Metric perturbations in noncommutative gravity

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Outline

Inducing noncommutativity

- Start with Lie algebra of diffeomorphisms $\mathfrak{g}$ on a spacetime manifold $\mathcal{M}$
- Embed $\mathfrak{g} \hookrightarrow (\mathcal{U}(\mathfrak{g}), m, \eta)$
- Consider Hopf algebra $(\mathcal{H}, m, \eta, \Delta, \epsilon, S)$
- Deform it using 2-cocycle (twist) $\mathcal{F} \in \mathcal{H} \otimes \mathcal{H}$
- $\mathcal{H} \to \mathcal{H}^{\mathcal{F}}$ and $\mathcal{R} \to \mathcal{R}^{\mathcal{F}}$ (might become nontrivial)
- $(C^\infty(\mathcal{M}), \cdot) \to (C^\infty_\star(\mathcal{M}), \star)$
- By imposing left/right $\mathcal{H}^{\mathcal{F}}$ module algebra property in various structures (Lie derivative, vector fields, forms, connection), one can use similar prescription for their deformation

Primitive coproduct in $\mathcal{H}$ is

$$\Delta(h) = 1 \otimes h + h \otimes 1.$$

Deformations turns $(\mathcal{H}, m, 1, \Delta, \epsilon, S)$ into $(\mathcal{H}^{\mathcal{F}}, m, 1, \Delta^{\mathcal{F}}, \epsilon^{\mathcal{F}}, S^{\mathcal{F}})$, where coproduct $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ deforms into $\Delta^{\mathcal{F}}$ as

$$\Delta^{\mathcal{F}}(h) = \mathcal{F}\Delta(h)\mathcal{F}^{-1}.$$

Module algebra properties for $\mathcal{H}$ and $\mathcal{H}^{\mathcal{F}}$ are

$$h \rhd (f \cdot g) = \cdot\Big(\Delta(h)(\rhd \otimes \rhd)(f \otimes g)\Big), \quad h \in \mathcal{H}, \quad f, g \in C^{\infty}(\mathcal{M}),$$

$$h \rhd (f \star g) = \star\Big(\Delta^{\mathcal{F}}(h)(\rhd \otimes \rhd)(f \otimes g)\Big), \quad h \in \mathcal{H}^{\mathcal{F}}, \quad f, g \in C^{\infty}_{\star}(\mathcal{M}).$$

Here the $\star$-product is

$$f \star g = \cdot\Big(\mathcal{F}^{-1}(\rhd \otimes \rhd)(f \otimes g)\Big)$$

Now consinder 2-dimensional abelian subalgebra of $\mathfrak{g}$ with $[K, X] = 0$. For $\mathcal{F} = \exp \frac{ia}{2}(K \otimes X - X \otimes K)$, $\star$-product covariant with respect to $\mathcal{H}^{\mathcal{F}}$ is

$$
\begin{aligned}
f \star g &= f \ \exp \frac{ia}{2}\left(\overleftarrow{\mathcal{L}}_K \overrightarrow{\mathcal{L}}_X - \overleftarrow{\mathcal{L}}_X \overrightarrow{\mathcal{L}}_K\right) g \\
&= f\, g + \frac{ia}{2}(\mathcal{L}_K f\, \mathcal{L}_X g - \mathcal{L}_X f\, \mathcal{L}_K g) + O(a^2),
\end{aligned}
$$

where f and g are smooth functions of the spacetime manifold.

