# The geometry of Flag manifolds I

## SRNI 45th School

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The slides are available at
`https://irma.math.unistra.fr/~guichard/srni`

# The linear algebra exercise of the day

Let $V$ be a real vector space of dimension $4n+2$ ($n$ is an integer) equipped with a quadratic form $q$ of signature $(2n+1, 2n+1)$. Let $E$ and $F$ be two maximal isotropic subspaces of $V$. This means that $q(v) = 0$ for every $v \in E \cup F$ and $\dim E = \dim F = 2n+1$.

There exists thus an element $g$ in the orthogonal group $\mathrm{O}(V, q)$ such that $g(E) = F$ (Witt's theorem).

## Exercise

If $E$ and $F$ are transverse (that is, if $E \cap F = \{0\}$), then $g \notin \mathrm{SO}(q)$ (that is, $\det(g) = -1$).

# Lie algebra setting

$G$ a semisimple Lie group ; $\mathfrak{g}$ its Lie algebra. For example, $G = \mathrm{O}(p, p+k)$ is the orthogonal group of a quadratic form $q$ of signature $(p, p+k)$ [$p$ and $k$ are positive integers]. For definiteness we will realize $\mathrm{O}(p, p+k)$ as a subgroup of $\mathrm{GL}_{2p+k}(\mathbf{R})$ and $q$ will be the form

$$q(x_1, \ldots, x_{2p+k}) = 2\sum_{i=1}^{p}(-1)^{i+p}x_i x_{2p+k+1-i} - \sum_{i=1}^{k} x_{p+i}^2.$$

$K$ is a maximal compact subgroup of $G$ ; $\mathfrak{k}$ its Lie algebra. One can take $K = G \cap \mathrm{O}(2p+k)$.

A *Cartan subspace* $\mathfrak{a}$ is a maximal (Abelian) subalgebra orthogonal to $\mathfrak{k}$ with respect to the Killing form.

One can take $\mathfrak{a}$ to be the space of matrices of the form

$$\mathrm{diag}(\lambda_1, \ldots, \lambda_p, 0, \ldots, 0, -\lambda_p, \ldots, -\lambda_1),\ (\lambda_1, \ldots, \lambda_p) \in \mathbf{R}^p$$

# Lie algebra setting (continued)

For $\beta \in \mathfrak{a}^*$, set $\mathfrak{g}_\beta = \{X \in \mathfrak{g} \mid [A, X] = \beta(A)X, \forall A \in \mathfrak{a}\}$ and $\Sigma = \{\beta \in \mathfrak{a}^* \smallsetminus \{0\} \mid \mathfrak{g}_\beta \neq 0\}$.
The maps $\varepsilon_i \colon \mathfrak{a} \to \mathbf{R}$ [$i$ varies from 1 to $p$] defined by $\varepsilon_i(\mathrm{diag}(\lambda_1, \ldots, \lambda_p, 0, \ldots, 0, -\lambda_p, \ldots, -\lambda_1)) = \lambda_i$ are linear and form a basis of $\mathfrak{a}^*$. the roots are the $\pm\varepsilon_i \pm \varepsilon_j$ (for $i < j$) and the $\pm\varepsilon_i$.

Choosing $<_{\mathfrak{a}^*}$ a total linear ordering (the lexicographic order), one defines $\Sigma^+ = \{\alpha \in \Sigma \mid 0 <_{\mathfrak{a}^*} \alpha\}$ the positive roots. Here $\varepsilon_i \pm \varepsilon_j$, $i < j$ and $+\varepsilon_i$.

Let $\alpha$ belongs to $\Sigma^+$, when there are $\beta, \gamma$ in $\Sigma^+$ such that $\alpha = \beta + \gamma$, one has $\mathfrak{g}_\alpha = [\mathfrak{g}_\beta, \mathfrak{g}_\gamma]$ and the root $\alpha$ is called *decomposable*, it is called *simple* otherwise. The simple roots are $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ and $\alpha_p = \varepsilon_p$.

Denote $\Delta \subset \Sigma^+$ the set of simple roots.
Every positive root decomposes $\beta = \sum_\Delta n_\alpha \alpha$ where $n_\alpha \geq 0$.

