An example of curved translation principle Srní 2024

(pure algebra, except this slide)

- Eastwood, S. 97
- Souček, S. to appear soon

Diff. operators on homogeneous bundles:

$P \subset G$ ... Lie groups, $\mathfrak{p} \subset \mathfrak{g}$ ... Lie algebras, $\mathbb{E}, \mathbb{F}$ ... P-modules

$\mathcal{E} = G \times_P \mathbb{E}$, $\mathcal{F} = G \times_P \mathbb{F}$

$$D : \Gamma(\mathcal{E}) \longrightarrow \Gamma(\mathcal{F})$$

linear differential operators

i.e. $D : J^k \mathcal{E} = G \times_P J^k \mathbb{E} \longrightarrow \mathcal{F}$

$\Gamma(\mathcal{E}) = C^\infty(G, \mathbb{E})_P$, $X_1, \dots, X_\ell \in \mathfrak{g}$, $X_\ell \cdots X_1 \cdot s$

$U(\mathfrak{g}) = T\mathfrak{g} / I$, $I = \langle XY - YX - [X,Y] \rangle$

(topological) dual $(J^\infty \mathbb{E})^* = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathbb{E}^*$

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$\Gamma(\mathcal{E}) = C^\infty(G, \mathbb{E})_P$, $X_1, \dots, X_\ell \in \mathfrak{g}$, $X_\ell \cdot \ldots \cdot X_1 \cdot s$ ← derivatives = action by left invariant field

$\mathcal{U}(\mathfrak{g}) = T\mathfrak{g} / I$, $I = \langle XY - YX - [X,Y] \rangle$

(topological) dual $(J^\infty \mathbb{E})^* = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} \mathbb{E}^* = V(\mathbb{E})$ ← the induced module

G semisimple, P parabolic $\Longrightarrow$ (generalized) Verma modules

well understood structure of homomorphisms!!

THEOREM 3 (Harish–Chandra). *Two Verma modules $V(\mathbb{E})$ and $V(\mathbb{F})$ have the same central character if and only if their highest weights are related under the affine action of the Weyl group of* $\mathfrak{g}$.

$\Leftarrow \left\{ \text{action of center of } U(\mathfrak{g}) \right.$

PROPOSITION 6. *There is a canonical isomorphism of $(\mathfrak{U}(\mathfrak{g}), P)$-modules*

$$V(\mathbb{E} \otimes \mathbf{W}) = V(\mathbb{E}) \otimes \mathbf{W}^*.$$

$\mathbb{W} = \mathbb{W}_1 \oplus \ldots$

$V(\mathbb{E} \otimes \mathbb{W}) =$

$\begin{pmatrix} \vdots \end{pmatrix} + \begin{pmatrix} \vdots \end{pmatrix} + \ldots$

$= \cdots E_{ij} \cdots$

Proof. We may view $\mathfrak{U}(\mathfrak{g}) \otimes \mathbb{E}^* \otimes \mathbf{W}^*$ as a $\mathfrak{g}$-module in two different ways:

1. $X(x \otimes e \otimes w) = Xx \otimes e \otimes w;$
2. $X(x \otimes e \otimes w) = Xx \otimes e \otimes w + x \otimes e \otimes Xw.$

There is a $\mathfrak{g}$-homomorphism between these two modules characterized as the identity on elements of the form $1 \otimes e \otimes w$ for $e \in \mathbb{E}^*$ and $w \in \mathbf{W}^*$. This descends to the required isomorphism of induced modules. ▮

THEOREM 3 (Harish–Chandra). *Two Verma modules $V(\mathbb{E})$ and $V(\mathbb{F})$ have the same central character if and only if their highest weights are related under the affine action of the Weyl group of* $\mathfrak{g}$.

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PROPOSITION 6. *There is a canonical isomorphism of* $(\mathfrak{U}(\mathfrak{g}), P)$*-modules*

$$V(\mathbb{E} \otimes \mathbf{W}) = V(\mathbb{E}) \otimes \mathbf{W}^*.$$

$W = W_1 \oplus \dots$

$V(\mathbb{E} \otimes W) =$

$\begin{pmatrix} \vdots \end{pmatrix} + \begin{pmatrix} \vdots \end{pmatrix} + \dots$

$= \dots E_{ij} \dots$

PROPOSITION 8. *Suppose that $V(\mathbb{E}')$ has distinct central character from all the other $V(\mathbb{E}_{ij})$ occurring on the right-hand side of (10). Then $V(\mathbb{E}')$ canonically splits off from $V(\mathbb{E} \otimes \mathbf{W})$ as a direct summand.*

Jantzen–Zuckermann translation principle:

series of couples of adjoint functors ← "all patterns look the same"!

