha} - i\left(z^T H_{\zeta_\alpha}(\zeta,\bar\zeta)\bar z + \Re(z^T S_{\zeta_\alpha}(\zeta,\bar\zeta)z)\right)\frac{\partial}{\partial \Im(w)}$$

$$- \sum_{\bar b,c}\left(z^T H_{\zeta_\alpha,\bar b}(\zeta,\bar\zeta) + \bar z^T \overline{S_{\bar\zeta_\alpha,\bar b}(\zeta,\bar\zeta)}\right)(H(\zeta,\bar\zeta)^{-1})_{c,\bar b} f_c,$$

## Levi-Tanaka algebra

$g, f_a, \overline{f_a}$ generate the complex $2s+1$ dimensional Heisenberg Lie algebra $\mathbb{C}\mathfrak{g}_- = \mathfrak{g}_{-2,0} \oplus \mathfrak{g}_{-1,1} \oplus \mathfrak{g}_{-1,-1}$

$e_\alpha, \overline{e_\alpha}$ generate the complexification of the Levi kernel (if the rank condition is satisfied)

# Bigraded and modified symbols I)

## 2-nondegeneracy

$$\iota(e_\beta)(\overline{f_k}) := \sum_{l=1}^{s} \left(\Xi_\beta\right)_{l,k} f_l \equiv [e_\alpha, \overline{f_k}] \pmod{\overline{f_a}, e_\alpha, \overline{e_\alpha}}$$

$$\Xi_\beta = (H(\zeta,\overline{\zeta})^T)^{-1}\overline{S_{\overline{\zeta_\beta}}(\zeta,\overline{\zeta})}, \quad \iota(e_\beta)(f_k) := 0$$

$\iota(e_\beta), \iota(\overline{e_\beta})$ span subspaces $\mathfrak{g}_{0,2} \oplus \mathfrak{g}_{0,-2} \subset \mathfrak{csp}(\mathbb{C}\mathfrak{g}_{-1})$ and the CR hypersurface is 2-nondegenerate if and only if $\iota$ is injective (The image of $\iota$ depends linearly on the admissible frame).

# Bigraded and modified symbols I)

## 2-nondegeneracy

$$\iota(e_\beta)(\overline{f_k}) := \sum_{l=1}^{s} \left(\Xi_\beta\right)_{l,k} f_l \equiv [e_\alpha, \overline{f_k}] \pmod{\overline{f_a}, e_\alpha, \overline{e_\alpha}}$$

$$\Xi_\beta = (H(\zeta,\overline{\zeta})^T)^{-1}\overline{S_{\overline{\zeta_\beta}}(\zeta,\overline{\zeta})}, \quad \iota(e_\beta)(f_k) := 0$$

$\iota(e_\beta), \iota(\overline{e_\beta})$ span subspaces $\mathfrak{g}_{0,2} \oplus \mathfrak{g}_{0,-2} \subset \mathfrak{csp}(\mathbb{C}\mathfrak{g}_{-1})$ and the CR hypersurface is 2-nondegenerate if and only if $\iota$ is injective (The image of $\iota$ depends linearly on the admissible frame).

## Bigraded symbol (with involution $\sigma$)

$$\mathbb{C}\mathfrak{g}_{\leq 0} := \mathfrak{g}_{-2,0} \oplus \mathfrak{g}_{-1,-1} \oplus \mathfrak{g}_{-1,1} \oplus \mathfrak{g}_{0,-2} \oplus \mathfrak{g}_{0,0} \oplus \mathfrak{g}_{0,2}$$

$$\mathfrak{g}_{0,0} := \{v \in \mathfrak{csp}(\mathbb{C}\mathfrak{g}_{-1}) \,|\, [v,w] \subset \mathfrak{g}_{j,k} \;\forall\, w \in \mathfrak{g}_{j,k}\}$$

