On 4D split-conformal structures with G_2-symmetric twistor distribution

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(Joint work in progress with
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A tale of three (parabolic) geometries

XXO-geometry (5D)

$(N^5 = \mathbb{T}^+(M); \ell, D)$

D non-int. $\rightsquigarrow$

$(2,3,5)$-geometry (5D)

$(N^5; D)$

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(oriented) split-conformal (4D)

$(M^4; [g])$

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Example (Rolling w/o twisting or slipping $\rightsquigarrow G_2$)

$M = \Sigma_1 \times \Sigma_2$ with $g = g_1 \oplus (-g_2)$ for Riemannian surfaces (Σ_i, g_i):

- Two 2-spheres with ratio of radii $1:3$;
- An–Nurowski (2013): new examples (conf. hom & non-hom).

$(2,3,5)$-geometry

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$$\mathcal{D}_x := \partial_x + p\partial_y + q\partial_p + f(x,y,p,q,z)\partial_z, \qquad \partial_q.$$

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Any $(2,3,5)$-distribution has at most 14-dim symmetry. Locally, $\exists!$ maximally symmetric model, and this has G_2-symmetry.

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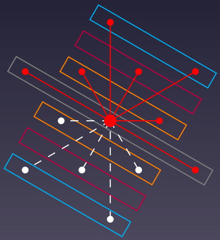
Example (Cartan 1893)

D_{q^2} has G_2-symmetry.



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... consists of a 5-mfld N with rank 3 dist. $H \subset TN$ satisfying:

- $[H, H] = TN$ and $H = \ell \oplus D$ (of ranks 1 and 2);
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Model: $\mathrm{Flag}_{1,2}(\mathbb{R}^4)$ $(\mathsf{SL}_4/P_{1,2})$

0	1	2	2
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Example (Pairs of 2nd order ODE as integrable XXO-str.)

$$\begin{cases} \ddot{x} = F(t, x, y, \dot{x}, \dot{y}) \\ \ddot{y} = G(t, x, y, \dot{x}, \dot{y}) \end{cases}$$

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Example (Enhancing $(2, 3, 5)$ to a non-integrable XXO-str.)

For $D = D_{q^2} = \langle \mathcal{D}_x, \partial_q \rangle$, we have $[D, D]/D = \langle \partial_p + 2q\partial_z \rangle$. We can define an XXO geometry via a choice of ℓ:

$$\ell = \langle \partial_p + 2q\partial_z + A\mathcal{D}_x + B\partial_q \rangle \quad \text{(non-int: } [D, D] = H := \ell \oplus D)$$

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General construction:

- Input: $(M^4, [g])$, with g a split-signature $(2,2)$-metric.
- Output: On "circle-twistor bundle" $N = \mathbb{T}^+(M) \to M$ (fibres are SD totally null 2-planes), get XXO structure $H = \ell \oplus D$.

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Theorem (Nurowski–Sagerschnig–T. 2024)

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Complexified summary:

Label	Petrov type	Comments
M9	N.O	$\mathfrak{p}_1^{\mathrm{op}} \subset G_2$
M8	D.O	$\mathfrak{sl}(3,\mathbb{R}), \mathfrak{su}(1,2) \subset G_2$
M7$_a$	$\begin{cases} \text{N.N}, & a^2 \neq \frac{4}{3}; \\ \text{N.O}, & a^2 = \frac{4}{3} \end{cases}$	new : $\mathbb{R}^2 \ltimes \mathfrak{heis}_5$
M6S	D.D	1:3 rolling spheres + variants
M6N	III.O	new : $\mathfrak{aff}(2)$

Example: 9-dim symmetry

Plebanski: $g = dwdx + dydz - \Theta_{xx} dz^2 - \Theta_{yy} dw^2 + 2\Theta_{xy} dwdz$

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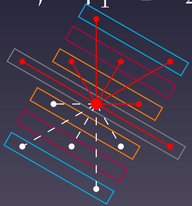

Harmonic curvatures

Structure	Hieroglyphic	Harmonic curvatures	Hom.
$(2,3,5)$	$-8 \quad 4$ 	Cartan quartic: $\mathcal{Q}$	$+4$
4D split-conf	$0 \quad -4 \quad 4$ 	ASD Weyl: $\mathcal{W}^-$	$+2$
	$4 \quad -4 \quad 0$ 	SD Weyl: $\mathcal{W}^+$	$+2$
XXO	$0 \quad -4 \quad 4$ 	$\mathcal{S}$	$+3$
	$-4 \quad 1 \quad 2$ 	$\mathcal{T}$	$+2$
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Remarks:

- $\mathcal{I}$ precisely obstructs integrability of D.
- Čap (2006) $\rightsquigarrow$ 4D split-conf. $\leftrightarrow$ XXO geometry with $\mathcal{T} \equiv 0$.

