Symplectic-Haantjes geometry of Hamiltonian integrable systems in magnetic fields

Ondřej Kubů

Czech Technical University in Prague

Collaboration: P. Tempesta, D. Reyes Nozaleda, and G. Tondo

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Motivation

Despite significant effort in the new millennium, integrable systems immersed in magnetic fields are not yet well understood, mainly because the 1:1 correspondence with separation of variables in the Hamilton-Jacobi (HJ) equation is broken. Moreover, the standard theory of separation of variables on the configuration space requires fixing of the gauge in a somewhat ad hoc manner.

In this talk we study the geometry of physically relevant integrable systems with magnetic fields using the recently proposed formalism based on symplectic-Haantjes ($\omega\mathscr{H}$) manifolds. In addition to the theoretical insight, the geometry naturally determines the gauge needed for separation thanks to its definition on the full phase space. We also obtain a new family of integrable systems on curved manifolds by generalizing the obtained geometries.

Contents

Nijenhuis and Haantjes operators

Let M be a differentiable manifold and $\boldsymbol{K} : TM \to TM$ be a (1,1) tensor field.

Definition (Nijenhuis 1951)

The **Nijenhuis torsion** of $\boldsymbol{K}$ is the vector-valued 2-form defined by

$$\mathcal{T}_{\mathbf{K}}(X,Y) := \boldsymbol{K}^2[X,Y] + [\boldsymbol{K}X, \boldsymbol{K}Y] - \boldsymbol{K}\Big([X, \boldsymbol{K}Y] + [\boldsymbol{K}X, Y]\Big),$$

where $X, Y \in TM$ and $[\ ,\ ]$ denotes the commutator of two vector fields.

Definition (Haantjes, 1955)

The **Haantjes torsion** of $\boldsymbol{K}$ is the vector-valued 2-form defined by

$$\mathcal{H}_{\mathbf{K}}(X,Y) := \mathbf{K}^2\mathcal{T}_{\mathbf{K}}(X,Y) + \mathcal{T}_{\mathbf{K}}(\mathbf{K}X, \mathbf{K}Y) - \mathbf{K}\Big(\mathcal{T}_{\mathbf{K}}(X, \mathbf{K}Y) + \mathcal{T}_{\mathbf{K}}(\mathbf{K}X, Y)\Big).$$

Definition

A **Nijenhuis/Haantjes operator** is an operator whose Nijenhuis/Haantjes torsion identically vanishes.

Examples of Nijenhuis and Haantjes operators

- **Every operator** on a 2-dimensional manifold is a **Haantjes operator**. In general, it is not a Nijenhuis one.
- Let M be an n-dimensional manifold and $(x_1, \ldots, x_n)$ a local chart on M. Let us consider the **simple** operators

$$\mathbf{K}_1 := \begin{pmatrix} \lambda_1(x_1) & 0 & \cdots & 0 \\ 0 & \lambda_2(x_2) & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n(x_n) \end{pmatrix},$$

$$\mathbf{K}_2 := \begin{pmatrix} \lambda_1(x_1, \ldots, x_n) & 0 & \cdots & 0 \\ 0 & \lambda_2(x_1, \ldots, x_n) & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n(x_1, \ldots, x_n) \end{pmatrix}$$

. $\mathbf{K}_1$ is a **Nijenhuis** operator, $\mathbf{K}_2$ is a **Haantjes** operator.

