Infinitesimal braidings and pre-Cartier bialgebras

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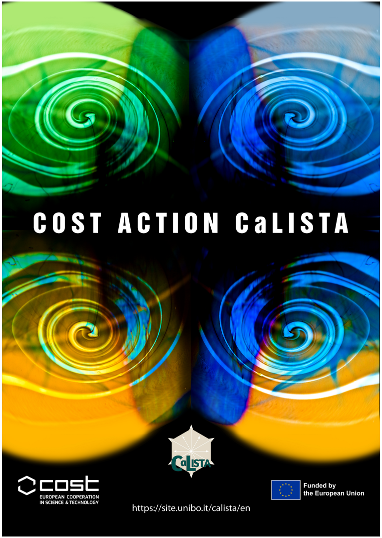
arXiv:2306.00558 with **A.Ardizzoni, L.Bottegoni, A.Sciandra**



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Category-Algebra correspondence

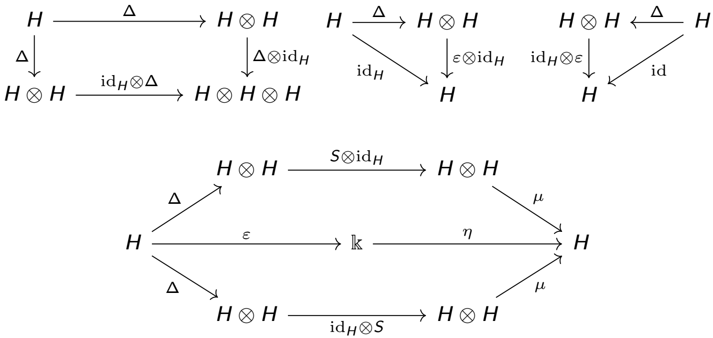
H associative algebra over field $\mathbb{k}$ and ${}_H\mathcal{M}$ category of left H-modules.

Category	**Algebra**
$({}_H\mathcal{M}, \otimes, \mathbb{k})$ is monoidal	(H, Δ, ε) is bialgebra $\Delta\colon H \to H \otimes H$ coproduct $\varepsilon\colon H \to \mathbb{k}$ counit
$({}_H\mathcal{M}, \otimes, \mathbb{k}, \sigma)$ is **braided** monoidal $\sigma^{\mathcal{R}}_{M,N}\colon M \otimes N \to N \otimes M$ braiding	$(H, \Delta, \varepsilon, \mathcal{R})$ is **quasitriangular** bialgebra $\mathcal{R} \in H \otimes H$ universal $\mathcal{R}$-matrix
$({}_H\mathcal{M}, \otimes, \mathbb{k}, \sigma, t)$ pre-Cartier category $t^{\chi}_{M,N}\colon M \otimes N \to M \otimes N$ **infinitesimal braiding**	$(H, \Delta, \varepsilon, \mathcal{R}, \chi)$ is pre-Cartier bialgebra $\chi \in H \otimes H$ **infinitesimal** $\mathcal{R}$**-matrix**

$$\sigma^{\mathcal{R}}_{M,N} := \sigma^{\mathrm{flip}}_{M,N} \circ \mathcal{R} \circ (\triangleright_M \otimes \triangleright_N)$$

$$t^{\chi}_{M,N} := \chi \circ (\triangleright_M \otimes \triangleright_N)$$

$$H \text{ Hopf algebra} \begin{cases} \mathbb{k}\text{-algebra } (H,\mu,\eta) \\ \mathbb{k}\text{-coalgebra} \begin{cases} \text{coproduct } \Delta\colon H \to H\otimes H \\ \text{counit } \varepsilon\colon H \to \mathbb{k} \end{cases} \text{algebra morphisms} \\ \text{antipode } S\colon H \to H \end{cases}$$

