A prolegomenon to renormalisation

or a (desperate?) attempt to make the infinite finite

45th WINTER SCHOOL GEOMETRY AND PHYSICS
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University of Potsdam

22-24 January 2025

Aims and scope

- Three lectures based on joint work with various coauthors, in particular Li Guo (Rutgers Univ., Newark) and Bin Zhang (Sichuan Univ., Chengdu),
- that aim to give a mathematical perspective on certain aspects of renormalisation.

Rough definition of renormalisation

Renormalisation comprises a set of techniques derived from quantum field theory, which are used to deal with infinities arising when calculating quantities by modifying their values to compensate for discrepancies.

Disclaimer

These lectures only provide a **prolegomenon** in that we do not claim to explain renormalisation in its full breath. In the language of perturbative quantum field theory, we are only dealing with a finite number of loops.

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1. **Exposition: from regularisation to renormalisation**
   1.1 Various **regularisation** techniques (cut-off, dimensional, zeta and heat-kernel regularisation) underlying renormalisation methods.
   1.2 Their **usage** in number theory, quantum field theory, microlocal analysis and index theory.
2. **Development: algebraic and analytic methods for renormalisation**
   2.1 From **simple** to **multiple** sums or integrals: **sub-divergences**
   2.2 Coombining **coproducts** with dimensional/ regularisation
   2.3 **Analytic regularisation** à la Speer and meromorphic functions
3. **Recapitulation: how locality comes to the rescue. Applications.**
   3.1 The concept of **locality** as a leading thread
   3.2 Meromorphic functions in **several variables** with **linear poles**
   3.3 How **locality** comes into play when "evaluating" them at poles.

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Lecture 1

Exposition: from regularisation to renormalisation

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How can we "extract" a finite part from

- the harmonic sum $\sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$?
- how does this divergent sum relate to the corresponding integral $\int_1^n \frac{1}{x} dx$?

Discrete sum versus integral

They relate via the **Euler-Mascheroni constant** = **Hadamard's finite part**

$$\gamma := \lim_{n \to \infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n \right) = \lim_{n \to \infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \int_1^n \frac{1}{x} dx \right)$$ called cut-off sum in QFT

This follows from the **Euler-Maclaurin** formula for a continuous function f on $[1, +\infty[$

$$\sum_{k=1}^{n} f(k) = \int_1^n f(x)\, dx + \frac{f(n) + f(1)}{2} + \sum_{\ell=2}^{m} \frac{B_\ell}{\ell!} [f^{(\ell-1)}]_1^n + R_m^n(f). \quad (1)$$

Here the B_ℓ's are the **Bernoulli numbers**. Note that for $f(x) = \frac{1}{x}$ we have $f^{(\ell-1)}(x) = (-1)^{\ell-1} (\ell-1)!\, x^{-\ell}$. Here, $R_m^n(f) = \frac{(-1)^{m+1}}{m!} \int_1^n f^{(m)}(x) P_m(x - [x])\, dx$.

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Riemann zeta function

The function (which we would like to evaluate at zero)

$$z \longmapsto \sum_{n=1}^{\infty} \frac{1}{n^{1+z}}$$

is well-defined and holomorphic on the upper half plane $\Re(z) > 0$. It uniquely extends to a **meromorphic function** on $\mathbb{C}$:

$$\zeta(1+\bullet) : z \longmapsto \sum_{n=1}^{\infty} \frac{1}{n^{1+z}} \qquad \left(\sum_{n=1}^{\infty} \text{is called canonical sum} \right)$$

which has a simple pole at zero with $\mathrm{Res}_{z=0}\zeta(1+\bullet) = 1$.

Hadamard (and Euler) versus Riemann (and Riesz)

$$\gamma = \lim_{z \to 0} \left(\zeta(1+z) - \frac{1}{z} \right) \quad \text{called minimal subtraction scheme in QFT.}$$

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$$z \longmapsto \sum_{n=1}^{\infty} \frac{1}{n^{1+z}}$$

is well-defined and holomorphic on the upper half plane $\Re(z) > 0$. It uniquely extends to a **meromorphic function** on $\mathbb{C}$:

$$\zeta(1+\bullet) : z \longmapsto \sum_{n=1}^{\infty} \frac{1}{n^{1+z}} \qquad \left(\sum_{n=1}^{\infty} \text{ is called canonical sum} \right)$$

which has a simple pole at zero with $\mathrm{Res}_{z=0}\zeta(1+\bullet) = 1$.

Hadamard (and Euler) versus Riemann (and Riesz)

$$\gamma = \lim_{z \to 0} \left(\zeta(1+z) - \frac{1}{z} \right)$$ called minimal subtraction scheme in QFT.