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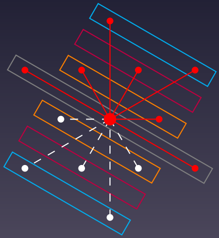

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(X.1) $\mathfrak{f} \hookrightarrow \mathfrak{g} = G_2$ as a filtered (wrt P) Lie subalgebra;

(X.2) $\mathrm{gr}_-(\mathfrak{f}) = \mathfrak{g}_-$ & $\dim(\mathfrak{f}^0) \geq 1$ ("multiply-transitive");

(X.3) On $\mathfrak{f}/\mathfrak{f}^0$, we have the data

$$\mathfrak{f}^{-1}/\mathfrak{f}^0 = D, \quad \mathfrak{f}^{-2}/\mathfrak{f}^0 = \ell \oplus D;$$

(X.4) $\mathcal{T} \equiv 0$;

(X.5) maximal wrt natural partial order: $\mathfrak{f} \leq \widetilde{\mathfrak{f}}$ iff $\mathfrak{f} \hookrightarrow \widetilde{\mathfrak{f}}$.

What should we classify?

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Classify admissible $(\mathfrak{f}, \mathfrak{f}^0; \ell, D)$ (up to the natural P-action).

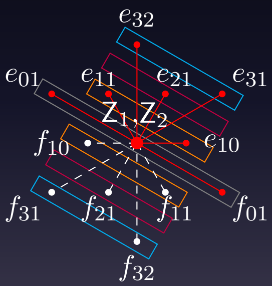
Classification steps

1. Classify complex admissible $(\mathfrak{f}, \mathfrak{f}^0; \ell, D)$ + real forms.
2. Integrate structure eqns to get local coordinate models. Check Petrov types.
3. Find "Cartan-theoretic" embeddings into $(\mathfrak{sl}_4, \mathfrak{p}_{1,2})$. (This also determines Petrov types, as well as holonomy.)

We'll sketch a few key ideas relating to the first step.

Examples

Let $\mathsf{Z}_1, \mathsf{Z}_2 \in \mathfrak{h}$ be dual to the simple roots $\alpha_1, \alpha_2 \in \mathfrak{h}^*$ of G_2.
Grading element: Z_1. Define $\mathsf{H} := [e_{01}, f_{01}] = -\mathsf{Z}_1 + 2\mathsf{Z}_2$.



	M9	M6S
$\mathfrak{f}^0$	$f_{01}, \mathsf{Z}_1, \mathsf{H}, e_{01}$	H
$D = \mathfrak{f}^{-1}/\mathfrak{f}^0$	f_{10}	$f_{10} + e_{11}$
	f_{11}	$f_{11} + e_{10}$
$\mathfrak{f}^{-2}/\mathfrak{f}^{-1}$	f_{21}	$f_{21} + e_{21}$
$\mathfrak{f}^{-3}/\mathfrak{f}^{-2}$	f_{31}	$f_{31} + e_{32}$
	f_{32}	$f_{32} + e_{31}$
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The graded subalgebra $\mathfrak{s} := \mathrm{gr}(\mathfrak{f}) \subset \mathfrak{g}$ determines leading parts of $\mathfrak{f}$. (Also need to determine "tails".) Start with the isotropy $\mathfrak{f}^0$.

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Lemma

1. $\mathfrak{f}^i = 0$ *for* $i \geq 1$
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P-action $\rightsquigarrow$ G_0-action on $\mathfrak{s}_0 \subseteq \mathfrak{g}_0$. Classification over $\mathbb{C}$:

$\dim$	$\mathfrak{gl}_2$ subalgebra
4	$\mathfrak{gl}_2$
3	$\mathfrak{sl}_2$, $\left\langle \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \right\rangle$
2	$\left\langle \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\rangle$, $\left\langle \begin{pmatrix} \lambda_1 & 0 \\ * & \lambda_2 \end{pmatrix} \right\rangle$
1	$\left\langle \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \right\rangle$, $\left\langle \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix} \right\rangle$

(Here, identify $Z_1 \leftrightarrow -\mathrm{id}$ and $H \leftrightarrow \mathrm{diag}(1,-1)$.)

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- Suppose $\exists S \in \mathfrak{f}^0$ with $S_0 = \mathrm{gr}_0(S) \in \mathfrak{s}_0$ semisimple.

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All cases with such semisimple elements in (normalized) $\mathfrak{s}_0$:

$S_0 \in \mathfrak{s}_0$	Constraints	Admissible models
Z_1	–	M9
H	$\mathsf{Z}_1 \notin \mathfrak{s}_0$	M8 & M6S
$\mathsf{Z}_1 + c\mathsf{H}$	$c \neq 0$, $\mathfrak{s}_0 \subseteq \langle \mathsf{Z}_1 + c\mathsf{H}, f_{01} \rangle$	$c \neq 1$: none $c = 1$: $\mathrm{M7}_a$

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- $r \neq 0$: No admissible model. (Easy.)
- $r = 0$: M6N. (Most challenging case; $\mathcal{T} \equiv 0$ is essential.)