# The Weyl group

It is the automorphism group $W$ of $\Sigma \subset \mathfrak{a}^*$. It is the group of signed permutation matrices, isomorphic to $\{\pm 1\}^p \rtimes S_p$.
For each $\alpha$ in $\Sigma$ there is a unique hyperplane reflection contained in $W$ such that $s_\alpha(\alpha) = -\alpha$. $s_i = s_{\alpha_i}$, $s_p$ changes the sign of the last coordinate and $s_i$ exchanges the coordinates in the indices $i$ and $i+1$.

$W$ is generated by $\{s_\alpha\}_{\alpha \in \Delta}$.
There is a unique element $w_{\max}$ of $W$ sending $\Sigma^+$ to $\Sigma^- = -\Sigma^+ = \Sigma \smallsetminus \Sigma^+$. It is the longest length element.
$w_{\max} = -\operatorname{Id}$.

The map $\iota \colon \alpha \mapsto -w_{\max}(\alpha)$ sends $\Sigma^+$ to $\Sigma^+$ and $\Delta$ to $\Delta$. It is called the *opposition involution*. The opposition involution is trivial.

$W$ is isomorphic to $N_K(\mathfrak{a})/Z_K(\mathfrak{a})$. For $w$ in $W$, we will sometimes denote $\dot{w}$ a representative of $w$ in $N_K(\mathfrak{a})$.

# $\mathfrak{sl}_2$-triples, fundamental weights

Those are triples $(x, y, h)$ in $\mathfrak{g}$ such that $[x, y] = h$, $[h, x] = 2x$

For example $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in $\mathfrak{sl}_2(\mathbf{R})$.

For all $\alpha$ in $\Delta$ we will choose an $\mathfrak{sl}_2$-triple $(x_\alpha, x_{-\alpha}, h_\alpha)$ with $x_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}$.

If $i < p$, one can set $x_i = E_{i,i+1} + E_{2p+k-i,2p+k+1-i}$ and $x_{-i} = {}^t x_i$, and $x_p = E_{p,p+1} + E_{p+1,p+k+1}$, $x_{-p} = {}^t x_p$.

The element $h_\alpha$ does not depends on the choices. The family $\{h_\alpha\}_{\alpha\in\Delta}$ is a basis of $\mathfrak{a}$. The dual basis $\{\omega_\alpha\}_{\alpha\in\Delta}$ of $\mathfrak{a}^*$ is called the *fundamental weights*. $\omega_i = \varepsilon_1 + \cdots + \varepsilon_i$.

Let $\exp\colon \mathfrak{g} \to G$ be the exponential. For every $\alpha$, one can choose $\dot{s}_\alpha = \exp\left(\pi/2(x_\alpha - x_{-\alpha})\right)$ to represent in $N_K(\mathfrak{a})$ the element $s_\alpha$.

