Generalised geometry for the group E7

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w/ O. Hulík, E. Malek, D. Waldram, 2308.01130

16.1.2024, Srní[1]

[1] this time with no hidden Monty Python references whatsoever

Idea: Generalised geometry

Generalised tangent bundle

- vector bundle $E = TM \oplus T^*M$
- inner product $\langle X + \alpha, Y + \beta \rangle = \alpha(Y) + \beta(X)$
- bracket $[X + \alpha, Y + \beta] = \mathcal{L}_X Y + \mathcal{L}_X \beta - i_Y d\alpha$

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- generalised Riemannian geometry, string effective action $S = \int \mathcal{R}\,\mu$

Idea [Klimčík–Ševera '95, Ševera '98, Chapman–Cleese–Gilliam–Idle–Jones–Palin '69]

generalised tangent bundle $\rightsquigarrow$ Courant algebroids

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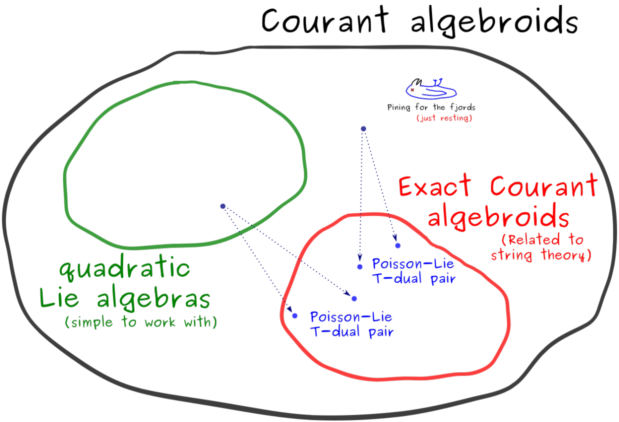

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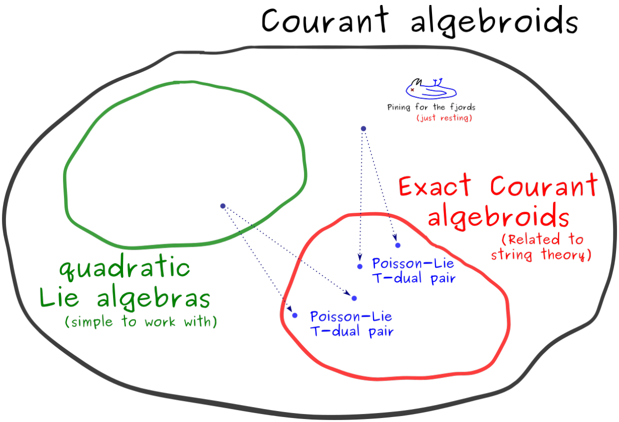

Task: Do the same for M-theory

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[Hull '07, Pacheco–Waldram '08, Coimbra–Strickland-Constable–Waldram '14, . . .]

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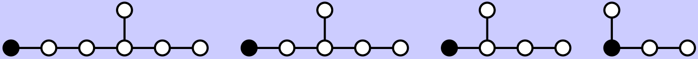

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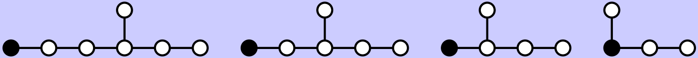

$$[\,\cdot\,,\,\cdot\,] = \mathcal{L}_X X' + (\mathcal{L}_X \sigma_2' - i_{X'} d\sigma_2) + (\mathcal{L}_X \sigma_5' - i_{X'} d\sigma_5 - \sigma_2' \wedge d\sigma_2) + (\mathcal{L}_X \sigma_{1,7}' - j\sigma_5' \wedge d\sigma_2 - j\sigma_2' \wedge d\sigma_5) + \text{twists}(F_1, F_4, F_7)$$

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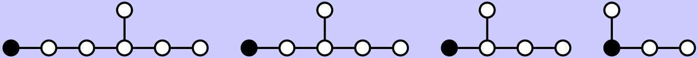

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- exceptional Riemannian geometry, action $S = \int \mathcal{R}\, \mu$

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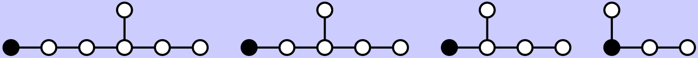

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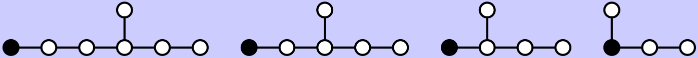

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General structure: $E \cong_{loc} \mathcal{U} \times E_0, \quad [u,\,\cdot\,] = \rho(u) - (\rho^* du)_{ad}$

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Central object: vector space R with a tensor $y\colon R^* \otimes R \to R^* \otimes R$

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- $y^{\alpha\beta}_{\gamma\delta} = 12\omega^{\alpha\epsilon}\omega^{\beta\zeta}q_{\epsilon\zeta\gamma\delta} + \delta^{(\alpha}_{\gamma}\delta^{\beta)}_{\delta} + \frac{1}{2}(\omega^{-1})^{\alpha\beta}\omega_{\gamma\delta}$

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A **Y-algebroid** *is a Leibniz algebroid with an invariant tensor* $Y: E^* \otimes E \to E^* \otimes E$ *such that* $\exists$ *a* Y*-compatible connection* ∇ *for which*

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Example (Y-algebroids of particular classes)

- $GL(n, \mathbb{R}) \rightsquigarrow$ Lie algebroids, $O(p, q) \rightsquigarrow$ Courant algebroids
- $E_n \times \mathbb{R}^+ \rightsquigarrow$ E_n-**algebroids**, $Spin(n, n) \times \mathbb{R}^+$, $SL(n, \mathbb{R}) \times \mathbb{R}^+$, ...

Results for E_7

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Proof.

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