Haantjes algebras: definition

A **Haantjes algebra** of rank m is a pair $(M, \mathscr{H})$ that satisfies the following conditions

- M is a differentiable manifold of dimension n;
- $\mathscr{H}$ is a set of Haantjes operators $\boldsymbol{K} : TM \to TM$, that generate
  i) a **free module** of rank m over the ring of smooth functions on M

  $$\mathcal{H}_{\left(f(\mathbf{x})\mathbf{K}_1 + g(\mathbf{x})\mathbf{K}_2\right)}(X, Y) = \mathbf{0}, \quad \forall \boldsymbol{K}_1, \boldsymbol{K}_2 \in \mathscr{H}, \quad \forall X, Y \in TM,$$

  where $f(\boldsymbol{x}), g(\boldsymbol{x})$ are arbitrary smooth functions on M. **Rank** m means that the module is generated by a basis with m elements.
  ii) a **ring** w.r.t. the composition operation

  $$\mathcal{H}_{\left(\mathbf{K}_1\,\mathbf{K}_2\right)}(X, Y) = \mathcal{H}_{\left(\mathbf{K}_2\,\mathbf{K}_1\right)}(X, Y) = \mathbf{0}, \forall \boldsymbol{K}_1, \boldsymbol{K}_2 \in \mathscr{H}, \forall X, Y \in TM.$$

- In addition, if $\boldsymbol{K}_1\,\boldsymbol{K}_2 = \boldsymbol{K}_2\,\boldsymbol{K}_1\ \ \forall \boldsymbol{K}_1, \boldsymbol{K}_2 \in \mathscr{H}$ the algebra $\mathscr{H}$ will be called an **Abelian Haantjes algebra**.

Basic examples

A natural Haantjes algebra is realized, in a local chart $\{U, \boldsymbol{x} = (x^1, \ldots, x^n)\}$, by any set of **diagonal operators** of the form

$$\mathbf{K} = \sum_{k=1}^{n} l_k(\boldsymbol{x}) \frac{\partial}{\partial x^k} \otimes \mathrm{d}x^k, \tag{1}$$

where the smooth functions $l_k(\boldsymbol{x})$ play the role of eigenvalue fields of $\mathbf{K}$. Such operators generate an algebraic structure that will be said to be a *diagonal* Haantjes algebra. More generally, we shall say that a Haantjes algebra is **semisimple** if each operator $\mathbf{K} \in \mathscr{H}$ is semisimple.

$\omega\mathscr{H}$ manifolds

P. Tempesta & G. Tondo, *Ann. Mat. Pura Appl.* (2021)

Definition

A $\omega\mathscr{H}$ **(or symplectic–Haantjes) manifold** of class m is a triple $(\mathbf{M}, \omega, \mathscr{H})$ which satisfies the following properties:

i) (M, ω) is a symplectic manifold of dimension $2\mathrm{n}$;

ii) ω is a symplectic form in M;

ii) $\mathscr{H}$ is an **Abelian Haantjes algebra** of rank m;

iii) $(\omega, \mathscr{H})$ are algebraically compatible, that is

$$\omega(X, \boldsymbol{K}Y) = \omega(\boldsymbol{K}X, Y) \qquad \forall \boldsymbol{K} \in \mathscr{H}.$$

For our purposes, the manifold M will be the phase space $M = T^*Q$.

Diagonalization of Haantjes algebras

P. Tempesta & G. Tondo. J. Geom. Phys. (2021).

Theorem (Simultaneous diagonalization)

Let $(M, \mathscr{H})$ *be an* **Abelian** *Haantjes algebra.*
i) If $\mathscr{H}$ *is a* **semisimple Haantjes algebra**, *then there exist local coordinate charts* $\{(x_1, \ldots, x_n)\}$ *where all* $\boldsymbol{K} \in \mathscr{H}$ *can be simultaneously diagonalized.*

Conversely, let $\{\boldsymbol{K}_1, \ldots, \boldsymbol{K}_m\}$ *be a commuting set of* m $C^\infty(M)$*–linearly independent operators. If they share a set of local coordinate charts in which they take a* **diagonal form**, *then they generate a semisimple Abelian Haantjes algebra (of rank not smaller than* m*).*

ii) More generally, if the Abelian Haantjes algebra $(M, \mathscr{H})$ *is* **non-semisimple**, *then there exist local coordinate charts* $\{(x_1, \ldots, x_n)\}$ *where all* $\boldsymbol{K} \in \mathscr{H}$ *take a* **block-diagonal form**.