In spherical coordinates for $X = \partial_r$ and $K = \alpha\partial_t + \beta\partial_\varphi$ we have

$$
\begin{aligned}
[t, r]_\star &= ia\alpha, \\
[\varphi, r]_\star &= ia\beta.
\end{aligned}
$$

Noncommutative differential geometry

$\star$-tensors are multilinear with respect to the $\star$-product, e.g.

$$T(f \star \partial_\mu, \partial_\nu) = f \star T(\partial_\mu, \partial_\nu).$$

$\star$-inverse of the metric satisfies

$$g_{\mu\alpha} \star g_\star^{\alpha\nu} = g_\star^{\nu\alpha} \star g_{\alpha\mu} = \delta_\mu^\nu.$$

Christoffel symbols, Riemann and Ricci tensor are

$$\begin{aligned}
\hat{\Gamma}^\mu_{\nu\rho} &= \frac{1}{2} g_\star^{\mu\alpha} \star (\partial_\nu g_{\rho\alpha} + \partial_\rho g_{\nu\alpha} - \partial_\alpha g_{\nu\rho}), \\
\hat{R}_{\mu\nu\rho}{}^\sigma &= \partial_\mu \hat{\Gamma}^\sigma_{\nu\rho} - \partial_\nu \hat{\Gamma}^\sigma_{\mu\rho} + \hat{\Gamma}^\beta_{\nu\rho} \star \hat{\Gamma}^\sigma_{\mu\beta} - \hat{\Gamma}^\beta_{\mu\rho} \star \hat{\Gamma}^\sigma_{\nu\beta}, \\
\hat{R}_{\nu\rho} &= \hat{R}_{\mu\nu\rho}{}^\mu.
\end{aligned}$$

Proposed vacuum Einstein equation is

$$\hat{R}_{(\mu\nu)} = 0.$$

These formulas are correct, given that fields X, K which generate the twist, commute with the basis vector fields.

Otherwise, they get more complicated, with frequent occurence of the $\mathcal{R}$-matrix and $\star$-pairing between vector fields and 1-forms.

In this more general notation, it is readily seen that symmetrization in the proposed Einstein equation is in fact $\mathcal{R}$-symmetrization.

Perturbations of the Schwarzschild metric

To study perturbations of the Schwarzschild spacetime we split the metric into the background part $\mathring{g}_{\mu\nu}$ and perturbation $h_{\mu\nu}$,

$$g_{\mu\nu} = \mathring{g}_{\mu\nu} + h_{\mu\nu}.$$

Due to spherical symmetry of the Schwarzschild background, decomposing the perturbation $h_{\mu\nu}$ into tensor spherical harmonics allows for independent consideration of axial (parity -1) and polar (parity +1) modes.

Generic component of the perturbation mode (ℓ, m, ω) is

$$h_{\mu\nu} = f(r,\theta)_{\mu\nu} e^{-i\omega t} e^{im\varphi}.$$

$\star$-inverse of the metric is

$$g_\star^{\mu\nu} = \mathring{g}^{\mu\nu} - \mathring{g}^{\mu\alpha} \star h_{\alpha\beta} \star \mathring{g}^{\beta\nu}.$$

We can now calculate the Christoffel symbols, Riemann and Ricci tensor up to the first order in $h_{\mu\nu}$ and noncommutativity parameter a.

Since $h_{\mu\nu} \propto e^{-i\omega t} e^{im\varphi}$, for $K = \alpha\partial_t + \beta\partial_\varphi$ we have

$$\mathcal{L}_K h_{\mu\nu} = i\lambda h_{\mu\nu},$$

where $\lambda = -\alpha\omega + \beta m$ is the eigenvalue of the Killing field's action on the perturbation mode.

Proposed Einstein equation $\hat{R}_{(\mu\nu)} = 0$ separates the angular and radial parts in all components. In the axial case, we initially get 3 distinct radial differential equations.

As in the classical spacetime, the system reduces to a single Schrödinger equation of the form