Sums and integrals of polyhomogeneous symbols 1

Polyhomogeneous (or classical) symbols

For $U \subset \mathbb{R}^n$ open, a function $(x,\xi) \mapsto \sigma(x,\xi)$ in $C^\infty(T^*U)$ is called a **polyhomogeneous symbol** of order α if it has the following asymptotic behaviour as ξ goes to infinity:

$$\sigma(x,\xi) = \sum_{j=0}^{N} \sigma_{\alpha-j}(x,\xi) + \sigma_{(N)}(x,\xi) \quad \forall\, (x,\xi) \in T^*U. \tag{2}$$

Here, $\sigma_{\alpha-j}$ is (quasi-) positively homogeneous of order $\alpha - j$, $\sigma_{(N)}$ is a symbol of order $r := \Re(\alpha) - N - 1$, namely $\partial_x^\mu \partial_\xi^\nu \sigma(x,\xi)$ is $O(1+|\xi|)^{r-|\nu|}$ uniformly in ξ and in x on compact subsets of U. We then write $\sigma(x,\xi) \sim \sum_{j=0}^\infty \sigma_{\alpha-j}(x,\xi)$.

Examples: symbols constant in x

$(n=1)$ $\sigma(x,\xi) = \chi(\xi)\,\frac{1}{\xi}$ of order-1; $(n \geq 1)$ $\sigma(x,\xi) = \frac{1}{|\xi|^2+1}$ of order-2.

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Regularisation: holomorphic families of classical symbols

$$\mathcal{R} : \sigma \longmapsto \sigma(z) \quad \text{of order } \alpha(z) = \alpha - q\, z, \quad \alpha(0) = \alpha = \mathrm{ord}(\sigma).$$

Cut-off sums and integrals (here of symbols constant in x)

The function (which we would like to evaluate at zero)

$$z \longmapsto \int_0^\infty \sigma(z)(\xi)\, d\xi \quad \text{and} \quad z \longmapsto \sum_{n=0}^\infty \sigma(z)(n)$$

is well-defined and holomorphic on the upper half plane $\Re(\alpha(z)) > 0$. It uniquely extends to a **meromorphic function** on $\mathbb{C}$:

$$\mathfrak{I}(\sigma) : z \longmapsto \int_{\mathbb{R}^n} \sigma(z)(\xi)\, d\xi \quad \text{and (here } n=1) \quad \mathfrak{S}(\sigma) : z \longmapsto \sum_{n=0}^\infty \sigma(z)(n)$$

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The residue of a symbol

$$\mathrm{res}(\sigma) = (2\pi)^{-n} \int_U \int_{|\xi|=1} \sigma_{-n}(x,\xi)\, d_S\xi\, dx.$$

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Two emblematic examples (q=1)

$(n=1)$ $\sigma(x,\xi) = \chi(\xi)\, \frac{1}{\xi} \Longrightarrow \mathrm{res}(\sigma) = 1 \Longrightarrow \mathrm{Res}_{z=0}\, \mathrm{Res}_{z=0}\, \zeta(1+z) = \sum_0^\infty \sigma(z) = 1.$

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The Wodzicki residue of a polyhomogeneous symbol

The residue of a symbol

$$\mathrm{res}(\sigma) = (2\pi)^{-n} \int_U \int_{|\xi|=1} \sigma_{-n}(x,\xi)\, d_S\xi\, dx.$$

The complex versus the Wodzicki residue

The mermorphic functions $\mathfrak{I}(\sigma)$ and $\mathfrak{S}(\sigma)$ on $\mathbb{C}$ have a simple pole at $z = 0$:

$$\mathrm{Res}_{z=0} \sum_0^\infty \sigma(z) = \mathrm{Res}_{z=0} ⨍_{\mathbb{R}^n} \sigma(z) = q\, \mathrm{res}(\sigma).$$

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- M an n-dimensional (Riemannian) smooth closed manifold;
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Characterisation and locality of the Wodzicki residue

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use The Wodzicki residue is the only (modulo a multiplicative factor) trace on $\Psi^{\rm cl}(M,E) \longrightarrow \mathbb{C}$ i.e., the only linear form $L : \Psi^{\rm cl}(M,E) \longrightarrow \mathbb{C}$ such that

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Consequently,

No go theorem

The ordinary trace $\mathrm{Tr} : \Psi^{\mathrm{cl}<-n}(M,E) \to \mathbb{C}$ on operators of order with real part $< -n$ does not linearly extend to a trace on $\Psi^{\rm cl}(M,E)$.

Locality

- Whereas A is a priori only pseudo-local (it preserves the singular support but not necessarily the support),
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Back to the ζ-function: the regularised trace as a Wodzicki residue

Regularisation: $\mathcal{R} : \Psi^{\mathrm{cl}}(M, E) \to \mathbb{C}; \quad A \longmapsto A\, Q^{-z}$ by means of an elliptic differential operator $Q \in \Psi^{cl}(M, E)$ of order $q > 0$ with spectral cut.