# Parabolic subgroups, flag manifolds

- The subspace $\mathfrak{u}_\Delta = \sum_{\beta \in \Sigma^+} \mathfrak{g}_\beta$ is a nilpotent subalgebra generated by $\bigcup_{\alpha \in \Delta} \mathfrak{g}_\alpha$. Similarly $\mathfrak{u}_\Delta^{\mathrm{opp}} = \sum_{\beta \in \Sigma^+} \mathfrak{g}_{-\beta}$.
- For every $\Theta \subset \Delta$ we let $\mathfrak{u}_\Theta$ to be the ideal of $\mathfrak{u}_\Delta$ generated by $\bigcup_{\alpha \in \Theta} \mathfrak{g}_\alpha$. One has $\mathfrak{u}_\Theta = \sum_{\alpha \in \Sigma^+ \smallsetminus \mathrm{Span}(\Delta \smallsetminus \Theta)} \mathfrak{g}_\alpha$. Similarly set $\mathfrak{u}_\Theta^{\mathrm{opp}} = \sum_{\alpha \in \Sigma^+ \smallsetminus \mathrm{Span}(\Delta \smallsetminus \Theta)} \mathfrak{g}_{-\alpha}$.
- The *standard parabolic subgroups* are $P_\Theta = N_G(\mathfrak{u}_\Theta)$, $P_\Theta^{\mathrm{opp}} = N_G(\mathfrak{u}_\Theta^{\mathrm{opp}})$.
- The unipotent radical of $P_\Theta$ (resp. $P_\Theta^{\mathrm{opp}}$) is $U_\Theta = \exp(\mathfrak{u}_\Theta)$ (resp. $U_\Theta^{\mathrm{opp}} = \exp(\mathfrak{u}_\Theta^{\mathrm{opp}})$).
- $L_\Theta = P_\Theta \cap P_\Theta^{\mathrm{opp}}$ is called a *Levi factor*. One has $P_\Theta = U_\Theta \rtimes L_\Theta$.
- $\mathcal{F}_\Theta$ is the space of parabolic groups conjugated to $P_\Theta$ ; $\mathcal{F}_\Theta^{\mathrm{opp}}$ is the space of parabolic groups conjugated to $P_\Theta^{\mathrm{opp}}$. As $P_\Theta^{\mathrm{opp}}$ is conjugated to $P_\Theta$ (by $\dot{w}_{\max}$), $\mathcal{F}_{\iota(\Theta)} = \mathcal{F}_\Theta^{\mathrm{opp}}$.
- As $P_\Theta = N_G(P_\Theta)$, $\mathcal{F}_\Theta \simeq G/P_\Theta$.

# Parabolic subgroups (continued)

For all $i \leq p$, $P_i$ (resp. $P_i^{\mathrm{opp}}$) is the stabilizer of the (isotropic) $i$-dimensional space generated by the $i$ first (resp. last) basis vectors.
$\mathcal{F}_i = \mathcal{F}_i^{\mathrm{opp}}$ is naturally isomorphic to the space of isotropic $i$-planes.
More generally, $\mathcal{F}_{i_1 < \cdots < i_\ell}$ is the space of partial flags $(E_1 \subset \cdots \subset E_\ell)$ with $\dim E_m = i_m$ and $E_\ell$ isotropic.

A pair $(P, Q)$ of parabolic subgroups is *transverse* if it is conjugated to $(P_\Theta, P_\Theta^{\mathrm{opp}})$. This is equivalent to $P \cap Q$ being reductive.
Two isotropic $i$-dimensional space $E$ and $F$ in $\mathcal{F}_i$ are transverse if and only if they are ...transverse! that is $E^{\perp_q} \cap F = 0$.

Lemma

*The map $\mathfrak{u}_\Theta^{\mathrm{opp}} \to \mathcal{F}_\Theta \mid X \mapsto \exp(X) \cdot P_\Theta$ is one-to-one onto the space of elements transverse to $P_\Theta^{\mathrm{opp}}$*

# Embeddings into projective space

Let $\eta = \sum_\Delta k_\alpha \omega_\alpha$ be a dominant weight and let $\tau\colon G \to \mathrm{GL}(V)$ be the associated irreducible representation.
If $\eta = \omega_i$ take $V = \bigwedge^i \mathbf{R}^{2p+k}$.
We denote by $V_\eta$ the eigenspace of $\mathfrak{a}$ (with respect to $\tau$) relative to the eigenvalue $\eta$. This is a line in $V$. Denote by $V_\eta^\circ$ the $\mathfrak{a}$-invariant supplementary hyperplane.

## Lemma

*Let $\Theta = \{\alpha \in \Delta \mid k_\alpha = 0\}$. Then the stabilizer of $V_\eta$ in $G$ is $P_\Theta$, the stabilizer of $V_\eta^\circ$ is $P_\Theta^{\mathrm{opp}}$.*

We can therefore build (one-to-one) maps

$$i_\Theta\colon \mathcal{F}_\Theta \longrightarrow \mathbb{P}(V) \mid g\cdot P_\Theta \longmapsto \tau(g)\cdot V_\eta$$
$$i_\Theta^{\mathrm{opp}}\colon \mathcal{F}_\Theta^{\mathrm{opp}} \longrightarrow \mathbb{P}^*(V) \mid g\cdot P_\Theta^{\mathrm{opp}} \longmapsto \tau(g)\cdot V_\eta^\circ$$

## Lemma

*$(P,Q) \in \mathcal{F}_\Theta \times \mathcal{F}_\Theta^{\mathrm{opp}}$ are transverse if and only if $(i_\Theta(P), i_\Theta^{\mathrm{opp}}(Q)) \in \mathbb{P}(V)\times\mathbb{P}^*(V)$ are transverse.*

# The geometry of Flag manifolds II

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# The linear algebra exercise of the day

Let $V = \mathrm{Sym}^{d-1}\mathbf{R}^2 \simeq \mathbf{R}_{d-1}[X,Y]$ be the space of homogenous polynomials in 2 variables.