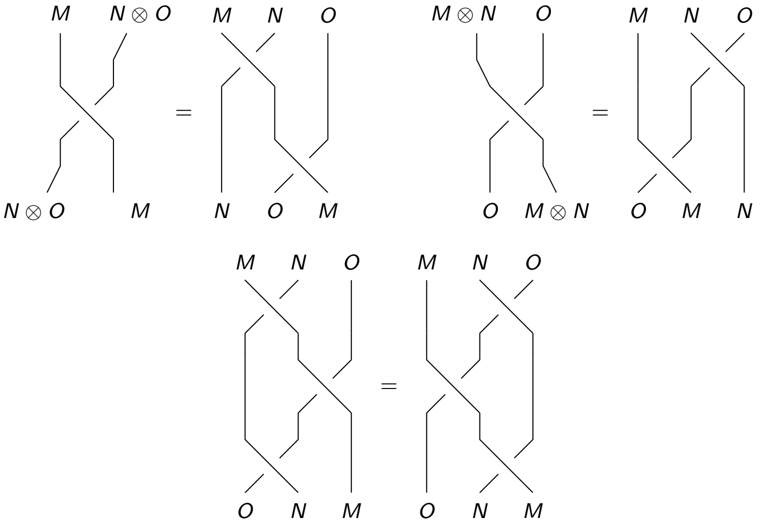


Sweedler's notation: $\Delta(h) =: h_1 \otimes h_2$,

$$(\Delta \otimes \mathrm{id}_H)(\Delta(h)) = (\mathrm{id}_H \otimes \Delta)(\Delta(h)) =: h_1 \otimes h_2 \otimes h_3$$

Braided monoidal categories

Category $\mathcal{C}$ with a monoidal structure $\otimes$, the "tensor product", and a natural isomorphism $\sigma_{M,N}\colon M \otimes N \to N \otimes M$ such that



Quasitriangular bialgebras and universal $\mathcal{R}$-matrices

Given a bialgebra (H, Δ, ε) its category of representations (let's say left H-modules) ${}_H\mathcal{M}$ is monoidal with

$$h \cdot (m \otimes n) := \Delta(h)(m \otimes n)$$

for $h \in H$, $m \in M$, $n \in N$, where $M, N \in {}_H\mathcal{M}$.

Theorem (Drinfel'd-Majid '90)

$({}_H\mathcal{M}, \otimes)$ is braided if and only if H is **quasitriangular**, *i.e. $\exists\, \mathcal{R} \in H \otimes H$ invertible s.t.*

$$(\Delta \otimes \mathrm{id}_H)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}, \qquad (\mathrm{id}_H \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}$$

and $\Delta^{\mathrm{op}}(\cdot) = \mathcal{R}\Delta(\cdot)\mathcal{R}^{-1}$. Then $\sigma^{\mathcal{R}}_{M,N}(m \otimes n) := \mathcal{R}^{\mathrm{op}} \cdot (n \otimes m)$.

Note: σ is **symmetric**, i.e. $\sigma^2 = \mathrm{id}$ if and only if $\mathcal{R}$ is **triangular**, i.e. $\mathcal{R}^{-1} = \mathcal{R}^{\mathrm{op}}$.

Example

i.) Every cocommutative (i.e. $\Delta = \Delta^{\mathrm{op}}$) bialgebra is triangular with $\mathcal{R} = 1 \otimes 1$. For example $\mathbb{k}[G]$ for any group G and $U\mathfrak{g}$ for any Lie algebra $\mathfrak{g}$.

ii.) Group algebra $\mathbb{C}[\mathbb{Z}_n]$ n-cyclic group $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$. Quasitriangular structure $\mathcal{R} = \frac{1}{n}\sum_{j,k=0}^{n-1} e^{-\frac{2\pi \mathrm{i} jk}{n}} g^j \otimes g^k$. For $n = 2$: "supergeometry braiding".

iii.) The Drinfel'd double $D(H) = H^* \otimes H$ of a finite-dim. Hopf algebra H is quasitriangular w.r.t. $\mathcal{R} = (1 \otimes e_i) \otimes (e^i \otimes 1)$.

The first order of an $\mathcal{R}$-matrix

Let H be a bialgebra.

$\Rightarrow$ The formal power series $\tilde{H} = H[[\hbar]]$ with formal parameter $\hbar$ become a **topological bialgebra** with $\hbar$-linearly extended (co)algebra structures.

Consider a quasitriangular structure $\tilde{\mathcal{R}} \in (H \otimes H)[[\hbar]] \cong \tilde{H} \hat{\otimes} \tilde{H}$ on $\tilde{H} = H[[\hbar]]$.

$\Rightarrow$ $\tilde{\mathcal{R}} = \mathcal{R} + \mathcal{O}(\hbar)$ gives quasitriangular bialgebra $(H, \mathcal{R})$ and we can write

$$\tilde{\mathcal{R}} = \mathcal{R}(1 \otimes 1 + \hbar\chi + \mathcal{O}(\hbar^2)).$$

What are the properties of $\chi \in H \otimes H$?