Spectral ζ-function

The holomorphic map $z \mapsto \mathrm{Tr}\left(A\, Q^{-z}\right)$ on the half-plane $\Re(z) > \frac{n+a}{q}$ extends to a meromorphic map

$$z \longmapsto \zeta_{A,Q}(z) := \underbrace{\mathrm{TR}\left(A\, Q^{-z}\right)}_{\text{canonical trace}}$$

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Q-regularised trace of a differential operator

A differential operator $\Longrightarrow$ holomorphicity of the map $z \longmapsto \zeta_{A,Q}(z)$ at zero:

$$\mathrm{Tr}^{Q}(A) := \zeta_{A,Q}(0) = -\frac{1}{q} \underbrace{\mathrm{Res}\,(A \log(Q))}_{\text{Wodzicki residue}} \quad (\text{defect formula [S. Scott, S.P. PLMS 2007]})$$

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Regularisation: $\mathcal{R}: \Psi^{\mathrm{cl}}(M, E) \to \mathbb{C}; \quad A \longmapsto A\, Q^{-z}$ by means of an elliptic differential operator $Q \in \Psi^{cl}(M, E)$ of order $q > 0$ with spectral cut.

Spectral ζ-function

The holomorphic map $z \mapsto \mathrm{Tr}\left(A\, Q^{-z}\right)$ on the half-plane $\Re(z) > \frac{n+a}{q}$ extends to a meromorphic map

$$z \longmapsto \zeta_{A,Q}(z) := \underbrace{\mathrm{TR}\left(A\, Q^{-z}\right)}_{\text{canonical trace}}$$

with a simple pole at zero and $\underbrace{\mathrm{Res}_{z=0}\mathrm{TR}\left(A\, Q^{-z}\right)}_{\text{complex residue}} = \frac{1}{q} \underbrace{\mathrm{Res}\,(A)}_{\text{Wodzicki residue}}$.

Q-regularised trace of a differential operator

A differential operator $\Longrightarrow$ holomorphicity of the map $z \longmapsto \zeta_{A,Q}(z)$ at zero:

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The index as a logarithmic Wodzicki residue

Notations

- $\pi : E = E_+ \oplus E_- \longrightarrow M$ a finite rank $\mathbb{Z}_2$-graded Clifford hermitian bundle;
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The index of D_+

$$
\begin{aligned}
\mathrm{ind}(D_+) &:= \dim(\mathrm{Ker}(D_+)) - \dim(\mathrm{Ker}(D_-)) \\
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&\qquad \text{when} \quad \Re(z) >> 0 \\
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1. We introduced various **regularisation** techniques:
   - cut-off regularisation,
   - dimensional regularisation,
   - zeta regularisation

   but not heat-kernel regularisation.
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   - number theory: ζ-functions,
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A prolegomenon to renormalisation

or a (desperate?) attempt to make the infinite finite

45th WINTER SCHOOL GEOMETRY AND PHYSICS
Czech Republic, Srni, 18-25 January 2025

Sylvie Paycha

University of Potsdam

22-24 January 2025

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From simple to multiple sums or integrals
From a single to several variables

From **simple** to **multiple** integrals: Feynman integrals

The Feynman integral for the **one loop graph** G_1 without external momenta reads

$$I(G_1) = \int_{\mathbb{R}^4} \frac{1}{k^2+m^2}\, dk = \int_{\mathbb{R}^4} \sigma(k)\, dk \quad \text{with} \quad \sigma(k) := \frac{1}{k^2+m^2}.$$

The Feynman integral for the **sunset graph** G_2 without external momenta reads

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It is an integral over the **hyperplane** $k_3 = k_1 + k_2$ in $\mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R}^4$:

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with $\sigma_j(k) := \frac{1}{k^2+m_j^2}$, which is a polyhomogeneous symbol of order -2.

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Multiple integrals with affine (resp. linear) constraints

Claim

Feynman integral are multiple integrals of tensor products of symbols on intersections of hyperpanes:

$$I(G) = \int_{\cap H_j \subset (\mathbb{R}^4)^k} \sigma_1 \otimes \cdots \otimes \sigma_k,$$

where $H_j, j \in J$ are affine (resp. linear) hyperplanes.

Two ways of regularising Feynman integrals

$$z \longmapsto I(G)(z) = \int_{\cap H_j \subset (\mathbb{R}^4)^k} \sigma_1(z) \otimes \cdots \otimes \sigma_k(z) \quad \text{(e.g. dimensional regularisation)}$$

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using **analytic** regularisation.

Multiple integrals with affine (resp. linear) constraints

Claim

Feynman integral are multiple integrals of tensor products of symbols on intersections of hyperpanes:

$$I(G) = \int_{\cap H_j \subset (\mathbb{R}^4)^k} \sigma_1 \otimes \cdots \otimes \sigma_k,$$

where $H_j, j \in J$ are affine (resp. linear) hyperplanes.