Proposition 9. *Suppose that $V(\mathbb{E})$ and $V(\mathbb{F})$ have the same central character. Suppose that $V(\mathbb{E}')$ and $V(\mathbb{F}')$ have the same central character. Let* **W** *be a finite-dimensional irreducible representation of G and suppose that*

- *$V(\mathbb{F}')$ occurs in the composition series for $V(\mathbb{F} \otimes \mathbf{W})$ and has distinct central character from all other factors;*
- *$V(\mathbb{E}')$ occurs in the composition series for $V(\mathbb{E} \otimes \mathbf{W})$ and has distinct central character from all other factors.*

It follows that $V(\mathbb{F})$ occurs in the composition series for $V(\mathbb{F}' \otimes \mathbf{W}^*)$ and that $V(\mathbb{E})$ occurs in the composition series for $V(\mathbb{E}' \otimes \mathbf{W}^*)$. We suppose further that

- *all other composition factors of $V(\mathbb{F}' \otimes \mathbf{W}^*)$ have central character distinct from $V(\mathbb{F})$;*
- *all other composition factors of $V(\mathbb{E}' \otimes \mathbf{W}^*)$ have central character distinct from $V(\mathbb{E})$.*

Then translation gives an isomorphism

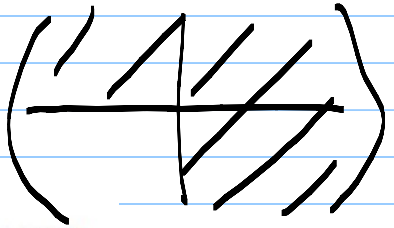
$$\mathrm{Hom}_{(\mathfrak{U}(\mathfrak{g}),\,P)}(V(\mathbb{F}), V(\mathbb{E})) \xrightarrow{\simeq} \mathrm{Hom}_{(\mathfrak{U}(\mathfrak{g}),\,P)}(V(\mathbb{F}'), V(\mathbb{E}'))$$

(whose inverse is given by translation using $\mathbf{W}^*$*).*

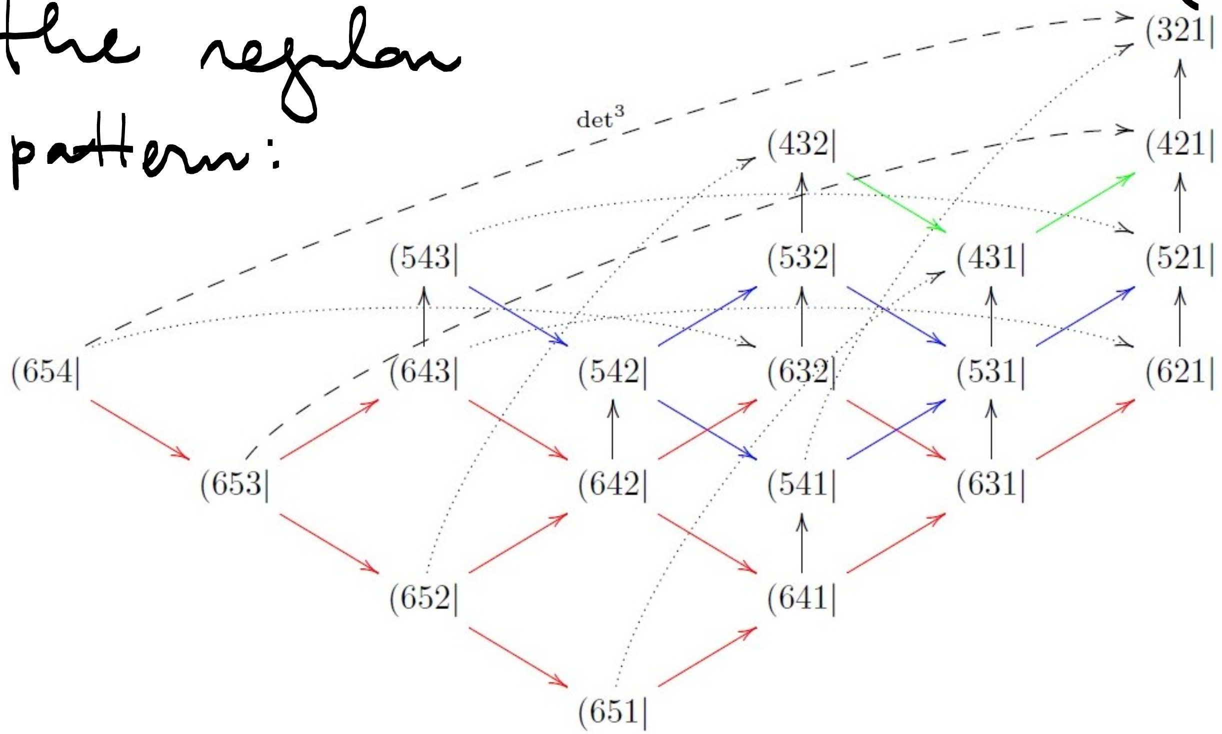
In particular:
taking the
highest weights
of $\mathbb{W}$
and $\mathbb{W}^*$
provides the
couples from
J–Z !

