Higher order symplectic structures on shape spaces of space curves

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Abstract: For $c \in \mathrm{Imm}(S^1, \mathbb{R}^3)$ the 2- form

$$\Omega_c^{MW}(h,k) = \int_{S^1} \det(D_s c, h, k) ds,$$

where $ds = |c'(\theta)| d\theta$ and $D_s = \frac{1}{|c'(\theta)|}\partial_\theta$, induces the Marsden-Weinstein symplectic structure[1] on the shape space $\mathrm{Imm}(S^1, \mathbb{R}^3)/\,\mathrm{Diff}(S^1)$, corresponding to a Kähler structure. The Hamiltonian flow for the length functional is the binormal flow. In this talk I will present other symplectic structures related to this.

[1] Marsden, J., and Weinstein, A. Coadjoint orbits, vortices, and Clebsch variables for in- compressible fluids. Physica D: Nonlinear Phenomena 7, 1 (1983), 305-323.

The space

of regular space curves

$$\mathrm{Imm} = \mathrm{Imm}(S^1, \mathbb{R}^3) := \left\{ c \in C^\infty(S^1, \mathbb{R}^3) : |c'| \neq 0 \right\}.$$

is open subset in $C^\infty(S^1, \mathbb{R}^3)$, a manifold with tangent space

$$T_c\mathrm{Imm} = T_c\mathrm{Imm}(S^1, \mathbb{R}^3) = C^\infty(S^1, \mathbb{R}^3).$$

Consider the action of the reparametrization group $\mathrm{Diff} = \mathrm{Diff}(S^1)$ by composition from the right and the quotient (shape) space

$$B_i = B_i(S^1, \mathbb{R}^3) := \mathrm{Imm}(S^1, \mathbb{R}^3)/\mathrm{Diff}(S^1),$$

which is an infinite dimensional orbifold; the isotropy groups are finite cyclic groups.[2]
The vertical fibers consist exactly of all fields h that are tangent to it's foot point c, i.e., $h = a.c'$ with $a \in C^\infty(S^1)$.

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[2] V.Cervera, F.Mascaro, PM: The action of the diffeomorphism group on the space of immersions. Diff. Geom. Appl. 1 (1991), 391–401

Reparametrization invariant Riemannian metrics

on Imm are Riemannian metrics of the form:

$$G_c^L(h,k) = \int_{S^1} \langle L_c h, k\rangle |c'| d\theta = \int_{S^1} \langle h, L_c k\rangle |c'| d\theta, \qquad \text{where}$$

$$L \in \Gamma(\mathrm{End}(T\mathrm{Imm}), \quad L_c : T_c\mathrm{Imm} = C^\infty(S^1, \mathbb{R}^3) \to T_c\mathrm{Imm}\,,$$

called *inertia operator*, is an elliptic, pseudo differential operator that is equivariant under the right action of $\mathrm{Diff}(S^1)$ and also under left action of of $SO(3)$, and which is also selfadjoint with respect to the L^2-metric, i.e.,

$$L_{c\circ\varphi}(h\circ\varphi) = (L_c(h))\circ\varphi \quad \text{and} \quad \int \langle L_c h, k\rangle ds = \int \langle h, L_c k\rangle ds\,.$$

The class of Sobolev H^k-metrics, where $L = (1-(-1)^k D_s^{2k})$. For $k = 0$ we have $G_c^{\mathrm{id}}(h,k) = \int \langle h, k\rangle ds$.

The induced Liouville one form

corresponding to the metric G^L is the one-form:

$$\Theta_c^L(h) := G_c^L(c \times D_s c, h) = \int \langle c \times D_s c, L_c h \rangle ds = \int \det(c, D_s c, L_c h) ds$$

for $h \in T_c\mathrm{Imm}$, where $\times$ denotes the vector cross product on $\mathbb{R}^3$.
We have:
For any inertia operator L, that is equivariant under the right action of Diff *and the left action of $SO(3)$, the induced Liouville one-form Θ^L is invariant under the right action of* Diff *and the left action of $SO(3)$, i.e., for any $c \in \mathrm{Imm}$, $h \in T_c\mathrm{Imm}$, $\varphi \in \mathrm{Diff}$ and $O \in SO(3)$ we have*

$$\Theta^L_{O \circ c \circ \varphi)}(O \circ h \circ \varphi)) = \Theta^L_c(h).$$