Two ways of regularising Feynman integrals

$$z \longmapsto I(G)(z) = \int_{\cap H_j \subset (\mathbb{R}^4)^k} \sigma_1(z) \otimes \cdots \otimes \sigma_k(z) \quad \text{(e.g. dimensional regularisation)}$$

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From **simple** to **multiple** sums: Multiple zeta functions

Recall that the zeta function reads

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} \sigma_s \quad \text{with} \quad \sigma_s(x) := \frac{\chi(x)}{x^s} \quad \text{of order } -s_j$$

It generalises to **multiple zeta functions**

$$\zeta(s_1, \cdots, s_k) = \sum_{n_1 > n_2 > \cdots > n_k > 0}^{\infty} \sigma_{s_1}(n_1) \cdots \sigma_{s_k}(n_k),$$

It is a discrete sum over the **half spaces** $0 < x_k < x_{k-1} \cdots < x_1$ in $\mathbb{R}_+^k$:

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$$= \sum_{n_k < n_{k-1} \cdots < n_1} \left(\sigma_{s_1} \otimes \sigma_{s_2} \otimes \cdots \otimes \sigma_{s_k}\right)(n_1, n_2, \cdots, n_k).$$

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Multiple sums with affine (resp. linear) constraints

Sums over interesections of half spaces

Multiple sums of tensor products of symbols with affine constraints:

$$\sum_{\cap H_j^+ \subset \mathbb{R}_+^k} \sigma_1 \otimes \cdots \otimes \sigma_k,$$

$H_j^+, j \in J$ are affine (resp. linear) half spaces delimited by a hyperplane H_j.

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II. Circumventing non multiplicativity: coalgebraic approach

Single parameter regularisations

Non multiplicativity of the finite part

Single parameter regularisation

Let $f(z) = \frac{a_{-1}}{z} + a_0 + a_1\, z + o(z); \quad g(z) = \frac{b_{-1}}{z} + b_0 + b_1\, z + o(z)$, then

$$f(z)\, g(z) = \underbrace{\frac{a_{-1}\, b_{-1}}{z^2} + \frac{a_{-1}\, b_0 + a_0\, b_{-1}}{z}}_{\text{singular part}} + \underbrace{a_0\, b_0 + a_{-1}\, b_1 + a_1\, b_{-1}}_{\mathrm{fp}_{z=0}(f(z)\, g(z))} + O(z).$$

The finite part is not multiplicative

$$\mathrm{fp}_{z=0}\left(f(z)\, g(z)\right) = \mathrm{fp}_{z=0}\left(f(z)\right)\, \mathrm{fp}_{z=0}\left(g(z)\right) + \underbrace{b_1\, \mathrm{Res}_{z=0} f(z) + a_1\, \mathrm{Res}_{z=0} g(z)}_{\text{extra terms}}$$

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The finite part is partially multiplicative... thanks to a locality relation.

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Single parameter regularisation: a coproduct comes to the rescue

Coalgebras

- They are dual-in the category-theoretic sense of reversing arrows- to unital associative algebras. **Turning all arrows around in the axioms of unital associative algebras**, one obtains a $\mathbb{K}$-vector space equipped with a
  - coproduct $\Delta : \mathcal{C} \longrightarrow \mathcal{C} \otimes \mathcal{C}$
  - counit $\epsilon : \mathcal{C} \longrightarrow \mathbb{K}$;

  that obey the axioms of counitarity and coassociativity.
- If $\mathcal{C}$ is equipped with
  - a product $m : \mathcal{C} \otimes \mathcal{C} \longrightarrow \mathcal{C}$
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  both of which obey some compatibility relations with the product and the counit, it is called a Hopf algebra.
- **Examples** are the Hopf algebras of Feynman graphs [Kreimer, Connes and Kreimer],of planar trees [Kreimer, Foissy,..],of convex polyhedral cones [Guo, S.P., Zhamg]....

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  - coproduct $\Delta : \mathcal{C} \longrightarrow \mathcal{C} \otimes \mathcal{C}$
  - counit $\epsilon : \mathcal{C} \longrightarrow \mathbb{K}$;

  that obey the axioms of counitarity and coassociativity.
- If $\mathcal{C}$ is equipped with
  - a product $m : \mathcal{C} \otimes \mathcal{C} \longrightarrow \mathcal{C}$
  - and a unit $u : \mathbb{K} \longrightarrow \mathcal{C}$,

  both of which obey some compatibility relations with the product and the counit, it is called a Hopf algebra.
- **Examples** are the Hopf algebras of Feynman graphs [Kreimer, Connes and Kreimer],of planar trees [Kreimer, Foissy,..],of convex polyhedral cones [Guo, S.P., Zhamg]....