The example: Grassmanians $Gr(3,3)$

$G = SL(3+3, \mathbb{R})$ $P =$



the regular pattern:



$\mathfrak{g}$-reps: $(a\,b\,c\,d\,e\,f)$

$a>b>c>\dots$

(up to a shift)

irred. $\mathfrak{p}$-reps $\mathbb{E}\otimes\mathbb{F}^*$

$(abc\,|\,def)$

$a>b>c, \quad d>e>f$

The example: Grassmanians $Gr(3,3)$

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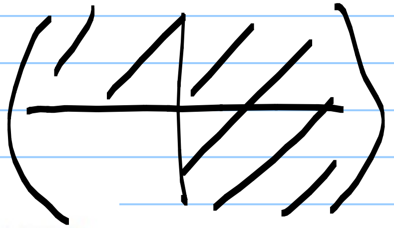
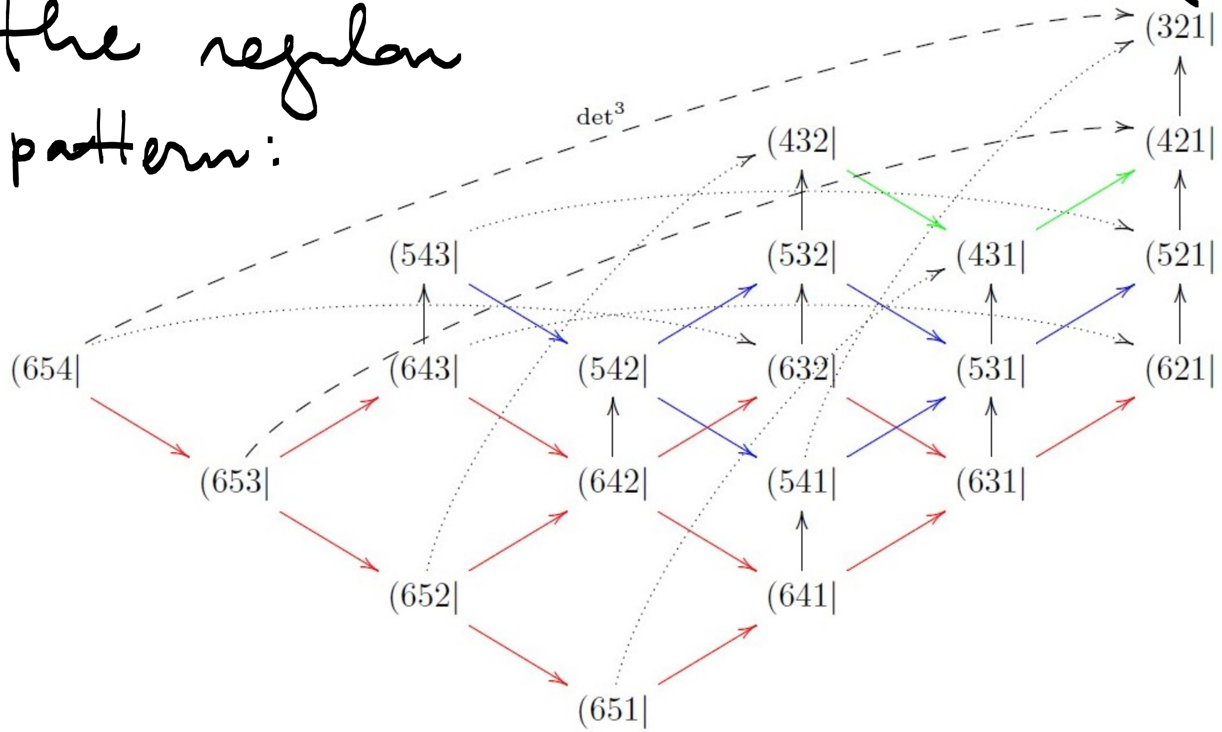


normalize by $f = 0$

the regular pattern:



$\mathfrak{g}$-reps: $(a\,b\,c\,d\,e\,f)$
$a > b > c > \ldots$ ← regular
(up to a shift)

irred. $\mathfrak{p}$-reps $\mathbb{E} \otimes \mathbb{F}^*$
$(a\,b\,c \mid d\,e\,f)$
$a > b > c$, $d > e > f$

action of grading element:
$\frac{1}{2}(a+b+c-d-e-f)$
provides the order!

Semiholonomic version (Eastwood, S., Souček, S.)

i.e. $D: \bar{J}^k\mathcal{E} = \underset{g}{G}\times_P \bar{J}^k\mathbb{E} \longrightarrow \mathcal{F}$ $\leftarrow$ $\left\{\begin{array}{l}\text{NOT true}\\ \text{without the bars}\end{array}\right.$!

but universal $\bar{J}^k$ EXISTS

$\Gamma(\mathcal{E}) = C^\infty(G, \mathbb{E})_P$, $X_1, \dots, X_\ell \in \mathfrak{g}$, $X_\ell \cdot \ldots \cdot X_1 \cdot s$

$\bar{U}(\mathfrak{g}) = T\mathfrak{g}/\bar{I}$, $\bar{I} = \langle XY - YX - [X,Y], X \in \mathfrak{p}, Y \in \mathfrak{g} \rangle$

(topological) dual $(\bar{J}^\infty\mathbb{E})^* = \bar{U}(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathbb{E}^* = \bar{V}(\mathbb{E})$

PROPOSITION 2. *A homomorphism $V(\mathbb{F}) \to V(\mathbb{E})$ of order 2 or less always lifts to a homomorphism of the corresponding semiholonomic modules.*

$$\begin{tikzcd} & \bar{V}(\mathbb{E}) \arrow[d] \\ \mathbb{F}^* \arrow[ur, dashed] \arrow[r] & V(\mathbb{E}) \end{tikzcd}$$

sometimes not true for higher order!

covering of singular vect

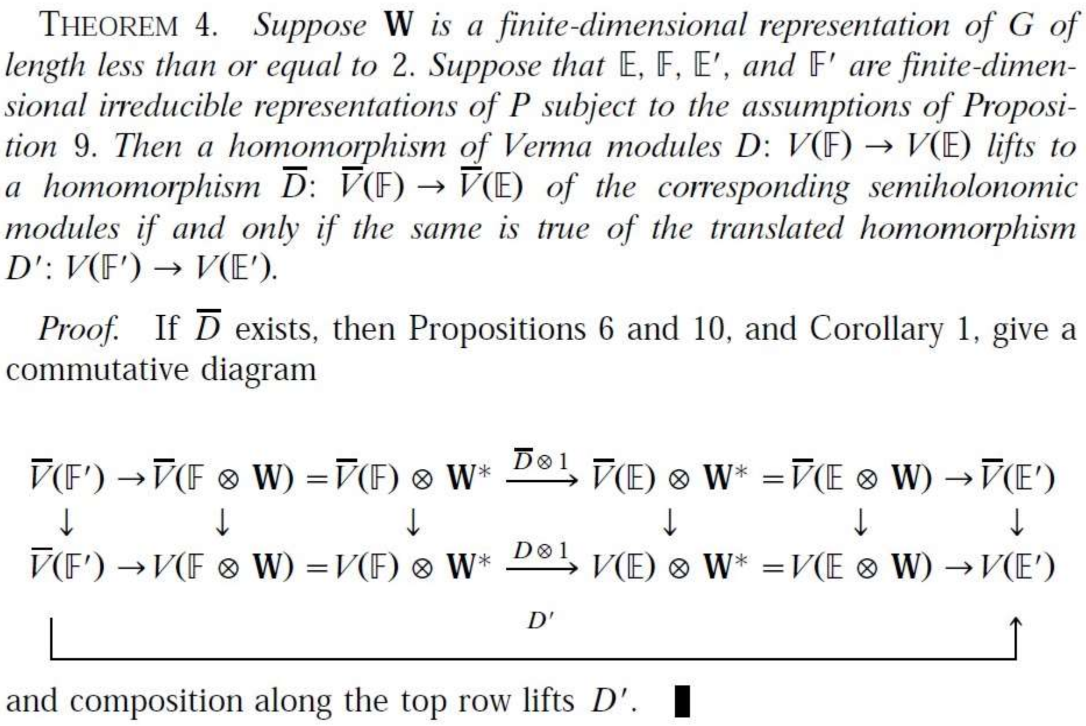
Remarks: (1) the construction of the translated homomorphisms includes splitting operators in the filtering of $\mathbb{W}$

semi-holomorphic existence has to be checked! — OK if up to order 2 (e.g. filtration short)

(2) the choice of highest weights in $\mathbb{W}$ allows to recover all "equisingular" patterns from the "lowest weight" ones, if we check the splittings for fundamental reps.

