Geometric approach to graded vector bundles

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Outline

$\mathbb{Z}$-graded geometry

We will use the word "graded" for $\mathbb{Z}$-graded. Under the definition that we use, graded objects (vector spaces, modules, algebras..) are defined as "sequences of objects". For example: a **graded vector space** V is defined as a sequence of vector spaces

$$V = (V_i)_{i \in \mathbb{Z}},$$

where for every $i \in \mathbb{Z}$, V_i is an ordinary vector space. We write $v \in V$ for $\exists! \ell \in \mathbb{Z},\ v \in V_\ell$ and we call this ℓ the **degree** of v and write

$$\ell =: |v|$$

A **graded linear map** between two graded vector spaces $\varphi : V \to W$ is defined as a sequence $\varphi = (\varphi_i)_{i \in \mathbb{Z}}$, where for every $i \in \mathbb{Z}$,

$$\varphi_i : V_i \to W_{i+|\varphi|},$$

is a linear map. Here $|\varphi| \in \mathbb{Z}$ is called the **degree** of the graded linear map.

Consider a typical example: let $(n_j)_{j\in\mathbb{Z}}$ be a sequence of non-negative integers. Then $\mathbb{R}^{(n_j)}$ denotes the vector space

$$\left(\mathbb{R}^{(n_j)}\right)_i = \mathbb{R}^{n_i}.$$

We say that $\{e_k\}$ is a **basis** for $\mathbb{R}^{(n_j)}$ if $\{e_k\}_{k:|e_k|=i}$ form a basis for $\mathbb{R}^{n_i}$ for every $i \in \mathbb{Z}$.

- We can view $\mathbb{R}$ as a graded vector space, where $\mathbb{R}_0 = \mathbb{R}$ and $\mathbb{R}_j = \{0\}$ for every $j \neq 0$. If $\{e_k\}$ is a basis for $\mathbb{R}^{(n_j)}$, then we can define vectors of the **dual basis** as graded linear maps

$$e^k : \mathbb{R}^{(n_j)} \to \mathbb{R}, \qquad e^k(e_\ell) = \delta^k{}_\ell,$$

where $|e^k| = -|e_k|$ and $|\delta^k{}_\ell| = |e_\ell| - |e_k|$.

Now let us move on to graded manifolds. Any choice of basis $\{e_1, \ldots, e_m\}$ of an ordinary vector space $\mathbb{R}^m$ makes $\mathbb{R}^m$ into a smooth manifold, with global coordinates $\{e^1, \ldots, e^m\}$.

Similarly, consider some sequence $(n_j)_{j\in\mathbb{Z}}$ of non-negative integers, only finitely many of which are non-zero. Then, by choosing a basis $\{e_1, \ldots, e_{n_0}, \xi_1, \ldots, \xi_n\}$ for $\mathbb{R}^{(n_{-j})}$, where $n = \sum_{i\neq 0} n_i$ and $|\xi_i| \neq 0$ for any $i \in \{1, \ldots, n\}$, we can view $\mathbb{R}^{(n_{-j})}$ as a **graded manifold**.

In particular, we call $\mathbb{R}^{n_0}$ the underlying smooth manifold, and for every $U \in \mathrm{op}(\mathbb{R}^{n_0})$ we say that **graded functions** f, of degree $|f|$, on U are formal power series

$$f = \sum_{\mathsf{p}} f_{\mathsf{p}}\, (\xi^1)^{p_1} \cdots (\xi^n)^{p_n},$$

where the sum ranges over the set

$$\{\mathsf{p} \in \mathbb{N}_0^n \mid \sum_{k=1}^{n} p_k |\xi^k| = |f| \,\text{and}\, p_k \in \{0,1\} \,\text{for}\, |\xi^k| \,\text{odd}\ \},$$

the coefficients f_{p} are ordinary smooth functions $f_{\mathsf{p}} \in C^\infty(U)$ and we impose the commutation relations

$$f\, g = (-1)^{|f||g|} g\, f,$$

for any graded functions f, g.

The **graded algebra of graded functions** over U is then denoted as

$$C^\infty_{(n_j)}(U)$$

The assignment $C^\infty_{(n_j)}$ is a sheaf on $\mathbb{R}^{n_0}$ called the structure sheaf or the sheaf of graded functions. Restrictions are just restrictions of the coefficient smooth functions. We call ξ^k the **graded coordinates** on $\mathbb{R}^{(n_{-j})}$, although sometimes we just call them coordinates, and denote $\{e^1, \ldots, e^{n_0}, \xi^1, \ldots, \xi^n\} =: \{x^i\}$.