$V$ has a natural basis $e_i = Y^{i-1}X^{d-i}$ $(i = 1, \ldots, d)$.

$V$ and $V^*$ bear a natural action of $\mathrm{SL}_2(\mathbf{R})$.

Exercise

$t \mapsto \left\langle \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \cdot e_1^*, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot e_1 \right\rangle$ is a non zero multiple of $t \mapsto t^d$.

# The cross-ratio on the projective line, the collar lemma

The cross-ratio on the projective line $\mathbb{P}^1(\mathbf{R}) = \mathbf{R} \cup \{\infty\}$ is defined by the formula $[x, y, X, Y] = \dfrac{X - x}{X - y}\dfrac{Y - y}{Y - x}$
The normalization is so that $[\infty, 0, 1, t] = t$. This means that $[x, y, X, Y]$ belongs to $[0, 1]$ if $Y$ is between $y$ and $X$, $[x, y, X, Y] \geq 1$ if $Y$ is between $X$ and $x$, etc.

Let $M$ be a complete hyperbolic surface.
Let $\alpha$, $\beta$ be intersecting geodesics on $M$ and $\ell(\alpha)$, $\ell(\beta)$ their lengths.

## Theorem (Collar lemma)

$$\frac{1}{\exp(\ell(\alpha))} + \frac{1}{\exp(\ell(\beta))} \leq 1.$$

# Collar lemma (the proof)

[Two nice drawings should go here]

Let $A$ and $B$ in $\mathrm{SL}_2(\mathbf{R})$ be the holonomies of $\alpha$ and $\beta$ respectively. Those are diagonalizable and thus admit attracting $a^+, b^+$ and repelling $a^-, b^-$ fixed points in $\mathbb{P}^1(\mathbf{R})$.

The relation between the length and the cross-ratio is the following $\exp(\ell(\alpha)) = [a^+, a^-, x, A(x)]$

Magical relation $[a^-, b^+, a^+, A(b^+)] + [a^-, a^+, b^+, A(b^+)] = 1$.

One has $[a^-, a^+, b^+, A(b^+)] = \exp(-\ell(\beta))$.

$$\begin{aligned}
[a^-, b^+, a^+, A(b^+)] &= [a^-, b^-, a^+, A(b^+)][b^-, b^+, a^+, A(b^+)] \\
&\geq [b^-, b^+, a^+, B(a^+)][b^-, b^+, B(a^+), A(b^+)] \\
&\geq [b^-, b^+, a^+, B(a^+)] = \exp(-\ell(\beta))
\end{aligned}$$

# Projective cross-ratios

$V$ a real vector space
On $\mathcal{O}_V = \{(x, y, X, Y) \in \mathbb{P}(V)^2 \times \mathbb{P}^*(V)^2 \mid X \pitchfork y \text{ and } Y \pitchfork x\}$,
we define

$$b_V(x, y, X, Y) = \frac{\langle \dot{X}, \dot{x} \rangle}{\langle \dot{X}, \dot{y} \rangle} \frac{\langle \dot{Y}, \dot{y} \rangle}{\langle \dot{Y}, \dot{x} \rangle}$$

Cocycle relations $b_V(x, y, X, Y) b_V(y, z, X, Y) = b_V(x, z, X, Y)$

For a loxodromic element $A \in \mathrm{GL}(V)$,
$b_V(a^+, a^-, X, A \cdot X) = \lambda_{\max}(A) / \lambda_{\min}(A)$

# Symplectic interpretation

Let $\omega^V$ be the natural symplectic form on $V \times V^* = T^*V$.
The map $\mu\colon V \times V^* \to \mathbf{R} \mid (v, \varphi) \mapsto \langle \varphi, v \rangle$ is a moment for the $\mathbf{R}^*$-action $\lambda \cdot (v, \varphi) = (\lambda v, \lambda^{-1}\varphi)$.