Single parameter regularisation: a coproduct comes to the rescue

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How the coproduct comes to the rescue

Assume that $\mathcal{C}$ is a Hopf algebra.

Building a character

From a character which stems from a single parameter regularisation:

$$\Phi : \mathcal{C} \longrightarrow \mathcal{M}_0(\mathbb{C})$$

($\mathcal{M}_0(\mathbb{C})$ is the space of meromorphic germs at $z = 0$) we want to build a character:

$$\varphi : \mathcal{C} \longrightarrow \mathbb{C}.$$

Warning

$\varphi := \mathrm{fp}_{z=0} \circ \Phi$ does not do the job due to the fact that $\mathrm{fp}_{z=0}$ is not a character.

However, the coproduct "undoes" the products which lead to the extra terms. One can then introduce adequate counterterms to cancel them.

Birkhoff-Hopf factorisation

$\mathcal{M}_0(\mathbb{C}) = \mathcal{M}_0^+(\mathbb{C}) \oplus \mathcal{M}_0^-(\mathbb{C}) \Longrightarrow \Phi = \Phi^+ \star \Phi^-$. Take $\varphi := \mathrm{fp}_{z=0} \circ \Phi^+$.

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III. Circumventing non multiplicativity: Locality

Multiple parameter regularisations

Functions of several variables in QFT

Speer's analytic renormalisation [JMP 1967] revisited

Eugene Speer considers Feynman amplitudes given by the coefficients of the perturbation-series expansion of the S matrix in a Lagrangian field theory (with non zero mass).

Excerpt of Speer's article

In this paper we apply a method of defining divergent quantities which was originated by Riesz and has been used in various contexts by many authors. [....]We find it necessary to consider functions of several complex variables $z_1, \cdots, z_k$, one associated with each line of the Feynman graph.The main difficulty is the extension of the above [Riesz's] treatment of poles to the more complicated singularities which occur in several complex variables...

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Brain teaser

(We assume the poles are at zero)
Speer shows that the divergent expressions lie in the filtered algebra $\mathcal{M}^{\mathrm{Feyn}}(\mathbb{C}^\infty) := \cup_{k=1}^{\infty} \mathcal{M}^{\mathrm{Feyn}}(\mathbb{C}^k)$ consisting of Feynman functions $f : \mathbb{C}^k \to \mathbb{C}$,

$$f = \frac{h(z_1, \cdots, z_k)}{{L_1}^{s_1} \cdots {L_m}^{s_m}}, \quad L_i = \sum_{j \in J_i} z_j, \quad J_i \subset \{1, \cdots, k\}, \ h \text{ holom. at zero.}$$

Questions:

1. How to evaluate f consistently at the poles $z_1 = \cdots = z_k = 0$?
2. What freedom of choice do we have for the evaluator?

Evaluating a fraction with a linear pole at zero

$$f(z_1, z_2) = \frac{z_1 - z_2}{z_1 + z_2}|_{z_1=0, z_2=0} = \begin{cases} 1 \text{ or } -1\,? \\ 0\,? \\ 10000\,? \end{cases}$$

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- $\mathcal{M}_0(\mathbb{C}^k) \ni f = \frac{h(\ell_1, \cdots, \ell_n)}{L_1^{s_1} \cdots L_n^{s_n}}$, h holomorphic germ, $s_i \in \mathbb{Z}_{\geq 0}$,
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- Dependence space $\mathrm{Dep}(f) := \langle \ell_1, \cdots, \ell_m, L_1, \cdots, L_n \rangle$.

Separation of variables: $\perp^Q$

On $\mathcal{M}_0(\mathbb{C}^\infty) = \bigcup_{k \in} \mathcal{M}_0(\mathbb{C}^k)$; $\quad f_1 \perp^Q f_2 :\Longleftrightarrow \mathrm{Dep}(f_1) \perp^Q \mathrm{Dep}(f_2)$.

$\mathcal{M}_0^-(\mathbb{C}^k)$ is the set of polar germs $f = \frac{h}{g}$ with $h \perp^Q g$.

Back to the brain teaser

$\ell := z_1 \perp z_2 =: L \Longrightarrow \frac{z_1}{z_2} \in \mathcal{M}_0^-(\mathbb{C}^2)$

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- Dependence space $\mathrm{Dep}(f) := \langle \ell_1, \cdots, \ell_m, L_1, \cdots, L_n \rangle$.

Separation of variables: $\perp^Q$

On $\mathcal{M}_0(\mathbb{C}^\infty) = \bigcup_{k \in} \mathcal{M}_0(\mathbb{C}^k)$; $\quad f_1 \perp^Q f_2 :\Longleftrightarrow \mathrm{Dep}(f_1) \perp^Q \mathrm{Dep}(f_2)$.