Haantjes chains, Darboux-Haantjes coordinates

Definition

Let $(M, \mathscr{H})$ be a Haantjes algebra of rank m. A smooth function H generates a **Haantjes chain** of 1-forms of length m if there exist a distinguished basis $\{\tilde{\boldsymbol{K}}_1, \ldots, \tilde{\boldsymbol{K}}_m\}$ of $\mathscr{H}$ such that $\tilde{\boldsymbol{K}}_\alpha^T \, \mathrm{d}H =: \mathrm{d}H_\alpha, \ \alpha = 1, \ldots, m$ are locally exact.

Theorem (D. Reyes, P. Tempesta & G. Tondo, 2022)

There is a 1:1 correspondence between complete **Louville integrability**, **separation of variables** *for the* **Hamilton–Jacobi equations** *associated with the integrals* $\{H_1, H_2, \ldots, H_n\}$, *and an* $\omega\mathcal{H}$ **manifold** *of class n with a Haantjes chain generated by* H_1 *with operators*

$$\boldsymbol{K}_\alpha = \sum_{i=1}^{n} \frac{\frac{\partial H_\alpha}{\partial p_i}}{\frac{\partial H_1}{\partial p_i}} \left(\frac{\partial}{\partial q^i} \otimes \mathrm{d}q^i + \frac{\partial}{\partial p_i} \otimes \mathrm{d}p_i \right) \qquad \alpha = 1, \ldots, n.$$

The **canonical coordinates** $(\boldsymbol{q}, \boldsymbol{p})$ *where* $\boldsymbol{K}_\alpha$ *take this diagonal form are called* **Darboux-Haantjes** *(DH) coordinates.*

Finding Darboux-Haantjes coordinates

P. Tempesta & G. Tondo. J. Geom. Phys. (2021).

1. Given an algebra $\mathscr{H}$, determine the nontrivial **joint eigen-distributions**

$$\mathcal{V}_a(\boldsymbol{x}) := \bigoplus_{i_1,\ldots,i_m}^{s_1,\ldots,s_m} \mathcal{D}_{i_1}^{(1)}(\boldsymbol{x}) \bigcap \ldots \bigcap \mathcal{D}_{i_m}^{(m)}(\boldsymbol{x}), \qquad a = 1, \ldots, v \leq n$$

   of the (generalized) eigen-distributions $\mathcal{D}_{i_m}^{(j)}(\boldsymbol{x}) := ker\big(\boldsymbol{K}^{(j)} - l_{i_m}\boldsymbol{I}\big)^{\rho_i}(\boldsymbol{x})$ corresponding to operators $\boldsymbol{K}^{(j)}$.
2. For each of the corresponding **annihilators** $(\bigoplus_{a=1,\ldots,\hat{i},\ldots,n} \mathcal{V}_a)^\circ$, construct a basis of closed one-forms.
3. Find the **characteristic and canonical coordinates** by integrating the one forms.

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Application to 3D constant magnetic field

Let us consider an electron in the **constant magnetic field**

$$\vec{B}(\vec{x}) = b\ \vec{e}_z$$

with no electric potential on 3D Euclidean space. The corresponding Hamiltonian reads

$$H = \frac{1}{2}\left(\vec{p} + \vec{A}(\vec{x})\right)^2 \equiv \frac{1}{2}\vec{\Pi}(\vec{x})^2.$$

$\vec{A}(\vec{x})$ is the vector potential generating the magnetic field $\vec{B} = \nabla \times \vec{A}$. The superintegrable system admits 4 independent **first-degree integrals**

$$H_1 = \Pi_x + by,\ H_2 = \Pi_y - bx,\ H_3 = \Pi_z,\ H_4 = (x\,\Pi_y - y\,\Pi_x) - \tfrac{b}{2}(x^2 + y^2)$$

and the fifth, nonpolynomial one (A. Marchesiello, L. Šnobl & P. Winternitz, 2015).