The symplectic reduction $\mu^{-1}(1)/\mathbf{R}^*$ carries a symplectic form $\omega$ and is isomorphic to $\mathcal{U}_V = \{(x, X) \in \mathbb{P}(V) \times \mathbb{P}^*(V) \mid X \pitchfork x\}$.

## Proposition

*Let $f\colon [0,1]^2 \to \mathcal{U}_V$ be a $C^1$ map such that, $\forall t \in [0,1]$, $f(0,t) = (x, *)$, $f(1,t) = (y, *)$ and $\forall s \in [0,1]$, $f(s,0) = (*, X)$, $f(s,1) = (*, Y)$, then*

$$b_V(x, y, X, Y) = \exp\left(\int_{[0,1]^2} f^*\omega\right)$$

# Constructing cross-ratio on flag manifolds

Let $G, \Theta, P_\Theta$, etc. as in lecture 1.
Suppose that $\tau\colon G \to \mathrm{GL}(V)$ is a (continuous) representation such that there exists a one-to-one $\tau$-equivariant map

$$i_\Theta\colon \mathcal{F}_\Theta \longrightarrow \mathbb{P}(V)$$

## Proposition

*Then there is* $i_\Theta^{\mathrm{opp}}\colon \mathcal{F}_\Theta^{\mathrm{opp}} \longrightarrow \mathbb{P}^*(V)$ *(one-to-one, equivariant) with the property that* $P \pitchfork Q \Rightarrow i_\Theta(P) \pitchfork i_\Theta^{\mathrm{opp}}(Q)$.

Define

$$\mathcal{O}_\Theta = \left\{(x, y, X, Y) \in (\mathcal{F}_\Theta)^2 \times (\mathcal{F}_\Theta^{\mathrm{opp}})^2 \mid X \pitchfork y \text{ and } Y \pitchfork x\right\}$$

$$b_\tau(x, y, X, Y) = b_V\left(i_\Theta(x), i_\Theta(y), i_\Theta^{\mathrm{opp}}(X), i_\Theta^{\mathrm{opp}}(Y)\right).$$

# Remembering some decompositions

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a}) \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha$$

$$\mathfrak{p}_\Delta = \mathfrak{a} \oplus \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a}) \oplus \left( \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha \right)$$

$$\mathfrak{p}_\Theta = \left( \bigoplus_{\alpha \in \Sigma^+ \cap \mathrm{Span}(\Delta \smallsetminus \Theta)} \mathfrak{g}_{-\alpha} \right) \oplus \left( \mathfrak{a} \oplus \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a}) \oplus \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha \right)$$

Let $\tau \colon G \to \mathrm{GL}(V)$, $i_\Theta \colon \mathcal{F}_\Theta \to \mathbb{P}(V)$ as above.

The line $L = i_\Theta(P_\Theta)$ is $P_\Theta$-invariant $\Rightarrow$ $L$ is an eigenline for the action of $\mathfrak{a}$, is cancelled by $\mathfrak{u}_\Delta$ and also by $\bigcup_{\alpha \in \Theta} \mathfrak{g}_{-\alpha}$.

The subspace $W = \langle \tau(G) \cdot L \rangle$ is an irreducible representation of $G$ with highest weight space $L$. Can (and will) assume $V = W$.

# Continuing the analysis of $V$

$V = \bigoplus_{\kappa \in P(V)} V_\kappa$, $P(V) \subset \sum_\Delta \mathbf{Z}\omega_\alpha$, denote $\eta$ the weight such that $L = V_\eta$. Then $\eta \in \sum_\Delta \mathbf{N}\omega_\alpha$ and uniquely determines $V$.

$V_\eta$ invariant by $\mathfrak{g}_{-\alpha} \Leftrightarrow \eta(h_\alpha) = 0$.