# Bigraded and modified symbols II)

## Modified symbol

$$[e_\beta, f_k] \equiv \sum_{l=1}^{s} (\Omega_\beta)_{l,k} f_l \pmod{e_\alpha, \overline{e_\alpha}}$$

$$\Omega_\beta = (H(\zeta,\bar{\zeta})^T)^{-1} H_{\zeta_\beta}(\zeta,\bar{\zeta})^T.$$

modifies the inclusions $\iota(e_a), \iota(\overline{e_a})$ into $\mathfrak{g}_{0,+}^{\mathrm{mod}} \oplus \mathfrak{g}_{0,-}^{\mathrm{mod}} \subset \mathfrak{csp}(\mathbb{C}\mathfrak{g}_{-1})$
$\mathfrak{g}_{\leq 0}^{\mathrm{mod}} = \mathbb{C}\mathfrak{g}_- \oplus \mathfrak{g}_{0,-}^{\mathrm{mod}} \oplus \mathfrak{g}_{0,0} \oplus \mathfrak{g}_{0,+}^{\mathrm{mod}}$... depends nonlinearly on the admissible frame.
For constant bigraded symbol $\mathbb{C}\mathfrak{g}_{\leq 0}$ is $\Omega_\beta$ well-defined modulo $\mathfrak{g}_{0,0}$
<=> choosing normal form of bigraded symbol.

# Bigraded and modified symbols II)

## Modified symbol

$$[e_\beta, f_k] \equiv \sum_{l=1}^{s} (\Omega_\beta)_{l,k} f_l \pmod{e_\alpha, \overline{e_\alpha}}$$

$$\Omega_\beta = (H(\zeta, \bar{\zeta})^T)^{-1} H_{\zeta_\beta}(\zeta, \bar{\zeta})^T.$$

modifies the inclusions $\iota(e_a), \iota(\overline{e_a})$ into $\mathfrak{g}_{0,+}^{\mathrm{mod}} \oplus \mathfrak{g}_{0,-}^{\mathrm{mod}} \subset \mathfrak{csp}(\mathbb{C}\mathfrak{g}_{-1})$
$\mathfrak{g}_{\leq 0}^{\mathrm{mod}} = \mathbb{C}\mathfrak{g}_{-} \oplus \mathfrak{g}_{0,-}^{\mathrm{mod}} \oplus \mathfrak{g}_{0,0} \oplus \mathfrak{g}_{0,+}^{\mathrm{mod}}$... depends nonlinearly on the admissible frame.
For constant bigraded symbol $\mathbb{C}\mathfrak{g}_{\leq 0}$ is $\Omega_\beta$ well-defined modulo $\mathfrak{g}_{0,0}$ <=> choosing normal form of bigraded symbol.

## Nonconstant bigraded symbol

First jet of the change of the bigraded symbol provides new CR invariant (obstruction to first order constancy). Bringing it to normal form makes modified symbol well-defined.

# Homogeneity assumptions

## Two symmetries

The weighted homogeneity and rigidity can be equivalently expressed by existence of two particular symmetries of the model.

# Homogeneity assumptions

## Two symmetries

The weighted homogeneity and rigidity can be equivalently expressed by existence of two particular symmetries of the model.

## Homogeneous models

We could assume that the models are homogeneous, but this provides restriction on which bigraded symbols can be realized. We adopt a weaker assumption.

# Homogeneity assumptions

## Two symmetries

The weighted homogeneity and rigidity can be equivalently expressed by existence of two particular symmetries of the model.

## Homogeneous models

We could assume that the models are homogeneous, but this provides restriction on which bigraded symbols can be realized. We adopt a weaker assumption.

## Transitive action on the leaf space of Levi kernel

Kernel of the Levi form is integrable distribution ... there is local leaf space and the CR symmetries descend on it. We assume that symmetries act locally transitively on the leaf space. This is action of $2s+1$ dimensional Heisenberg Lie group for the models.

- No restriction on bigraded symbols, but modified symbols are still restricted (not all obstruction tensors can be realized).

# The main theorems

## Theorem

*Suppose,* $\mathbf{H}$ *is nondegenerate Hermitian matrix and* $S^{0,2}$ *is symmetric matrix of holomorphic functions of* $\zeta_1, \dots, \zeta_{n-s}$ *that vanish at* $0$*, but with* $S^{0,2}_{\zeta_1}, \dots, S^{0,2}_{\zeta_{n-s}}$ *linearly independent at* $0$*. Then* $\Re(w) = z^T H(\zeta, \bar{\zeta})\bar{z} + \Re(z^T S(\zeta, \bar{\zeta})z)$ *with*

$$H(\zeta, \bar{\zeta}) = \frac{1}{2}(\mathbf{H}(\mathrm{Id} - \overline{S^{0,2}}\mathbf{H}^T S^{0,2}\mathbf{H})^{-1} + (\mathrm{Id} - \mathbf{H}\overline{S^{0,2}}\mathbf{H}^T S^{0,2})^{-1}\mathbf{H})$$

$$S(\zeta, \bar{\zeta}) = \mathbf{H}(\mathrm{Id} - \overline{S^{0,2}}\mathbf{H}^T S^{0,2}\mathbf{H})^{-1}\overline{S^{0,2}}\mathbf{H}^T$$