$\mathcal{M}_0^-(\mathbb{C}^k)$ is the set of polar germs $f = \frac{h}{g}$ with $h \perp^Q g$.

Back to the brain teaser

$\ell := z_1 \perp z_2 =: L \Longrightarrow \frac{z_1}{z_2} \in \mathcal{M}_0^-(\mathbb{C}^2)$

$(\ell := z_1 - z_2) \perp (z_1 + z_2 =: L) \Longrightarrow \frac{z_1 - z_2}{z_1 + z_2} \in \mathcal{M}_0^-(\mathbb{C}^2)$.

Multiparameter meromorphic germs

Multiparameter meromorphic germs with linear poles

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Decomposition of $\mathcal{M}_0(\mathbb{C}^k)$

Recall that $\mathcal{M}_0^-(\mathbb{C}^k)$ is the set of polar germs $f = \frac{h}{g}$ with $h \perp^Q g$.

Orthogonal projection [Berline and Vergne 2005, Guo, Zhang, S.P. 2015]

$\perp^Q$ induces a splitting and the induced projection onto the holomorphic part:

$$\mathcal{M}_0(\mathbb{C}^k) = \mathcal{M}_0^+(\mathbb{C}^k) \oplus^Q \mathcal{M}_0^-(\mathbb{C}^k) \quad \text{and} \quad {\pi_+}^Q : \mathcal{M}^\bullet \longrightarrow \mathcal{M}_+,$$

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Question

Assume that $\mathcal{C}$ is an algebra.

Building a character

From a character which stems from a multiple parameter regularisation:

$$\Phi : \mathcal{C} \longrightarrow \mathcal{M}_0(\mathbb{C}^k)$$

($\mathcal{M}_0(\mathbb{C}^k)$ is the space of meromorphic germs at $z = 0$) we want to build a character:

$$\varphi : \mathcal{C} \longrightarrow \mathbb{C}.$$

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Does $\varphi := \mathrm{fp}_{z=0} \circ {\pi_+}^Q \circ \Phi$ define a character?

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Development: algebraic and analytic methods for renormalisation

1. From **simple** to **multiple** sums or integrals: **sub-divergences**.
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A prolegomenon to renormalisation

or a (desperate?) attempt to make the infinite finite

45th WINTER SCHOOL GEOMETRY AND PHYSICS
Czech Republic, Srni, 18-25 January 2025

Sylvie Paycha

University of Potsdam

22-24 January 2025

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Lecture 3

Recapitulation: how locality comes to the rescue.

- The concept of **locality** as a leading thread
- **Locality** on meromorphic functions in **several variables** with **linear poles**
- How **locality** comes into play when "evaluating" them at poles.

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The concept of locality as a leading thread

Locality principle

The principle of locality (or locality principle) states that an object is influenced directly only by its immediate surroundings.

Thus, one can separate events located in different regions of space-time and should be able to measure them independently.

Our aim

- Propose a mathematical framework which encompasses the main features of the locality principle in QFT;
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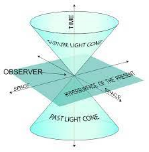
Causal separation

Light cone, past and future

In the Minkowski space $(\mathbb{R}^d, g)$, where $g(x,y) = -x_0 y_0 + \sum_{j=1}^{d-1} x_j y_j$ is the Lorentzian scalar product, there is a notion of "past" and "future":



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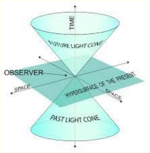

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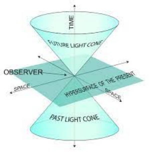

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Locality in axiomatic QFT

The Wightman field $\varphi : \mathcal{S}(\mathbb{R}^d) \to \mathcal{O}(H)$ obeys the locality axiom

$$\mathrm{Supp}(f_1) || \mathrm{Supp}(f_2) \Longrightarrow [\varphi(f_1), \varphi(f_2)] = 0. \tag{1}$$

The (relative) scattering matrix S_f satisfies the locality condition

$$\begin{aligned} \mathrm{Supp}(f_1) || \mathrm{Supp}(f_2) \quad &\Longrightarrow \quad S_f(f_1 + f_2) = S_f(f_1)\, S_f(f_2) \\ &\Longrightarrow \quad [S_f(f_1), S_f(f_2)] = 0. \end{aligned} \tag{2}$$

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Mathematical interpretation

We introduce two binary relations

- on operators:

$$O_1 \top' O_2 :\Longleftrightarrow [O_1, O_2] = 0, \tag{3}$$

- on test functions:

$$f_1 \top f_2 :\Longleftrightarrow \mathrm{Supp}(f_1) \| \mathrm{Supp}(f_2). \tag{4}$$