(3) the translations sometimes work "one-way" from "more singular" to "less singular":

(4) whatever is obtained by translations, whose splittings work, extends!



THEOREM 4. *Suppose* **W** *is a finite-dimensional representation of* G *of length less than or equal to* 2. *Suppose that* $\mathbb{E}$, $\mathbb{F}$, $\mathbb{E}'$, *and* $\mathbb{F}'$ *are finite-dimensional irreducible representations of* P *subject to the assumptions of Proposition* 9. *Then a homomorphism of Verma modules* $D\colon V(\mathbb{F}) \to V(\mathbb{E})$ *lifts to a homomorphism* $\overline{D}\colon \overline{V}(\mathbb{F}) \to \overline{V}(\mathbb{E})$ *of the corresponding semiholonomic modules if and only if the same is true of the translated homomorphism* $D'\colon V(\mathbb{F}') \to V(\mathbb{E}')$.

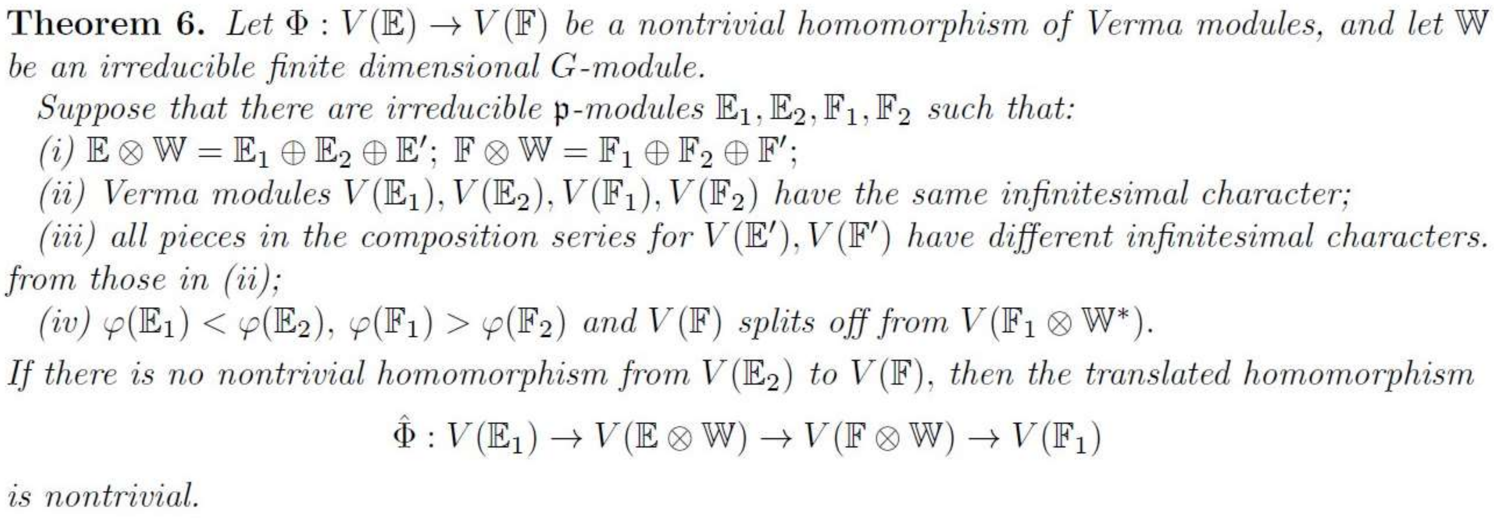
Proof. If $\overline{D}$ exists, then Propositions 6 and 10, and Corollary 1, give a commutative diagram