*is real analytic uniformly* $2$*-nondegenerate CR hypersurface with transitive action on the leaf space of the Levi kernel.*

# The main theorems

## Theorem

*Suppose,* $\mathbf{H}$ *is nondegenerate Hermitian matrix and* $S^{0,2}$ *is symmetric matrix of holomorphic functions of* $\zeta_1, \ldots, \zeta_{n-s}$ *that vanish at* 0*, but with* $S^{0,2}_{\zeta_1}, \ldots, S^{0,2}_{\zeta_{n-s}}$ *linearly independent at* 0*. Then* $\Re(w) = z^T H(\zeta, \bar{\zeta})\bar{z} + \Re(z^T S(\zeta, \bar{\zeta}) z)$ *with*

$$H(\zeta, \bar{\zeta}) = \frac{1}{2}(\mathbf{H}(\mathrm{Id} - \overline{S^{0,2}}\mathbf{H}^T S^{0,2}\mathbf{H})^{-1} + (\mathrm{Id} - \mathbf{H}\overline{S^{0,2}}\mathbf{H}^T S^{0,2})^{-1}\mathbf{H})$$

$$S(\zeta, \bar{\zeta}) = \mathbf{H}(\mathrm{Id} - \overline{S^{0,2}}\mathbf{H}^T S^{0,2}\mathbf{H})^{-1}\overline{S^{0,2}}\mathbf{H}^T$$

*is real analytic uniformly* 2*-nondegenerate CR hypersurface with transitive action on the leaf space of the Levi kernel.*

## Theorem

*Any smooth uniformly* 2*-nondegenerate CR hypersurface given by* $\Re(w) = z^T H(\zeta, \bar{\zeta})\bar{z} + \Re(z^T S(\zeta, \bar{\zeta}) z)$ *with transitive action on leaves of the Levi kernel is locally equivalent to the above.*

# Classification of realizable modified symbols in $\mathbb{C}^4$

| **H** | $S^{0,2}_{\zeta_1}(0)$ | $\mathfrak{g}'_{0,0}$ | realizable modification $\Omega_1$ |
|---|---|---|---|
| $\begin{pmatrix}1 & 0\\ 0 & \epsilon\end{pmatrix}$ | $\begin{pmatrix}1 & 0\\ 0 & \epsilon\lambda\end{pmatrix}$ for $\lambda > 1$ | $\left\{\begin{pmatrix}a & 0\\ 0 & a\end{pmatrix}\right\}$ | $\begin{pmatrix}0 & -\tau\epsilon\lambda\\ \tau & 0\end{pmatrix}$ |
| $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ | $\begin{pmatrix}e^{i\theta} & 0\\ 0 & 1\end{pmatrix}$ for $0 < \theta < \pi$ | $\left\{\begin{pmatrix}a & 0\\ 0 & a\end{pmatrix}\right\}$ | $\begin{pmatrix}0 & -\tau e^{-i\theta}\\ \tau & 0\end{pmatrix}$ |
| $\begin{pmatrix}1 & 0\\ 0 & \epsilon\end{pmatrix}$ | $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ | $\left\{\begin{pmatrix}a & -b\\ b & a\end{pmatrix}\right\}$ | $\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$ |
| $\begin{pmatrix}1 & 0\\ 0 & \epsilon\end{pmatrix}$ | $\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$ | $\left\{\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}\right\}$ | $\begin{pmatrix}0 & 0\\ \tau & 0\end{pmatrix}$ |
| $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ | $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ | $\left\{\begin{pmatrix}a & b\\ b & a\end{pmatrix}\right\}$ | $\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$ |
| $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ | $\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}$ | $\left\{\begin{pmatrix}a & 0\\ 0 & a\end{pmatrix}\right\}$ | $\begin{pmatrix}0 & 0\\ \tau & 2\tau\end{pmatrix}$ |
| $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ | $\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$ | $\left\{\begin{pmatrix}a & 0\\ b & c\end{pmatrix}\right\}$ | $\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$ |