Interpretation of (1): compatibility with the locality relation

$$f_1 \top f_2 \Longrightarrow \varphi(f_1) \top' \varphi(f_2). \tag{5}$$

Interpretation of (2): partial additivity

$$f_1 \top f_2 \Longrightarrow S_f(f_1 + f_2) = S_f(f_1)\, S_f(f_2). \tag{6}$$

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Locality as a symmetric binary relation

Definition of locality

A **set** is a couple $(X, \top)$ where X is a set and $\top \subseteq X \times X$ is a symmetric relation on X, called locality **relation** (or independence **relation**) of the locality set:

$$x_1 \top x_2 \Longleftrightarrow (x_1, x_2) \in \top, \quad \forall x_1, x_2 \in X.$$

First examples of

- $X \top_\cap Y :\Longleftrightarrow X \cap Y = \emptyset$ on subsets X, Y of a set Z.
- $X \top Y :\Longleftrightarrow X \perp Y$ on subsets X, Y of an euclidean vector space $(V, \perp)$.

(ϵ-)Separation of supports

Let $U \subset \mathbb{R}^n$ be an open subset and $\epsilon \geq 0$. Two functions ϕ, ψ in $\mathcal{D}(U)$ are independent i.e., $\phi \top_\epsilon \psi$ whenever $d\left(\mathrm{Supp}(\phi), \mathrm{Supp}(\psi)\right) > \epsilon$.

For $\epsilon = 0$, this amounts to disjointness of supports, otherwise to ϵ-separation of supports.

Algebraic locality

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$$x_1 \top x_2 \Longleftrightarrow (x_1, x_2) \in \top, \quad \forall x_1, x_2 \in X.$$

First examples of

- $X \top_\cap Y :\Longleftrightarrow X \cap Y = \emptyset$ on subsets X, Y of a set Z.
- $X \top Y :\Longleftrightarrow X \perp Y$ on subsets X, Y of an euclidean vector space $(V, \perp)$.

(ϵ-)Separation of supports

Let $U \subset \mathbb{R}^n$ be an open subset and $\epsilon \geq 0$. Two functions ϕ, ψ in $\mathcal{D}(U)$ are independent i.e., $\phi \top_\epsilon \psi$ whenever $d\left(\mathrm{Supp}(\phi), \mathrm{Supp}(\psi)\right) > \epsilon$.

For $\epsilon = 0$, this amounts to disjointness of supports, otherwise to ϵ-separation of supports.

Further examples

Probability theory: independence of events

Given a probability space $\mathcal{P} := (\Omega, \Sigma, P)$ and two events $A, B \in \Sigma$:

$$A \top B \Longleftrightarrow P(A \cap B) = P(A)\, P(B).$$

Geometry: transversal manifolds

Given two submanifolds L_1 and L_2 of a manifold M:

$$L_1 \top L_2 :\Longleftrightarrow L_1 \pitchfork L_2 \Longleftrightarrow T_x L_1 + T_x L_2 = T_x M \quad \forall x \in L_1 \cap L_2.$$

Number theory: coprime numbers

Given two positive integers m, n in $\mathbb{N}$:

$$m \top n \Longleftrightarrow m \wedge n = 1.$$

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Locality category

Locality structures

- set $X \rightsquigarrow$ locality set $(X, \top)$; the polar set of U is $U^\top := \{x \in X, x \top u \quad \forall u \in U\}$
- semi-group $(G, m_G) \rightsquigarrow$ locality semi-group $(G, m_G, \top)$
  $(U \subset G \Longrightarrow U^\top$ semi-group);
- vector space $(V, +, \cdot) \rightsquigarrow$ locality vector space $(V, +, \cdot, \top)$ $(U \subset V \Longrightarrow U^\top$ vector space);
- algebra $(A, +, \cdot, m_A) \rightsquigarrow$ locality algebra $(A, +, \cdot, m_A, \top)$.

Locality morphisms: $f : (X, \top_X) \to (Y, \top_Y)$

- locality map: $(f \times f)(\top_X) \subset \top_Y$ or equivalently $x_1 \top_X x_2 \Longrightarrow f(x_1) \top_Y f(x_2)$;
- locality semi-group morphism $f : (X, m_X, \top_X) \to (Y, m_Y, \top_Y)$:
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Example and counterexamples

Example: orthogonality

$(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$ equipped with the locality relation $u \top y \Longleftrightarrow \langle u, v\rangle = 0$. $(\mathbb{R}^n, \top, +)$ is a locality semi-group:

$$\langle u, w\rangle = 0 \wedge \langle v, w\rangle = 0 \Longrightarrow \langle u+v, w\rangle = 0.$$

Counterexample

$\mathbb{C}$ equipped with the locality relation $x \top^{\notin\mathbb{Z}} y \Longleftrightarrow x + y \notin \mathbb{Z}$.
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Indeed, for $U = \{1/3\}$, the polar set $U^\top$ is not stable under addition: for $x = y = 1/3 \in U$, we have $x \top y$, $x \in U^\top$ and $y \in U^\top$ but $x + y = 1/3 + 1/3 = 2/3 \notin U^\top$.