A general **graded manifold** is defined as a pair

$$\mathcal{M} = (M, C^\infty_{\mathcal{M}}),$$

where M is a second-countable Hausdorff topological space and the structure sheaf $C^\infty_{\mathcal{M}}$ on M is valued in graded commutative associative unital algebras. Furthermore, $\mathcal{M}$ is **locally isomorphic** to $\mathbb{R}^{(n_{-j})}|_U$ for some fixed finite sequence of non-negative integers $(n_j)_{j\in\mathbb{Z}}$, called the **graded dimension** of $\mathcal{M}$.

Finally we need the notion of a morphism in the category of graded manifolds: a **graded smooth map**

$$\varphi : \mathcal{M} \to \mathcal{N},$$

is a pair $\varphi \equiv (\underline{\varphi}, \varphi^*)$, where

$$\underline{\varphi} : M \to N$$

is an ordinary smooth map, and

$$\varphi^* : C^\infty_{\mathcal{N}} \to \underline{\varphi}_* C^\infty_{\mathcal{M}}$$

is a morphism of sheaves of graded commutative algebras (which gives rise to local ring morphisms between stalks).

Graded vector bundles as sheaves of modules

Consider the well known "almost unique" correspondence between non-graded vector bundles and their sheaves of sections.

$$\boxed{\pi : E \to M \quad \longleftrightarrow \quad \Gamma_E : \mathsf{Op}(M)^{\mathrm{op}} \to \mathsf{Vec}}$$

Where Γ_E is a locally freely and finitely generated sheaf of C^∞_M-modules of a constant rank. Our aim is to show the same correspondence in the graded setting.

One approach to define graded vector bundles, taken by J. Vysoký in [1], is to "define them as their sheaves of sections".

- A graded vector bundle over a graded manifold $\mathcal{M}$ is any locally freely and finitely generated sheaf $\mathcal{E}$ of $C^\infty_{\mathcal{M}}$-modules, of a constant rank.
- If one has two graded vector bundles $\mathcal{E}$ over $\mathcal{M}$ and $\mathcal{E}'$ over $\mathcal{M}'$, then morphisms $\Phi : \mathcal{E} \to \mathcal{E}'$ are defined as pairs $\Phi = (\varphi, A)$, where
  - $\varphi : \mathcal{M} \to \mathcal{M}'$ is a graded smooth map.
  - $A : (\mathcal{E}')^* \to \varphi_*(\mathcal{E}^*)$ is a morphism of sheaves of $C^\infty_{\mathcal{E}'}$-modules.

  Graded vector bundles form a category.

Graded vector bundles as graded manifolds

Let us attempt to indroduce graded vector bundles in a more conventional way.

- Consider a graded vector space $V := \mathbb{R}^{(n_{-j})}$, where only finitely many of n_j are non-zero. By choosing its total basis $\{k_a\}$ we may consider it as a graded manifold of graded dimension (n_j) with coordinates $\{k^a\}$.
- For any $\lambda \in \mathbb{R}$ we may consider the graded linear map $H_\lambda : V \to V$, defined on vectors as

  $$H_\lambda : v \to \lambda v.$$

  We can define the corresponding graded smooth map $H_\lambda \equiv (\underline{H}_\lambda, H_\lambda^*) : V \to V$ by taking $\underline{H}_\lambda : \mathbb{R}^{n_0} \to \mathbb{R}^{n_0}$ as the multiplication by lambda, and H_λ^* defined on coordinates as

  $$H_\lambda^*(k^a) = \lambda\, k^a.$$

A graded vector bundle is defined as $(\mathcal{E}, \mathcal{M}, \pi, \{H^\lambda_{\mathcal{E}}\}_{\lambda\in\mathbb{R}})$, where $\mathcal{E}, \mathcal{M}$ are graded manifolds, $\pi : \mathcal{E} \to \mathcal{M}$, $H^\lambda_{\mathcal{E}} : \mathcal{E} \to \mathcal{E}$ are graded smooth maps, such that $\underline{\pi}$ is surjective and there exists an open cover $\{U_\alpha\}_{\alpha\in I}$ of M together with graded diffeomorphisms

$$\psi_\alpha : \mathcal{M}|_U \times \mathbb{R}^{(n_{-j})} \to \mathcal{E}|_{\underline{\pi}^{-1}(U_\alpha)},$$