One then get $\eta \in \sum \alpha \in \Theta \mathbf{N}^* \omega_\alpha$

The hyperplane $V_\eta^\circ = \sum_{\kappa \in P(V) \smallsetminus \{\eta\}} V_\kappa$ is tranverse to $V_\eta$ and is $P_\Theta^{\mathrm{opp}}$-stable.

The maps

$$i_\Theta \colon \mathcal{F}_\Theta \longrightarrow \mathbb{P}(V) \mid g \cdot P_\Theta \longmapsto \tau(g) \cdot V_\eta$$
$$i_\Theta^{\mathrm{opp}} \colon \mathcal{F}_\Theta^{\mathrm{opp}} \longrightarrow \mathbb{P}^*(V) \mid g \cdot P_\Theta^{\mathrm{opp}} \longmapsto \tau(g) \cdot V_\eta^\circ$$

satisfy all the wanted properties.

Denote $b^\eta(x, y, X, Y) = b_V\big(i_\Theta(x), i_\Theta(y), i_\Theta^{\mathrm{opp}}(X), i_\Theta^{\mathrm{opp}}(Y)\big)$

# Naturality properties

**Lemma**

*One has* $b^{\eta_1+\eta_2} = b^{\eta_1} b^{\eta_2}$, $b^{k\eta} = (b^{\eta})^k$.

Proof:

The map $\mathbb{P}(V) \times \mathbb{P}(W) \longrightarrow \mathbb{P}(V \otimes W)$ sends $b_V\, b_W$ to $b_{V \otimes W}$

# Symplectic interpretation (bis)

There is a symplectic form $\omega^\eta$ on $\mathcal{U}_\Theta \subset \mathcal{F}_\Theta \times \mathcal{F}_\Theta^{\mathrm{opp}}$ such that

## Proposition

*Let* $f\colon [0,1]^2 \to \mathcal{U}_\Theta$ *be a* $C^1$ *map such that,* $\forall t \in [0,1]$, $f(0,t) = (x,*)$, $f(1,t) = (y,*)$ *and* $\forall s \in [0,1]$, $f(s,0) = (*,X)$, $f(s,1) = (*,Y)$, *then*

$$b^\eta(x,y,X,Y) = \exp\Big(\int_{[0,1]^2} f^*\omega^\eta\Big)$$

The tangent space at a point $(x,X)$ to $\mathcal{U}_\Theta$ identifies with $\mathfrak{u}_\Theta^{\mathrm{opp}} \times \mathfrak{u}_\Theta$.

The formula is

$$\omega^\eta\big((v_1,v_2),(w_1,w_2)\big) = \eta\big([v_1,w_2] - [v_2,w_1]\big).$$

# The geometry of Flag manifolds III

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# The linear algebra exercise of the day

$$\begin{pmatrix} 1 & & & & & \\ & \ddots & & & tF & \\ & & 1 & & & \\ & & & 1 & & \\ & A & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

$t \in \mathbf{R}$, where $A \in \mathrm{M}_{\ell,k}(\mathbf{R})$ and where $F \in \mathrm{M}_{k,\ell}(\mathbf{R})$ has rank one

## Exercise

There is a unique $t \in \mathbf{R}$ such that this matrix is singular.

## Geometric interpretation:

The first $k$ columns represent an $k$-plane $x$, the last $\ell$ columns represent a $\ell$-plane $y_t$;

The conclusion says that there is a unique $t$ such that $y_t$ is not tranverse to $x$.

# The main result (I)

## Theorem

*This happens in every flag manifold.*

$G$, $\mathcal{F}_\Theta$, $\mathcal{F}_\Theta^{\mathrm{opp}}$, $P_\Theta$, $P_\Theta^{\mathrm{opp}}$, $U_\Theta$, $U_\Theta^{\mathrm{opp}}$, $L_\Theta$, $\mathfrak{p}_\Theta$, $\mathfrak{u}_\Theta$, $\mathfrak{a}$, ...

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a}) \oplus \bigoplus_{\alpha\in\Sigma} \mathfrak{g}_\alpha$$

$\mathfrak{a}_L :=$ the centralizer of $\mathfrak{l}$ in $\mathfrak{a}$; $\mathfrak{a}_L = \bigcap_{\alpha\in\Delta\smallsetminus\Theta} \ker\alpha$.