$$\begin{tikzcd}
\overline{V}(\mathbb{F}') \arrow[r] \arrow[d] & \overline{V}(\mathbb{F} \otimes \mathbf{W}) \arrow[r, phantom, "="] \arrow[d] & \overline{V}(\mathbb{F}) \otimes \mathbf{W}^* \arrow[r, "\overline{D} \otimes 1"] \arrow[d] & \overline{V}(\mathbb{E}) \otimes \mathbf{W}^* \arrow[r, phantom, "="] \arrow[d] & \overline{V}(\mathbb{E} \otimes \mathbf{W}) \arrow[r] \arrow[d] & \overline{V}(\mathbb{E}') \arrow[d] \\
\overline{V}(\mathbb{F}') \arrow[r] \arrow[rrrrr, bend right, "D'"'] & V(\mathbb{F} \otimes \mathbf{W}) \arrow[r, phantom, "="] & V(\mathbb{F}) \otimes \mathbf{W}^* \arrow[r, "D \otimes 1"] & V(\mathbb{E}) \otimes \mathbf{W}^* \arrow[r, phantom, "="] & V(\mathbb{E} \otimes \mathbf{W}) \arrow[r] & V(\mathbb{E}')
\end{tikzcd}$$

and composition along the top row lifts D'. ∎

← we need the splittings to exist in semi-holonomi

Similarly for the "one-way" translations:



Theorem 6. *Let $\Phi : V(\mathbb{E}) \to V(\mathbb{F})$ be a nontrivial homomorphism of Verma modules, and let $\mathbb{W}$ be an irreducible finite dimensional G-module.*

Suppose that there are irreducible $\mathfrak{p}$-modules $\mathbb{E}_1, \mathbb{E}_2, \mathbb{F}_1, \mathbb{F}_2$ such that:

(i) $\mathbb{E} \otimes \mathbb{W} = \mathbb{E}_1 \oplus \mathbb{E}_2 \oplus \mathbb{E}'$; $\mathbb{F} \otimes \mathbb{W} = \mathbb{F}_1 \oplus \mathbb{F}_2 \oplus \mathbb{F}'$;

(ii) Verma modules $V(\mathbb{E}_1), V(\mathbb{E}_2), V(\mathbb{F}_1), V(\mathbb{F}_2)$ have the same infinitesimal character;

(iii) all pieces in the composition series for $V(\mathbb{E}'), V(\mathbb{F}')$ have different infinitesimal characters. from those in (ii);

(iv) $\varphi(\mathbb{E}_1) < \varphi(\mathbb{E}_2)$, $\varphi(\mathbb{F}_1) > \varphi(\mathbb{F}_2)$ and $V(\mathbb{F})$ splits off from $V(\mathbb{F}_1 \otimes \mathbb{W}^*)$.

If there is no nontrivial homomorphism from $V(\mathbb{E}_2)$ to $V(\mathbb{F})$, then the translated homomorphism

$$\hat{\Phi} : V(\mathbb{E}_1) \to V(\mathbb{E} \otimes \mathbb{W}) \to V(\mathbb{F} \otimes \mathbb{W}) \to V(\mathbb{F}_1)$$

is nontrivial.

Translations 2-singular: in 6 patterns (all covered)

1-singular: one obtained by a one-way translation, other by invertible within the patterns

and so on ....

All work, up to 3 non-standard touching the ends!

a $|1|$-singular case:

$$V(321|210) \to V(210|321) \implies V(431|210)) \to V(210|431).$$

Consider modules

$$\begin{aligned}
\mathbb{E} &= (321|210); \mathbb{E}_1 = (431|210); \mathbb{E}_2 = (421|310)\\
\mathbb{F} &= (210|321);\ \mathbb{F}_1 = (210|431), \mathbb{F}_2 = (310|421)\\
\mathbb{W} &= (110000) = (110|000) + (100|100) + (000|110);\\
\mathbb{W}^* &= (111100) = (111|100) + (110|110) + (100|111);
\end{aligned}$$

We have

$$\begin{aligned}
\mathbb{E} \otimes \mathbb{W} &= \boxed{(431|210)} + \boxed{(421|310)} + (321|320)\\
\mathbb{F} \otimes \mathbb{W} &= (320|321) + \boxed{(310|421)} + \boxed{(210|431)}\\
\mathbb{E}_1 \otimes \mathbb{W}^* &= (542|310) + (541|320) + (532|320) + (531|321) + \boxed{(432|321)}\\
\mathbb{F}_1 \otimes \mathbb{W}^* &= (321|531) + \boxed{(321|43)} + (320|541) + (320|532) + (310|532).
\end{aligned}$$

There is no homomorphism from $(421|310)$ to $(210|431)$, hence we can use Theorem 6.

Thanks for attention!