The induced pre-symplectic form on Imm is

$$\Omega^L_c(h,k) := -d\Theta^L_c(h,k) = -D_{c,h}\Theta^L_c(k) + D_{c,k}\Theta^L_c(h) + \Theta^L_c([h,k])\,,$$

$$\boxed{\begin{aligned}\Omega^L_c(h,k) = \int\Big(&\langle D_s c, L_c h\times k + h\times L_c k\rangle - \langle c, D_s h\times L_c k - D_s k\times L_c h\rangle\\ &- \langle c\times D_s c, (D_{c,h}L_c)k - (D_{c,k}L_c)h\rangle\Big)ds\,.\end{aligned}}$$

Ω^L is invariant under the actions of Diff and $SO(3)$. For the invariant L^2-metric, i.e., $L = \mathrm{id}$, one obtains three times the Marsden-Weinstein pre-symplectic structure, i.e.,

$$\Omega^{\mathrm{id}}_c(h,k) = 3\int_{S^1}\langle D_s c\times h, k\rangle ds = 3\int \det(D_s c, h, k)ds =: \Omega^{\mathrm{MW}}_c(h,k).$$

Its kernel consists exactly of all vector fields along c which are tangent to c, so by reduction it induces a pre-symplectic structure on shape space $\mathrm{Imm}(S^1,\mathbb{R}^3)/\mathrm{Diff}$ which is easily seen to be non-degenerate and thus is a symplectic structure there.

Theorem

The form Ω^L *factors to a (pre)-symplectic form* $\bar{\Omega}^L$ *on* B_i *if the inertia operator* L *maps vertical tangent vectors to* $\operatorname{span}\{c, c'\}$, *i.e., if for all* $c \in \operatorname{Imm}$ *and* $a \in C^\infty(S^1)$ *we have* $L_c(a.c') = a_1 c' + a_2 c$ *for some functions* $a_i \in C^\infty(S^1)$.

Proof. The 1-form Θ^L on Imm factors to a 1-form $\bar{\Theta}^L$ on shape space B_i with $\Theta^L = \pi^* \bar{\Theta}^L$ if and only if Θ^L is invariant under under Diff and is *horizontal* in the sense that it vanishes on each vertical tangent vector $h = a.c'$ for a in $C^\infty(S^1, \mathbb{R})$.
The condition on L such that Θ^L vanishes on all vertical h is

$$\Theta^L_c(ac') = \int \langle c \times D_s c, L_c(ac') \rangle ds = 0.$$

From here it is clear that this holds if $L_c(a.c') = a_1 c' + a_2 c$ for some functions $a_i \in C^\infty(S^1)$.
In that case also its exterior derivative satisfies

$$\Omega^L = -d\Theta^L = -d\pi^* \bar{\Theta}^L = -\pi^* d\bar{\Theta}^L =: \pi^* \bar{\Omega}^L$$

for the presymplectic form $\bar{\Omega}^L = -d\bar{\Theta}^L$ on B_i.

Inertia operators that satisfy these conditions are

$$L_c(h) = F(c).h \text{ for } F \in C^\infty(\mathrm{Imm}, \mathbb{R}_{>0}), \text{ for example}$$
$$L_c(h) = \Phi(\ell_c)h, \quad \text{or } L_c(h) = (1 + A\kappa_c^2)h.$$

Sobolev metrics do not satisfy the conditions. By using a projection in the definiton one can, however, modify them to still respect the vertical bundle:

$$L_c h = \Big(\operatorname{pr}_c(1-(-1)^k D_s^{2k}) \operatorname{pr}_c + (1-\operatorname{pr}_c)(1-(-1)^k D_s^{2k})(1-\operatorname{pr}_c)\Big) h,$$

where $\operatorname{pr}_c h = \langle D_s c, h\rangle D_s c$ is the L^2-orthogonal projection to the vertical bundle; see [Bauer, Harms, 2015], where metrics of this form were studied.

Horizontal Ω^L-Hamiltonian vector fields.