## Proposition (Kostant, 2010)

*The weight decomposition of $\mathfrak{u}_\Theta$ w.r.t. the action of $\mathfrak{a}_L$ coincides with the decomposition into irreducible $L$-summands:*

$$\mathfrak{u}_\Theta = \bigoplus_{\aleph\in P} \mathfrak{u}_\aleph, \quad P \subset \mathfrak{a}_L^*, \quad [\mathfrak{u}_\aleph, \mathfrak{u}_\beth] = \mathfrak{u}_{\aleph+\beth}.$$

# Photons

$$\mathfrak{u}_\Theta = \bigoplus_{\aleph \in P} \mathfrak{u}_\aleph, \quad P \subset \mathfrak{a}_L^*$$

In fact $P = \{\alpha|_{\mathfrak{a}_L}\}_{\alpha \in \Sigma^+ \smallsetminus \mathrm{Span}(\Delta \smallsetminus \Theta)}$; indecomposable weights $P \smallsetminus (P+P)$ naturally identifies with $\Theta$ [via $\Theta \to P \mid \alpha \mapsto \alpha|_{\mathfrak{a}_L}$]

For every $\alpha \in \Theta$, $\mathfrak{u}_\alpha \supset \mathfrak{u}_\alpha^{\text{high}} = \mathfrak{g}_\alpha$ (w.r.t. the action of $\mathfrak{a}$)

Consider $x_\alpha \in \mathfrak{u}_\alpha^{\text{high}}$

## Definition

$\Phi_\alpha := \overline{\{\exp(tx_\alpha) \cdot P_\Theta^{\text{opp}}\}} \subset \mathcal{F}_\Theta^{\text{opp}}$ is the *$\alpha$-photon* ; An *$\alpha$-photon* is $\Phi = g \cdot \Phi_\alpha$ for some $g \in G$.

## Lemma

*$\Phi_\alpha$ is homogenous under the action of $\mathrm{SL}_2(\mathbf{R})_\alpha$ [the subgroup tangent to $\langle x_\alpha, x_{-\alpha}, h_\alpha \rangle$] and is $\simeq$ to $\mathbb{P}^1(\mathbf{R})$.*

# Properties of Photons

## Lemma

*For all* $x \in \mathcal{F}_\Theta^{\mathrm{opp}}$*, so that* $\mathrm{T}_x\mathcal{F}_\Theta^{\mathrm{opp}} \simeq \mathfrak{u}_\Theta$ *and for all non zero* $v$ *in this tangent space*

- *There is* $\Phi$ *such that* $x \in \Phi$ *and* $v \in \mathrm{T}_x\Phi \iff v \in L_\Theta \cdot \mathfrak{u}_\alpha^{\mathrm{high}} \subset \mathfrak{u}_\alpha \subset \mathfrak{u}_\Theta \simeq \mathrm{T}_x\mathcal{F}_\Theta^{\mathrm{opp}}$.
- *In this case, there is a unique such* $\Phi$*.*

## Remark

$Z_\alpha = \mathbb{P}\big(L_\Theta \cdot x_\alpha\big) \subset \mathbb{P}(\mathfrak{u}_\alpha)$ is closed
$\Rightarrow$ the space of $\alpha$-photons is closed.

# Example(s)

$G = \mathrm{O}(p, p+k)$, $\Delta = \{\alpha_1, \dots, \alpha_p\}$, choose $\Theta = \{\alpha_1, \dots, \alpha_{p-1}\}$
Then
$\mathcal{F}_\Theta = \mathcal{F}_\Theta^{\mathrm{opp}} = \big\{(E_1 \subset \cdots \subset E_{p-1}) \mid \dim E_i = i, E_{p-1} \text{ isotropic}\big\}$.