for some fixed finite (n_j), such that for every $\alpha \in I$,

- $\pi|_{\underline{\pi}^{-1}(U_\alpha)} \circ \psi_\alpha = p_1$, where p_1 is the canonical product projection.
- The diagram

$$\begin{array}{ccc}
\mathcal{M}|_U \times \mathbb{R}^{(n_{-j})} & \xrightarrow{\psi_\alpha} & \mathcal{E}|_{\underline{\pi}^{-1}(U_\alpha)} \\
\downarrow{\scriptstyle 1\times H_\lambda} & & \downarrow{\scriptstyle H^\lambda_{\mathcal{E}}|_{\underline{\pi}^{-1}(U_\alpha)}} \\
\mathcal{M}|_U \times \mathbb{R}^{(n_{-j})} & \xrightarrow{\psi_\alpha} & \mathcal{E}|_{\underline{\pi}^{-1}(U_\alpha)}
\end{array}$$

  commutes for every $\lambda \in \mathbb{R}$.

We have objects, we need arrows: consider two graded vector bundles $\pi : \mathcal{E} \to \mathcal{M}$ and $\pi' : \mathcal{E}' \to \mathcal{M}'$. We say that $\Phi = (\varphi, \phi)$ is a morphism of vector bundles from $\mathcal{E}$ to $\mathcal{E}'$, if $\varphi : \mathcal{M} \to \mathcal{M}'$ and $\phi : \mathcal{E} \to \mathcal{E}'$ are graded smooth maps, s.t.

- The diagram

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\phi} & \mathcal{E}' \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ \mathcal{M} & \xrightarrow{\varphi} & \mathcal{M}' \end{array}$$

  commutes, and
- The diagram

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\phi} & \mathcal{E}' \\ {\scriptstyle H^\lambda_{\mathcal{E}}}\downarrow & & \downarrow{\scriptstyle H^\lambda_{\mathcal{E}'}} \\ \mathcal{E} & \xrightarrow{\phi} & \mathcal{E}' \end{array}$$

  commutes for every $\lambda \in \mathbb{R}$.

Their equivalence

The two definitions of graded VBs agree upto "some isomorphism", similarly as in the non-graded case. To put it precisely:

Theorem

The categories of graded vector bundles defined as sheaves of modules and graded vector bundles defined as graded manifolds, are equivalent.

We show this equivalence in the usual way - by constructing a functor $\Gamma : \mathsf{gVbun} \to \mathsf{gVbun}$ and showing that it is fully faithful and essentially surjective.

- For a graded vector bundle $\mathcal{E} \in \mathsf{gVbun}$, define (the dual of) $\Gamma_{\mathcal{E}}$ as

$$\Gamma^*_{\mathcal{E}} = \bigcap_{\lambda \in \mathbb{R}} \ker(H^{\lambda,*}_{\mathcal{E}} - \lambda),$$

which can be viewed as a subsheaf of $C^\infty_{\mathcal{M}}$-modules $\Gamma^*_{\mathcal{E}} \subseteq \pi_*(C^\infty_{\mathcal{E}})$.

- Basically, $f \in \Gamma_{\mathcal{E}}(U)^*$ are those graded functions $f \in C^\infty_{\mathcal{E}}(\underline{\pi}^{-1}(U))$ which satisfy
$$H^{\lambda,*}_{\mathcal{E}} f = \lambda f$$
for every $\lambda \in \mathbb{R}$. We call them functions linear in the fiber.
- Next, for every arrow $(\varphi, \phi) : \mathcal{E} \to \mathcal{E}'$ we notice that the pullback ϕ^* can be considered as
$$\phi^* : \pi'_* C^\infty_{\mathcal{E}'} \to \varphi_*(\pi_* C^\infty_{\mathcal{E}}),$$
which furthermore preserves linearity in the fiber, hence
$$\phi^* : \Gamma^*_{\mathcal{E}'} \to \varphi_* \Gamma^*_{\mathcal{E}},$$
which means we can take $\Gamma(\varphi, \phi) = (\varphi, \phi^*)$.
- Full faithfulness of Γ is then verified in coordinates, as is essential surjectivity, with the help of a construction theorem from [1, Vysoký 22].

Thank you.

Vysoký, J. (2022) Global theory of graded manifolds. *Reviews in Mathematical Physics.* 34(10), 2250035.

Vysoký, J. (2022) Graded generalized geometry. *Journal of Geometry and Physics.* 182, 104683.