If the inertia operator $L \in \Gamma(\text{End}(T\text{Imm}(S^1, \mathbb{R}^3))$ induces a weak symplectic structure on B_i, and is weakly non-degenerate in the sense that $\bar{\Omega}^L : T\text{Imm} \to T^*\text{Imm}$ is injectiven then the kernel of $\Omega^L_c : T_c\text{Imm} \to T^*_c\text{Imm}$ equals $T_c(c \circ \text{Diff})$ for all c. Thus Ω^L_c restricted to the G^L-orthogonal complement of $T_c(c \circ \text{Diff})$ is injective. Let us suppose that H is a Diff-invariant smooth function on Imm. The 2-form Ω^L on Imm is still only presymplectic, but if each dH_c lies in the image of $\Omega^L : T\text{Imm} \to T^*\text{Imm}$, then a unique smooth *horizontal Hamiltonian vector field* $X \in \mathfrak{X}(\text{Imm})$ is determined by

$$dH = i_X \Omega^L = \Omega^L(X,\ ), \qquad G^L_c(X_c, Tc.Y) = 0 \forall Y \in \mathfrak{X}(S^1)$$

which we will denote by $\text{hgrad}^{\Omega^L}(H)$. Obviously we then have

$$\text{grad}^{\bar{\Omega}^L}(\bar{H}) \circ \pi = T\pi \circ \text{hgrad}^{\Omega^L}(H), \quad \text{where } \bar{H} \circ \pi = H\,.$$

Momentum mappings

Since Θ^L is invariant for a pre-symplectic group action with fundamental vector field mapping $\zeta : \mathfrak{g} \to \mathfrak{X}(\mathrm{Imm}(S^1, \mathbb{R}^3))$, the momentum mapping is given as follows, for $Y \in \mathfrak{g}$,

$$\langle J(c), Y \rangle = \Theta^L(\zeta_Y)_c = \int \langle c \times D_s c, L_c(Y \circ c) \rangle ds,$$

where we denote the duality as $\langle \quad , \quad \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$. Namely,

$$d\Theta^L(\zeta_Y) = di_{\zeta_Y}\Theta^L = \mathcal{L}_{\zeta_Y}\Theta^L - i_{\zeta_Y} d\Theta^L = 0 - i_{\zeta_Y}\Omega^L.$$

Thus we have for $X \in \mathfrak{X}(S^1) = C^\infty(S^1)\partial_s$ and $Y \in \mathfrak{so}(3)$

$$\langle J^{\mathrm{Diff}}(c), X.D_s c \rangle = \Theta^L(X.D_s c) = \int \langle c \times D_s c, L_c(X.D_s c) \rangle ds$$

reparameterization momentum.

$$\langle J^{SO(3)}(c), Y \rangle = \Theta^L(Y \circ c) = \int \langle c \times D_s c, L_c(Y \circ c) \rangle ds$$

angular momentum.

$Y \in \mathbb{R}^3 \cong \mathfrak{so}(3) \cong L_{\mathrm{skew}}(\mathbb{R}^3, \mathbb{R}^3)$ acts via $X \mapsto 2Y \times X$.

Special case: length weighted metrics

Let $\Phi : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ and $L_c = \Phi(\ell_c)$, where $\ell_c = \int_{S^1} |c'| d\theta$.

Theorem. *Let $\Phi \in C^1(R_{>0}, R_{\geq 0})$. The induced (pre)-symplectic structure of the $G^{\Phi(\ell)}$-metric is given by:*

$$\Omega^L_c(h,k) = \int \Big(\langle D_s c, L_c h \times k + h \times L_c k \rangle - \langle c, D_s h \times L_c k + L_c h \times D_s k \rangle \\ + \langle c \times D_s c, (D_{c,k} L_c) h - (D_{c,h} L_c) k \rangle \Big) ds$$

Furthermore, we have:

1. *If $\Phi(\ell) \neq \text{Const.}\, \ell^{-3}$ then the presymplectic structure $\bar{\Omega}^{\Phi(\ell)}$ on $B_i(S^1, \mathbb{R}^3)$ is non-degenerate and thus symplectic.*
2. *In the scale invariant case, i.e., if $\Phi(\ell) = C\ell^{-3}$ with $C > 0$, then the Liouville form $\Theta^{C\ell^{-3}}$ is also invariant under the scaling action $c \mapsto a.c$ for $a \in \mathbb{R}_{>0}$ and vanishes on all scaling vectors. In that case $\Omega^{C\ell^{-3}} = \pi^*(d\bar{\Theta}^{C\ell^{-3}})$, and $\bar{\Omega}^{C\ell^{-3}} = d\bar{\Theta}^{C\ell^{-3}}$ turns out to be a symplectic structure on $\text{Imm}(S^1, \mathbb{R}^3)/\text{Diff} \times \mathbb{R}_{>0}$.*

