Geometric decomposition and its applications

@Srni 2024

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14.01.2024

Outline

1. Geometric decomposition
2. Applications
   - Covariant exterior derivative
   - Variational calculus
   - Topological duals
3. Summary
4. Bibliography

Part 1 - Geometric decomposition

The Poincaré lemma

Homotopy Invariance Formula (for linear homotopy)

$$Hd + dH = I^* - s^*_{x_0}, \qquad (1)$$

where

$$H\omega := \int_0^1 \mathcal{K} \lrcorner \omega_{F(t,x)} t^{k-1} dt, \quad H : \Lambda^*(U) \to \Lambda^{*-1}(U), \qquad (2)$$

for $\omega \in \Lambda^k(U)$, $\mathcal{K} := (x - x_0)^i \partial_i$, $k = deg(\omega)$, U - star-shaped, and linear homotopy $F(t,x) = x_0 + t(x - x_0)$ interpolates between Id and the constant map $s_{x_0} : x \to x_0$.

Why H is so interesting?

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$$(Hd)^2 = Hd, \quad (dH)^2 = dH. \tag{4}$$

We have projectors Hd and dH into

- Exact/closed vector space $\mathcal{E}(U) = im(dH) = ker(d)$,
- Antiexact module $\mathcal{A}(U) = im(Hd) = ker(H)$,
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We have cohomotopy operator $h : \Lambda^{k+1} \to \Lambda^k$. with $h^2 = 0$, and $h\delta h = h$, $\delta h\delta = \delta$.
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We therefore have the projectors $h\delta$ and δh onto

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In addition, on Riemannian manifolds:

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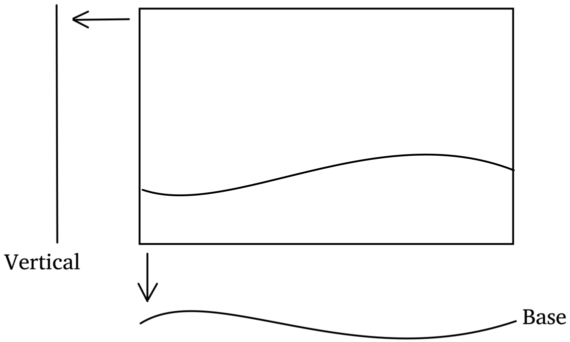
Part 2 - Applications

Covariant exterior derivative

Fibered set



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We are interested in V-valued differential forms: $\Lambda(U, V)$.

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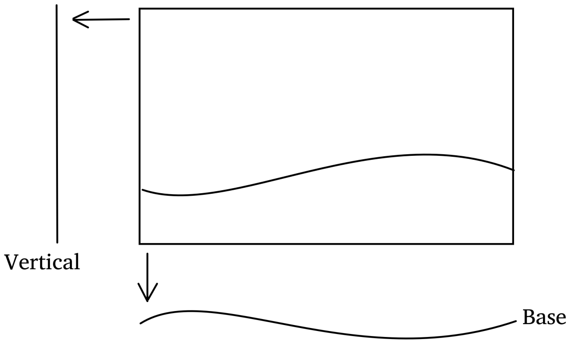

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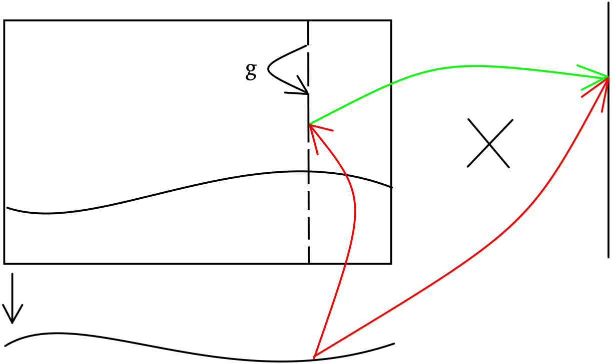
Vector-valued differential forms



Sections of associated vector bundle are in 1:1 correspondence with equivariant horizontal forms.

Covariant exterior derivative

$$d^{\nabla} := d + A \wedge \_, \tag{8}$$

for $A \in \Lambda^1(U, End(V))$ (usually with additional properties related to underlying bundle).

(Homogenous) parallel transport equation

Theorem

The unique nontrivial solution to the equation

$$d^{\nabla}\phi = 0,\ k > 0 \tag{9}$$

with the condition $dH\phi = c \in \mathcal{E}(U,V) \setminus ker(A \wedge \_)$, $c \neq 0$, is given by

$$\phi = \sum_{l=0}^{\infty} (-1)^l (H(A \wedge \_))^l c, \tag{10}$$

where c is an arbitrary form, $(H(A \wedge \_))^0 = Id$, and

$$(H(A \wedge \_))^l = \underbrace{H(A \wedge (\ldots (H(A \wedge \_) \ldots)}_{l}, \tag{11}$$

is the l-fold composition of the operator $H \circ A \wedge \_$.