$$H_5 = -\Pi_x \cos\left(\frac{bz}{\Pi_z}\right) - \Pi_y \sin\left(\frac{bz}{\Pi_z}\right). \qquad (2)$$

Cartesian $\omega\mathscr{H}$ structure I

In the following, we shall provide a set of **semisimple** Haantjes operators that solve the chain equations $\boldsymbol{K}_i^T dH = dH_i$, jointly with their spectral properties. We focus on the Cartesian $\omega\mathscr{H}$ structure $\mathscr{H}_1 = \{\boldsymbol{I}_{6\times 6}, \boldsymbol{K}_1, \boldsymbol{K}_3\}$.

$$\boldsymbol{K}_1 = \frac{1}{\Pi_x}\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & \partial_x A_y & \partial_x A_z & 1 & 0 & 0 \\ -\partial_x A_y & 0 & 0 & 0 & 0 & 0 \\ -\partial_x A_z & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\lambda_1^{(1)} = 0,\ \mathcal{D}_1 = \left\langle \partial_x A_z \frac{\partial}{\partial y} - \partial_x A_y \frac{\partial}{\partial z}, \frac{\partial}{\partial y} - \partial_x A_y \frac{\partial}{\partial p_x}, \frac{\partial}{\partial p_y}, \frac{\partial}{\partial p_z} \right\rangle,$$

$$\lambda_2^{(1)} = \frac{1}{\Pi_x},\ \mathcal{D}_2 = \left\langle \frac{\partial}{\partial x} - \partial_x A_y \frac{\partial}{\partial p_y} - \partial_x A_z \frac{\partial}{\partial p_z}, \frac{\partial}{\partial p_x} \right\rangle.$$

Cartesian $\omega\mathscr{H}$ structure II

$$\boldsymbol{K}_3 = \frac{1}{\Pi_z}\left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & -\partial_z A_x & 0 & 0 & 0 \\ 0 & 0 & -\partial_z A_y & 0 & 0 & 0 \\ \partial_z A_x & \partial_z A_y & 0 & 0 & 0 & 1 \end{array}\right],$$

$$\lambda_1^{(3)} = 0,\ \mathcal{D}_3 = \left\langle \partial_z A_y \frac{\partial}{\partial x} - \partial_z A_x \frac{\partial}{\partial y}, \frac{\partial}{\partial x} - \partial_z A_x \frac{\partial}{\partial p_z}, \frac{\partial}{\partial p_x}, \frac{\partial}{\partial p_y} \right\rangle,$$

$$\lambda_2^{(3)} = \frac{1}{\Pi_z},\ \mathcal{D}_4 = \left\langle \frac{\partial}{\partial z} - \partial_z A_x \frac{\partial}{\partial p_x} - \partial_z A_y \frac{\partial}{\partial p_y}, \frac{\partial}{\partial p_z} \right\rangle.$$

DH coordinates for the Cartesian $\omega\mathscr{H}$ structure I

We now determine the **Darboux-Haantjes coordinates** for $\mathscr{H}_1$.
Defining the distributions

$$\mathcal{V}_1 = \mathcal{D}_1 \cap \mathcal{D}_3, \quad \mathcal{V}_2 = \mathcal{D}_1 \cap \mathcal{D}_4, \quad \mathcal{V}_3 = \mathcal{D}_2 \cap \mathcal{D}_3, \quad \mathcal{V}_4 = \mathcal{D}_2 \cap \mathcal{D}_4 = \emptyset,$$

we can work out the annihilators

$$\begin{aligned}
(\mathcal{V}_1 \oplus \mathcal{V}_2)^\circ &= \langle dx, \partial_x A_y dy + \partial_z A_x dz + dp_x \rangle, \\
(\mathcal{V}_1 \oplus \mathcal{V}_3)^\circ &= \langle dz, \partial_x A_z dx + \partial_y A_z dy + dp_z \rangle, \\
(\mathcal{V}_2 \oplus \mathcal{V}_3)^\circ &= \langle dy, \partial_x A_y dx + \partial_z A_y dz + dp_y \rangle.
\end{aligned}$$