Definition

We call $(H, \mathcal{R}, \chi)$ a **pre-Cartier bialgebra** with infinitesimal $\mathcal{R}$-matrix $\chi \in H \otimes H$ if $(H, \mathcal{R})$ is a quasitriangular bialgebra such that $\chi\Delta(\cdot) = \Delta(\cdot)\chi$ and

$$(\mathrm{id}_H \otimes \Delta)(\chi) = \chi_{12} + \mathcal{R}_{12}^{-1}\chi_{13}\mathcal{R}_{12}, \qquad (\Delta \otimes \mathrm{id}_H)(\chi) = \chi_{23} + \mathcal{R}_{23}^{-1}\chi_{13}\mathcal{R}_{23}.$$

If also $\mathcal{R}\chi = \chi^{\mathrm{op}}\mathcal{R}$, we call $(H, \mathcal{R}, \chi)$ a **Cartier bialgebra**.

The categorical counterpart

Proposition

$(H, \mathcal{R}, \chi)$ is a pre-Cartier bialgebra iff ${}_H\mathcal{M}$ is a braided monoidal category and there is a natural transformation $t\colon M \otimes N \to M \otimes N$ on ${}_H\mathcal{M}$ such that

$$t_{M,N\otimes L} = t_{M,N} \otimes L + (\sigma_{M,N}^{-1} \otimes L)(N \otimes t_{M,L})(\sigma_{M,N} \otimes L)$$

$$t_{M\otimes N,L} = M \otimes t_{N,L} + (M \otimes \sigma_{N,L}^{-1})(t_{M,L} \otimes N)(M \otimes \sigma_{N,L}).$$

In this case $(H, \mathcal{R}, \chi)$ is a Cartier bialgebra, i.e. $\mathcal{R}\chi = \chi^{\mathrm{op}}\mathcal{R}$, iff

$$\sigma_{M,N} \circ t_{M,N} = t_{N,M} \circ \sigma_{M,N}.$$

Definition

A braided monoidal category $(\mathcal{C}, \sigma)$ with a natural transformation t as above is called **(pre-)Cartier category** and we call t an **infinitesimal braiding**.

For σ a symmetric braiding ($\sigma^2 = \mathrm{id}$) with $\sigma \circ t = t \circ \sigma$ we recover the known notion of **Cartier category**. [Cartier '93, Kassel '95, Heckenberger-Vendramin '22]

Let's see some examples...

Example (Trivial example)

The infinitesimal $\mathcal{R}$-matrices χ of a quasitriangular bialgebra $(H,\mathcal{R})$ form a vector space. In particular, $\chi = 0$ is a trivial solution which makes $(H,\mathcal{R})$ Cartier.

Example

If $(\mathfrak{g},[\cdot,\cdot],r)$ is a **quasitriangular Lie bialgebra**, i.e.

- $(\mathfrak{g},[\cdot,\cdot])$ is Lie algebra and
- $r \in \mathfrak{g}\otimes\mathfrak{g}$ is ad-invariant ($\mathrm{ad}_x^{(2)}(r) = 0$ for all $x \in \mathfrak{g}$) and satisfies the **classical Yang-Baxter equation**

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$$

then $\chi := r + r^{\mathrm{op}}$ is an infinitesimal $\mathcal{R}$-matrix of $(U\mathfrak{g}, 1\otimes 1)$.

Example (The (co)commutative case: primitive elements)

If H is commutative, then χ is an infinitesimal $\mathcal{R}$-matrix iff $\chi \in P(H)\otimes P(H)$, where $P(H)$ are the primitive elements of H. $(H,\mathcal{R},\chi)$ is Cartier if furthermore $\chi^{\mathrm{op}} = \chi$.

If H is cocommutative and $\mathcal{R} = 1\otimes 1$, then χ is an infinitesimal $\mathcal{R}$-matrix iff $\chi \in P(H)\otimes P(H)$.