$$\frac{d^2}{d\hat{r}_*^2}\psi + (\omega^2 - V(r))\psi = 0,$$

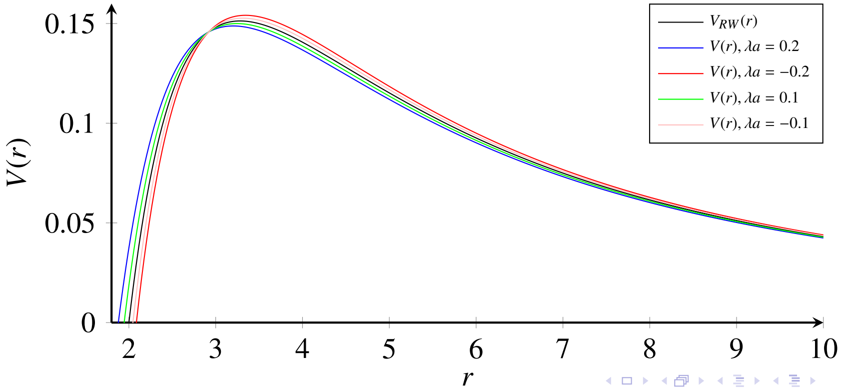
where

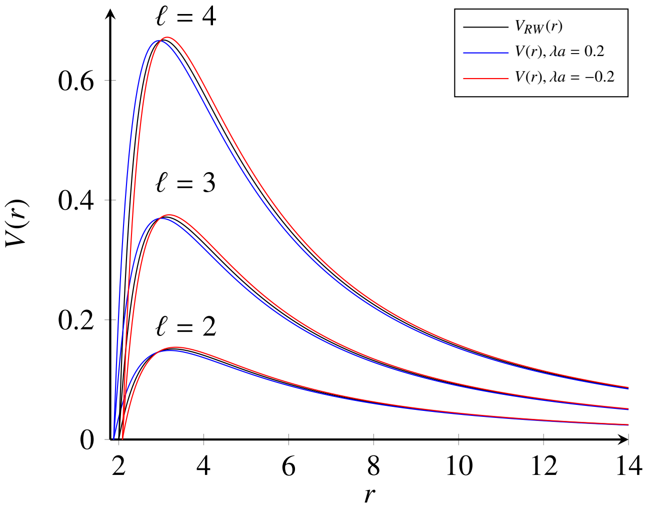
$$\hat{r}_* = \underbrace{r + R\log\frac{r-R}{R}}_{\text{usual tortoise coordinate}} + \underbrace{\frac{\lambda a}{2}\frac{R}{r-R}}_{\text{NC correction}}$$

is a $\star$-tortoise coordinate.

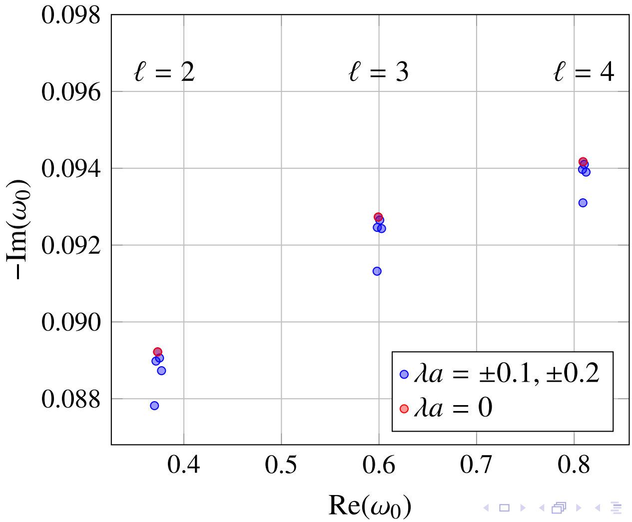
Effective potential is

$$V(r) = \underbrace{\frac{(r-R)(\ell(\ell+1)r - 3R)}{r^4}}_{\text{Regge-Wheeler potential}} + \underbrace{\frac{\lambda a}{2}\frac{r(3R-2r)(\ell(\ell+1) + R(5r-8R))}{r^5}}_{\text{NC correction}}.$$




$\ell = 4$
$\ell = 3$
$\ell = 2$
$V_{RW}(r)$
$V(r), \lambda a = 0.2$
$V(r), \lambda a = -0.2$
0.6
0.4
0.2
0
$V(r)$
2
4
6
8
10
12
14
r

Noncommutative Quasinormal Mode Frequencies



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