Notes about convergence

$$\phi = \sum_{l=0}^{\infty} (-1)^l (H(A \wedge \_))^l c, \quad c \in \mathcal{E}(U, V) \setminus ker(A \wedge \_), \qquad (12)$$

since

$$\begin{aligned} ||H(A \wedge \omega)||_\infty &= || \int_0^1 i_{\mathcal{K}}(A \wedge \omega)(x_0 + t(x - x_0)) t^{k-1} dt||_\infty \\ &\leq \int_0^1 ||x - x_0|| ||A||_\infty ||\omega||_\infty t^{k-1} dt \\ &= ||x - x_0|| ||A||_\infty ||\omega||_\infty \tfrac{1}{k}. \end{aligned}$$

We therefore have

$$||\phi||_\infty = ||(1 - H(A \wedge \_) + H(A \wedge (H(A \wedge \_))) - \ldots)c||_\infty \leq \left(1 + ||x - x_0|| \frac{||A||_\infty}{k} + \left(||x - x_0|| \frac{||A||_\infty}{k}\right)^2 + \ldots\right) ||c||_\infty,$$

Uniformly convergent series of smooth functions can converge to a continuous function!! Analytic functions have better uniform convergence properties (Cauchy–Kovalevskaya theorem).

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Geometry-based differential equations

Using Geometric decomposition we can solve (locally) all equations containing the operators

$$D^A = d^\nabla = d + A \wedge \_, \;\; A \in \Lambda^1(U, End(V)), \tag{13}$$

$$\reflectbox{D}^X = \delta + X \lrcorner \_. \tag{14}$$

using methods that mimics those for ODEs:

$$\begin{array}{ccc} d & \leftrightarrow & \frac{d}{dx}, \\ \int & \leftrightarrow & H. \end{array} \tag{15}$$

For series solutions convergence issues matters - analyticity is the best property - it preserves under uniform convergence of functional series.

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Variational calculus

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We have $\Lambda^*(J^\infty(M,\mathbb{R}))$ with local coordinates $x^i, y, y_j, \ldots$.
$\bar{d}$ - vertical exterior derivative

Inverse problem in the calculus of variations

Having $\mathcal{E}[y] \in C^\infty(J^\infty(M,\mathbb{R}))$ find such $L \in C^\infty(J^\infty(M,\mathbb{R}))$, such that

$$\bar{d}L = \mathcal{E}[y]dy \tag{16}$$

Obvious constraint: $\bar{d}(\mathcal{E}dy) = \bar{d}\bar{d}L = 0$.

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When $\bar{d}(\mathcal{E}dy) \neq 0$, then we can apply Geometric decomposition (vertical star-shaped set) to get

$$\mathcal{E}dy = \bar{d}L \oplus \lambda, \tag{17}$$

where $\lambda \in \mathcal{A}$ is an vertical atiexact form $\lambda = \bar{H}\bar{d}(\mathcal{E}dy)$.
The constraint $\lambda = 0$ restricts DE manifold and makes $\mathcal{E}$ variational on it (hybrid variational problem).

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Topological duals

Topological duals

We restrict ourselves to the forms with compact support. The support is a star-shaped set of a Riemannian manifold (use of homotopy operator). Define

$$< c, \omega >:= \int_c \omega \tag{18}$$

where c is a chain, and ω a form.

Notable topological dual (Stokes theorem)

$$< \partial c, \omega >=< c, d\omega > . \tag{19}$$

We can dualize homotopy operator

$$< H_{\#} c, \omega >=< c, H\omega >, \tag{20}$$

and then construct Geometric decomposition on chains in order to solve chain equations, e.g., $\partial c = e$ for chain c and a fixed chain e.

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We can also dualize inner product

$$< E_X c, \omega > = < c, i_X \omega >, \tag{21}$$

where E_X is the extrusion operator along the flow of $X \in \Gamma(M)$. Moreover, the dual of $A \wedge \_$ operator is given by

$$< E^{\dagger}_{A^{\sharp}} c, \omega > = < c, A \wedge \omega >, \tag{22}$$

where $E^{\dagger}$ is adjoint of extrusion with respect to the metric on chains, and $\sharp$ is the musical isomorphism on Riemannian manifold. All the geometry-based differential equations have their (topological) duals, e.g.

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- Geometric decomposition is the essential method to solve exterior differential equations locally (e.g. in a star-shaped subset) as simple as for ODEs.
  - In the worst (typical) case scenario we can construct a functional series solution - lack of smoothness in general case.
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Acknowledgement

This research was supported by the GACR grant GA22-00091S and Masaryk University grant MUNI/A/1099/2022.

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