Fix $x = (E_1, \dots, E_{p-1})$

For every $i < p-1$, there is a unique $\alpha_i$-photon through $x$ :
$\Phi_i = \big\{(F_1, \dots, F_{p-1}) \in \mathcal{F}_\Theta \mid \forall j \neq i, F_j = E_j\big\}$
The isomorphism with the projective line is concrete:
$\Phi_i \to \mathbb{P}(E_{i+1}/E_{i-1}) \mid (F_1, \dots, F_{p-1}) \mapsto F_i/E_{i-1}$

For every isotropic $p$-plane $E_p$ containing $E_{p-1}$,
$\Phi_{p-1} = \big\{(F_1, \dots, F_{p-1}) \in \mathcal{F}_\Theta \mid \forall j \neq p, F_j = E_j, F_{p-1} \subset E_p\big\}$ is a $\alpha_p$-photon through $x$ (and all $\alpha_p$-photon has this form)
$\Phi_{p-1} \to \mathbb{P}(E_p/E_{p-2}) \mid (F_1, \dots, F_{p-1}) \mapsto F_{p-1}/E_{p-2}$

# Photon projection

Define $\mathcal{V}_\Phi = \left\{ x \in \mathcal{F}_\Theta \mid \exists y \in \Phi, x \pitchfork y \right\}$

## Theorem

*For every $x$ in $\mathcal{V}_\Phi$, there is a unique $y$ in $\Phi$ such that $y$ is* not *transverse to $x$. Set $p_\Phi(x) = y$.*
*The map $p_\Phi \colon \mathcal{V}_\Phi \to \Phi$ has connected fibers.*

## Proof.

Up to $G$-action can assume $x = P_\Theta$, $P_\Theta^{\mathrm{opp}} \in \Phi$ and $\Phi = \Phi_\alpha$.
Then one needs to have $y = \dot{s}_\alpha \cdot P_\Theta^{\mathrm{opp}}$.

Let $U = \{\exp(t x_{-\alpha})\} \subset \mathrm{SL}_2(\mathbf{R})_\alpha$, one "sees" that
$\mathcal{V}_y = \{z \in \mathcal{F}_\Theta \mid z \pitchfork y\} \simeq U \times p_{\Phi_\alpha}^{-1}(\dot{s}_\alpha \cdot y)$. □

# Example(s) (continued)

$(E_1, \ldots, E_{p-1}) \in \mathcal{F}_{1,\ldots,p-1}$ [and choose also an isotropic $p$-plane $E_p$ containing $E_{p-1}$ in order to treat the case $i = p-1$ on an equal footing]

$$\Phi_i = \left\{ (F_1, \ldots, F_{p-1}) \in \mathcal{F}_{1,\ldots,p-1} \mid \forall j \neq i, F_j = E_j,\ F_i \subset E_p \right\}$$

$$\mathcal{V}_{\Phi_i} = \left\{ (F_1, \ldots, F_{p-1}) \in \mathcal{F}_{1,\ldots,p-1} \mid \forall j \neq i, F_j \pitchfork E_j \right\}$$

$$p_{\Phi_i} \colon \mathcal{V}_{\Phi_i} \to \Phi_i$$

$$(F_1, \ldots, F_{p-1}) \mapsto \left( \ldots, E_{i-1}, E_{i-1} \oplus F_i^{\perp} \cap E_{i+1}, E_{i+1}, \ldots \right)$$

# The main result (II)

Choose $\eta = \sum_{\alpha\in\Theta} n_\alpha \omega_\alpha$ ($n_\alpha \in \mathbf{N}$) so that $b^\eta$ is defined on $\mathcal{O}_\Theta \subset \mathcal{F}_\Theta \times \mathcal{F}_\Theta \times \mathcal{F}_\Theta^{\mathrm{opp}} \times \mathcal{F}_\Theta^{\mathrm{opp}}$

Fix $\alpha \in \Theta$ and an $\alpha$-photon $\Phi$.

## Theorem

*Let $x, y \in \Phi$. For all $z, w$ in $\mathcal{V}_\Phi$ such that $p_\Phi(z) = p_\Phi(w) \notin \{x, y\}$, then $b^\eta(x, y, z, w) = 1$.*

*Let $x, y \in \Phi$. For all $z, w$ in $\mathcal{V}_\Phi$, with $p_\Phi(z) \neq y$ and $p_\Phi(w) \neq x$, then*

$$b^\eta(x, y, z, w) = [x, y, p_\Phi(z), p_\Phi(w)]^{n_\alpha}$$