Proposition

If $\Phi(\ell) \neq \mathrm{Const.}\, \ell^{-3}$, *For a* Diff*-invariant Hamiltonian* $H \colon \mathrm{Imm}(S^1, \mathbb{R}^3) \to \mathbb{R}^3$, *we have*

$$\mathrm{hgrad}^{\Omega^{\Phi(\ell)}}(H) = \frac{1}{3\Phi(\ell_c)} \Bigg\{ -D_s c \times \mathrm{grad}^{G^{\mathrm{id}}} H$$

$$+ \frac{\Phi'(\ell_c)}{3\Phi(\ell_c) + \Phi'(\ell_c)\ell_c} \Big[\langle D_s^2 c, D_s c \times \mathrm{grad}^{G^{\mathrm{id}}} H \rangle_{L^2_{ds}(S^1)} D_s c \times (D_s c \times c)$$

$$+ \langle D_s c \times c, D_s c \times \mathrm{grad}^{G^{\mathrm{id}}} H \rangle_{L^2_{ds}(S^1)} D_s c \times D_s^2 c \Big] \Bigg\}$$

The scale invariant case

$$\Phi(\ell) = \mathrm{Const.}\, \ell^{-3} \iff 3\Phi(\ell_c) + \Phi'(\ell_c)\ell_c = 0$$

still offers difficulties.

Example: Length

Suppose $H(c) = f \circ \ell$ for some function f. Then:

$$dH_c(k) = f'(\ell_c) \int \langle D_s k, D_s c \rangle ds = -f'(\ell_c) \int \langle D_s^2 c, k \rangle ds,$$

$$\operatorname{grad}^{G^{\text{id}}} H = -f'(\ell_c) D_s^2 c.$$

$$\operatorname{hgrad}^{\Omega^{\Phi(\ell)}} H = \frac{f'(\ell_c)}{3\Phi(\ell_c)} \left(1 + \frac{\Phi'(\ell_c)\ell_c}{3\Phi(\ell_c) + \Phi'(\ell_c)\ell_c} \right) D_s c \times D_s^2 c.$$

If $f'(\ell_c) = 0$ then c is a fixed point of the Hamitonian flow. If $f'(\ell_c) \neq 0$, then the length ℓ_c is conserved along the flow as $H = f \circ \ell$ is conserved. Note that $\operatorname{hgrad}^{\Omega^{\Phi(\ell)}} H$ is a constant multiple of the binormal equation (also known as the vortex filament equation),

$$\operatorname{hgrad}^{\Omega^{\text{MW}}} \ell = D_s c \times D_s^2 c$$

using the Marsden-Weinstein symplectic structure.

Example: Kinetic energy

as a Hamiltonian function: $E(c)\frac{1}{2}\int|c|^2 ds$, Using $\operatorname{grad}^{G^{\mathrm{id}}} E = c$, the horizontal Hamiltonian field is:

$$\operatorname{hgrad}^{\Omega^{\Phi(\ell)}} E = \frac{1}{3\Phi(\ell_c)}\left\{-D_s c \times c + \frac{\Phi'(\ell_c)}{3\Phi(\ell_c)+\Phi'(\ell_c)\ell_c}\cdot\right.$$

$$\left.\cdot\left[\Theta^{D_s^2 c}(c) D_s c \times D_s c \times c - \int |D_s c \times c|^2 ds D_s c \times D_s^2 c\right]\right.$$

$$= \frac{1}{3\Phi(\ell_c)}\left\{\operatorname{hgrad}^{\Omega^{\mathrm{MW}}} E + \frac{\Phi'(\ell_c)}{3\Phi(\ell_c)+\Phi'(\ell_c)\ell_c}\cdot\right.$$

$$\left.\cdot\left[\Theta^{D_s^2 c}(c)(\mathrm{pr}_c - 1)\operatorname{grad}^{G^{\mathrm{id}}} E - \int |D_s c \times c|^2 ds \operatorname{hgrad}^{\Omega^{\mathrm{MW}}} \ell\right]\right..$$

Special case: curvature weighted metric

The presymplectic form $\Omega^{1+\kappa^2} = -d\Theta^{1+\kappa^2}$ descends to $\bar{\Omega}^{1+\kappa^2}$ by the general Theorem above. It seems that $\bar{\Omega}^{1+\kappa^2}$ is non-degenerate, but the computations are not completely done.

Thank you for your attention