Integrating them, we find the set of Darboux-Haantjes coordinates

$$\begin{aligned}
q^1 &= x, \qquad & p_1 &= \Pi_x + b\, y, \\
q^2 &= y, \qquad & p_2 &= \Pi_y, \\
q^3 &= z, \qquad & p_3 &= \Pi_z.
\end{aligned} \tag{3}$$

DH coordinates for the Cartesian $\omega\mathscr{H}$ structure II

In this chart, the operators of the algebra $\mathscr{H}_1$ read

$$\boldsymbol{K}_1 = \frac{1}{p_1 - b\,q^2}\ \mathrm{diag}(1,0,0,1,0,0),$$
$$\boldsymbol{K}_3 = \frac{1}{p_3}\ \mathrm{diag}(0,0,1,0,0,1);$$

the corresponding Hamiltonians H, H_1, H_3 are indeed separable.

$$H = \frac{1}{2}\left[(p_1 - bq^2)^2 + p_2^2 + p_3^2\right],$$
$$H_1 = p_1, \quad H_2 = p_2 - b\,q^1, \quad H_3 = p_3,$$
$$H_4 = q^1 p_2 - q^2 p_1 + \frac{b}{2}\left((q^2)^2 - (q^1)^2\right).$$

The form of H is the one we would obtain by choosing the gauge $\vec{A} = (-bq^2, 0, 0)$ in the original Hamiltonian (2).

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Stäckel geometry

In a suitable choice of coordinates, a large class of separable systems satisfy the **Stäckel equation**

$$\begin{bmatrix} \tilde{H}_1 \\ \tilde{H}_2 \\ \tilde{H}_3 \end{bmatrix} = S^{-1} \begin{bmatrix} f_1(q^1, p_1) \\ f_2(q^2, p_2) \\ f_3(q^3, p_3) \end{bmatrix}, \qquad S = \begin{bmatrix} S_{11}(q^1) & S_{12}(q^1) & S_{13}(q^1) \\ S_{21}(q^2) & S_{22}(q^2) & S_{23}(q^2) \\ S_{31}(q^3) & S_{31}(q^3) & S_{31}(q^3) \end{bmatrix},$$

where $\tilde{H}_i$ are the commuting Hamiltonians. Note the dependence for the rows of the **Stäckel matrix** S and of the generalized **Stäckel functions** f_i (Arnold, Kozlov and Neishtadt 1997).

The classical Stäckel functions for systems without magnetic field are always quadratic in momenta $f_k := \frac{1}{2}p_k^2 + W_k(q^k)$ and the associated Haantjes operator can be projected to a **Killing tensor**,

$$\boldsymbol{K}_{j-1} := \sum_{r=1}^{n} \frac{\tilde{S}_{jr}}{\tilde{S}_{1r}} \left(\frac{\partial}{\partial q^r} \otimes \mathrm{d}q^r + \frac{\partial}{\partial p_r} \otimes \mathrm{d}p_r \right) \xrightarrow{\pi} \tilde{\boldsymbol{K}}_{j-1} := \sum_{r=1}^{n} \frac{\tilde{S}_{jr}}{\tilde{S}_{1r}} \frac{\partial}{\partial q^r} \otimes \mathrm{d}q^r.$$

Here $\tilde{S}_{jk}$ denotes the cofactor of the element S_{kj}.