$\epsilon = \pm 1$, $\lambda, \theta$ are real, $\tau \geq 0$

# The models in $\mathbb{C}^4$ I)

$$S^{0,2} = \ln(\exp((S^{0,2}_{\zeta_1}(0) + \Omega_1)\zeta_1)\exp(-\Omega_1\zeta_1)), S^{0,2}(\zeta_1) = \begin{pmatrix} f_1(\zeta_1) & f_2(\zeta_1) \\ f_2(\zeta_1) & f_3(\zeta_1) \end{pmatrix}$$

For row 1, for $\tau > 0$ (after reparametrization),

$$f_1(\zeta_1) = \epsilon \frac{\left(\lambda^2+\epsilon\right)\zeta_1 - \left(\lambda^2-\epsilon\right)\cos(\zeta_1)\sin(\zeta_1)}{2\tau\sqrt{\lambda}},$$

$$f_2(\zeta_1) = \frac{\sin(\zeta_1)^2\left(\lambda^2-\epsilon\right)}{2\tau\lambda}$$

$$f_3(\zeta_1) = \epsilon \frac{\left(\lambda^2+\epsilon\right)\zeta_1 + \left(\lambda^2-\epsilon\right)\cos(\zeta_1)\sin(\zeta_1)}{2\tau\sqrt{\lambda^3}}.$$

For $\tau = 0$,

$$f_1(\zeta_1) = \zeta_1, \quad f_2(\zeta_1) = 0 \quad \text{and} \quad f_3(\zeta_1) = \epsilon\lambda\zeta_1.$$

Always have non-constant bigraded symbols.

# The models in $\mathbb{C}^4$ II)

Row 2, for $\tau > 0$ (after reparametrization),

$$f_1(\zeta_1) = -\frac{ie^{i\frac{\theta}{2}}}{\tau}\left(i\cos(\theta)\zeta_1 - \sin(\theta)\sin(\zeta_1)\cos(\zeta_1)\right),$$

$$f_2(\zeta_1) = \frac{ie^{i\theta}}{\tau}\sin(\theta)\sin(\zeta_1)^2$$

$$f_3(\zeta_1) = -\frac{ie^{i\frac{3\theta}{2}}}{\tau}\left(i\cos(\theta)\zeta_1 + \sin(\theta)\sin(\zeta_1)\cos(\zeta_1)\right).$$

For $\tau = 0$,

$$f_1(\zeta_1) = \zeta_1 e^{i\theta}, \quad f_2(\zeta_1) = 0, \quad \text{and} \quad f_3(\zeta_1) = \zeta_1.$$

Bigraded symbol is not constant if either $\theta \neq \frac{\pi}{2}$ or $\tau \neq \frac{1}{\sqrt{2}}$. For $\theta = \frac{\pi}{2}$ and $\tau = \frac{1}{\sqrt{2}}$ homogeneous model (has better parametrization).

Row 3, homogeneous model

$$\Re(w) = \frac{|z_1|^2 + \epsilon|z_2|^2 + \Re(\zeta_1\overline{z_1}^2) + \Re(\zeta_1\overline{z_2}^2)}{1 - |\zeta_1|^2}$$

Row 4,

$$f_1 = \zeta_1, \quad f_2 = \frac{\zeta_1^2\tau}{2}, \quad \text{and} \quad f_3 = \frac{\zeta_1^3\tau^2}{3}.$$

Have always constant bigraded symbol. When $\tau = 0$, or ($\tau = \frac{\sqrt{3}}{2}$ and $\epsilon = -1$), homogeneous models. Otherwise, nonconstant modified symbol.

Row 5, homogeneous model $\Re(w) = \frac{z_1\overline{z_2}+\overline{z_1}z_2+\Re(\zeta_1\overline{z_1}^2)-\Re(\zeta_1\overline{z_2}^2)}{1+|\zeta_1|^2}$.

Row 6, for $\tau > 0$ (after reparametrization)

$$f_1(\zeta_1) = 0, \quad f_2(\zeta_1) = \frac{\zeta_1}{2\tau}, \quad \text{and} \quad f_3(\zeta_1) = \frac{\zeta_1(\zeta_1+1)}{2\tau}$$

For $\tau = 0$,

$$f_1(\zeta_1) = 0 \quad \text{and} \quad f_2(\zeta_1) = f_3(\zeta_1) = \zeta_1.$$

Have always constant bigraded symbol. When $\tau = \frac{1}{2}$, homogeneous model. Otherwise, nonconstant modified symbol.

Row 7, homogeneous model $\Re(w) = \Re(z_1\overline{z_2} + z_1^2\overline{\zeta_1})$.