A noncommutative and non-cocommutative example

Example (Sweedler's 4-dim Hopf algebra H)

Generators g, x with $g^2 = 1$, $x^2 = 0$, $xg = -gx$, $\Delta(g) = g \otimes g$, $\Delta(x) = x \otimes 1 + g \otimes x$.
All quasitriangular structures on H are triangular and of the form

$$\mathcal{R}_\lambda = \frac{1}{2}\Big(1 \otimes 1 + g \otimes 1 + 1 \otimes g - g \otimes g\Big) + \frac{\lambda}{2}\Big(x \otimes x - xg \otimes x + x \otimes xg + xg \otimes xg\Big)$$

for $\lambda \in \mathbb{C}$.
One can show that there exists an exhaustive 1-parameter family

$$\chi_\alpha := \alpha xg \otimes x$$

of infinitesimal $\mathcal{R}$-matrices of $(H, \mathcal{R}_\lambda)$. They are Cartier iff $\alpha = 0$.

This generalizes to a full classification of infinitesimal $\mathcal{R}$-matrices on all **Radford Hopf algebras** $E(n) = \mathbb{C}\langle g, x^1, \ldots, x^n \rangle$. [Bottegoni-Renda-Sciandra, to appear soon...]

We can obtain many non-trivial examples via an infinitesimal FRT construction!
$\rightsquigarrow$ infinitesimal $\mathcal{R}$-matrices for $\mathrm{GL}_q(2)$, etc...

The role of Hochschild cohomology

Theorem (Ardizzoni-Bottegoni-Sciandra-TW '23)

Let $(H, \mathcal{R}, \chi)$ be a pre-Cartier quasitriangular bialgebra. Then

- i.) χ *is a* **Hochschild** 2-**cocycle**, *i.e.* $\chi_{12} + (\Delta \otimes \mathrm{id})(\chi) = \chi_{23} + (\mathrm{id} \otimes \Delta)(\chi)$.
- ii.) χ *satisfies the* **infinitesimal quantum Yang-Baxter equation**

$$\mathcal{R}_{12}\chi_{12}\mathcal{R}_{13}\mathcal{R}_{23} + \mathcal{R}_{12}\mathcal{R}_{13}\chi_{13}\mathcal{R}_{23} + \mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23}\chi_{23} = \mathcal{R}_{23}\chi_{23}\mathcal{R}_{13}\mathcal{R}_{12} + \mathcal{R}_{23}\mathcal{R}_{13}\chi_{13}\mathcal{R}_{12} + \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}\chi_{12}.$$

This generalizes a result of [Majid '97] for $(U\mathfrak{g}, 1 \otimes 1)$ with $(\mathfrak{g}, [\cdot,\cdot], r)$ quasitriangular Lie bialgebras as before.

If $(H, \mathcal{R}, \chi)$ is a pre-Cartier quasitriangular **Hopf** algebra, define $\gamma := S(\chi^i)\chi_i \in H$, the **Casimir element**, where $\chi = \chi^i \otimes \chi_i$. It follows that γ is central!

Proposition (Ardizzoni-Bottegoni-Sciandra-TW '23)

If $(H, \mathcal{R}, \chi)$ is a **Cartier triangular Hopf** *algebra, then $\chi = b^1(\frac{\gamma}{2})$ is a Hochschild 2-coboundary, where $b^1(h) := 1 \otimes h - \Delta(h) + h \otimes 1$.*

Future project

$\tilde{\mathcal{R}} \in (H \otimes H)[[\hbar]]$ induces $(H, \mathcal{R}, \chi)$ via

$$\tilde{\mathcal{R}} = \mathcal{R}(1 \otimes 1 + \hbar\chi + \mathcal{O}(\hbar^2)). \tag{1}$$

Deformation problem: Given $(H, \mathcal{R}, \chi)$ is there a quasitriangular structure $\tilde{\mathcal{R}}$ on $H[[\hbar]]$ such that (1) holds, i.e. such that $\tilde{\mathcal{R}}$ "quantizes" χ?

- Preliminary results $\tilde{\mathcal{R}} = \mathcal{R}\exp(\hbar\chi)$ for toy examples
- Related to Gerstenhaber-Schack cohomology
- Related to Etingof-Kazhdan quantization of quasitriangular Lie bialgebras

$\rightsquigarrow$ Work in progress with **A.Ardizzoni**, **L.Bottegoni** and **A.Sciandra**.
...recently joined forces with **C.Esposito**, **F.Renda**, **A.Rivezzi** and **J.Schnitzer**.

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Thank you for your attention!

Figure: Wolves attending a conference in Srní, 14.01.2024