Stäckel analysis for the system in a constant magnetic field

To show that for our constant magnetic field, we shall work in the coordinates (3) that diagonalize the Haantjes algebra $\mathscr{H}_1$ and use a suitable functional combination of the Hamiltonians:

$$\tilde{H}_1 = 2H - H_1^2 = -2bq^2 p_1 + b^2(q^2)^2 + p_2^2 + p_3^2,$$
$$\tilde{H}_2 = H_1 = p_1, \quad \tilde{H}_3 = H_3^2 = p_3^2. \tag{4}$$

(This does not change the underlying foliations by invariant tori.)
Then it follows that the Stäckel equation is satisfied by choosing

$$S = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2bq^2 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \tag{5}$$

and therefore $f_1(q^1, p_1) = p_1$, $f_2(q^2, p_2) = p_2^2 + b^2(q^2)^2$ and $f_3(q^3, p_3) = p_3^2$. Note that f_1 is **linear, i.e. generalized.**

New integrable system I

We obtain a **new integrable system with magnetic field on a curved background** by generalizing the Stäckel matrix and functions above. Namely, let us consider the Stäckel matrix

$$S = \begin{bmatrix} 0 & 1 & 0 \\ 1 & b\mu(q^2) & -1 \\ 0 & 0 & 1 \end{bmatrix}, \tag{6}$$

where $\mu(q^2)$ is an arbitrary function, and the Stäckel functions

$$f_1 = \mu_1(q^1)p_1, \quad f_2 = \mu_2(q^2)p_2^2 + b^2\mu_4(q^2), \quad f_3 = \mu_3(q^3)p_3^2,$$

where functions $\mu_1, \ldots, \mu_4$ are also arbitrary. This yields the following **integrable system**

$$\begin{aligned} \hat{H} &= -b\mu(q^2)\mu_1(q^1)p_1 + \mu_2(q^2)p_2^2 + \mu_3(q^3)p_3^2 + b^2\mu_4(q^2), \\ \hat{H}_2 &= \mu_1(q^1)p_1, \quad \hat{H}_3 = \mu_3(q^3)p_3^2. \end{aligned} \tag{7}$$

New integrable system - magnetic version

In order to construct the 'magnetic' version of the system (7), we first represent it in an algebraically equivalent way

$$(\hat{H} + \hat{H}_2^2, \hat{H}_2, \hat{H}_3) \tag{8}$$

and use the transformation

$$\begin{aligned} q^1 &= x, & p_1 &= \Pi_x + \frac{b\mu(y)}{2\mu_1(x)}, \\ q^2 &= y, & p_2 &= \Pi_y, \\ q^3 &= z, & p_3 &= \Pi_z. \end{aligned}$$

These coordinates are canonical if we take into account the magnetic field

$$\boldsymbol{B} = \mathrm{d}\boldsymbol{A} = \frac{b}{2}\frac{\mu'(y)}{\mu_1(x)}\mathrm{d}x \wedge \mathrm{d}y.$$

New integrable system - magnetic version

With this transformation, system (8) provides us with

$$H = \mu_1^2(x)\,\Pi_x^2 + \mu_2(y)\,\Pi_y^2 + \mu_3(z)\,\Pi_z^2 + b^2\left(\mu_4(y) - \frac{\mu(y)^2}{4}\right),$$

$$H_1 = \mu_1(x)\left(\Pi_x + \frac{b}{2}\frac{\mu(y)}{\mu_1(x)}\right), \qquad H_2 = \mu_3(z)\,\Pi_z^2,$$

i.e. a family of integrable magnetic models with a nontrivial Riemannian metric

$$g = diag(\mu_1^2(x), \mu_2(y), \mu_3(z)).$$

Summary

- We have reviewed the notions of $\omega\mathscr{H}$ **manifold** as needed for finite-dimensional Hamiltonian systems.
- The geometry enables search for **Darboux-Haantjes coordinates**, where the Haantjes algebra diagonalizes and the system is Liouville integrable through separation of variables.
- We have applied the Haantjes method to systems with vector potentials for the first time.
- We have shown on an example that for systems with magnetic fields this procedure, inherently working on the full phase space, **determines the gauge** needed for separation, in contrast to configuration space methods. We have so far failed to find new separation coordinates for magnetic systems.
- We have also used the connection to **Stäckel geometry** to generate a **new family of integrable systems** with magnetic field on curved spaces.

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