Category of Quantum L_∞ Algebras

w/ Branislav Jurčo, Ján Pulmann, arXiv:2401.06110

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Winter School of Geometry and Physics

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- Equivalently, S is a Maurer-Cartan element of $(\mathcal{F}V, \{-.-\}, \hbar\Delta)$ or the algebra over the twisted modular operad $\mathsf{F}(\mathrm{Mod}(\mathcal{C}om))$ [Markl'97].

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Prop: Let U, V be (-1)-symplectic vector spaces.

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Linear Lagrangian Relations

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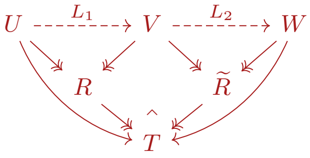
2. $\operatorname{Ker} L_1^T \perp \operatorname{Ker} L_2 \iff$ the composition $L_2 \circ L_1$ coincides with composition of the cospans along pushouts in the category of reductions.



Theorem: Let $L\colon V \twoheadrightarrow R$ be a reduction and S a quantum L_∞ algebra with the quadratic part $S_2^0 \equiv S_{\text{free}}$ non-degenerate on $\operatorname{Ker} L$. Then there exists a unique (up to normalization) **perturbative Gaussian integral**

$$\int_{\operatorname{Ker} L} e^{S_2^0/\hbar}\colon \mathcal{F}V \otimes |V|^{\frac{1}{2}} \longrightarrow \mathcal{F}R \otimes |R|^{\frac{1}{2}}$$

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- The prescription of Wick's lemma—used to prove uniqueness.
- The results of the homological perturbation lemma—used to prove existence.

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$$\mathsf{Hom}(U,V) := \begin{cases} C \subseteq \overline{U} \times V \text{ coisotropic,} \\ f \in \mathcal{F}(C/C^{\perp}) \otimes |C/C^{\perp}|^{\frac{1}{2}}, \end{cases}$$

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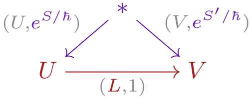
Example: For $C = U \times V$, $\mathcal{F}(C/C^{\perp}) = \mathcal{F}(U \times V)$. Let $U = *$, then $C = V$, $\mathcal{F}(C/C^{\perp}) = \mathcal{F}(V)$ and a quantum L_∞ algebra S defines a morphism by setting $f = e^{S/\hbar}$.

$$* \xrightarrow{(V, e^{S/\hbar})} V$$

For a reduction $L\colon U \twoheadrightarrow V$, the triangle commutes iff $\int_{\mathrm{Ker}\, L} e^{S/\hbar} = e^{S'/\hbar}$.



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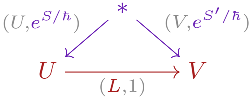

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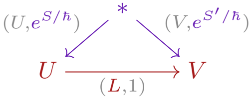

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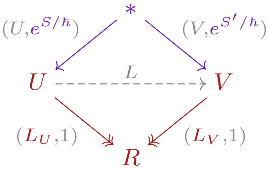

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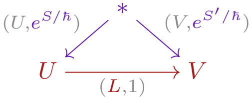

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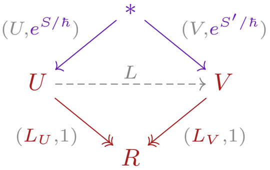

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