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Locality relations are ubiquitous

Local functionals

These are functions (fields) φ of the form $F(\varphi) = \int_M f\left(j_x^k(\varphi)\right)\, dx$ (here $j_x^k(\phi)$ is the k-th jet of ϕ at x): The localised version at φ:

$$F(\varphi+\psi) = F(\varphi) + \int_M f\left(j_x^k(\psi)\right)\, dx \quad \forall \psi. \tag{7}$$

Hammerstein property partial additivity

It is similar to a causality condition on S-matrices of [Epstein, Glaser (1973)], [Bogoliubov, Shirkov (1959))], [Stückelberg (1950, 1951)]

$$\varphi_1 \top \cap \varphi_2 \Longrightarrow F(\varphi_1+\varphi+\varphi_2) = F(\varphi_1+\varphi) - F(\varphi) + F(\varphi+\varphi_2) \quad \forall \varphi. \tag{8}$$

Comparing the two [Brouder, Dang, Laurent-Gengoux, Rejzner (2018)]

Provided the Gâteaux derivative $D_\varphi F$ of F in the direction φ can be represented as a function $\nabla_\varphi F$ such that the map $\varphi \mapsto \nabla_\varphi F$ is smooth, then (7) $\Longleftrightarrow$ (8).

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Local linear forms on pseudodifferential operators

$\Psi_{\rm phg}(M) \supset \Psi^{\Gamma}_{\rm phg}(M)$ polyhomog. pseudodiff. operators on M with order in $\Gamma \subset \mathbb{C}$.

Locality of linear forms

A linear form $\Lambda : \Psi^{\Gamma}_{\rm phg}(M) \longrightarrow \mathbb{C}$ is local if and only if

$$\chi \top\cap \chi' \Longrightarrow \Lambda(\chi\, A\, \chi') = 0 \quad \forall A \in \Psi^{\Gamma}_{\rm phg}(M).$$

A local linear form: Wodzicki residue

$\mathrm{res} : \Psi^{\mathbb{Z}}_{\rm phg}(M) \longrightarrow \mathbb{C}$ defined as an integral of the trace of the homogeneous component of the symbol of degree $-$ dim, is local.

The index as a Wodzicki residue

The index of a differential operator D of Dirac-type is local since

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Separation of wavefront sets

We define two locality relations on on $\mathcal{D}'(U)$, $U \subset \mathbb{R}^n$:

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Distributions can be independent for $\top^{\mathrm{WF}}$ and not for $\top^{\mathrm{sing}}$. We have $v_1 \top^{\mathrm{sing}} v_2 \Longrightarrow v_1 \top^{\mathrm{WF}} v_2$ but not conversely. The wavefront sets of $\nu_1(\phi) := \int_{\mathbb{R}^2} \phi(0, y)\, dy$ and $\nu_2(\phi) := \int_{\mathbb{R}^2} \phi(x, 0)\, dx$ read

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Partial product and locality

Partial product of distributions

(Hörmander) $\nu_1 \top^{\mathrm{WF}} \nu_2 \Rightarrow$ (the product $\nu_1 \cdot \nu_2$ is well-defined.)

Partial product of pseudodifferential operators of non-integer order

We equip $\Psi^{\notin \mathbb{Z}}_{\mathrm{phg}}$ (the canonical trace TR is well defined) with the locality relation
$A_1 \top^{\notin \mathbb{Z}} A_2 :\Leftrightarrow (\mathrm{ord}(A_1) + \mathrm{ord}(A_2) \notin \mathbb{Z}) \Rightarrow (\mathrm{TR}([A_1, A_2]) = 0)$.

Counterexample

Yet $\mathbb{C}$ equipped with the locality relation $x \top^{\notin \mathbb{Z}} y \Longleftrightarrow x + y \notin \mathbb{Z}$.
$(\mathbb{C}, \top, +)$ is NOT a locality semi-group:for $U = \{1/3\}$ we have
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Evaluating meromorphic germs at poles

Locality on meromorphic germs comes to the rescue

Where renormalisation comes into play: Speer's generalised evaluators

Reminder: Meromorphic germs in $\mathcal{M}^{\mathrm{Feyn}}(\mathbb{C}^k)$ have linear poles $L_i = \sum_{j_i \in J_i} j_i$.
Speer introduces evaluators, which consist of a family $\mathcal{E} = \{\mathcal{E}_k, k \in \mathbb{N}\}$ of linear forms $\mathcal{E}_k : \mathcal{M}^{\mathrm{Feyn}}(\mathbb{C}^k) \to \mathbb{C}$, compatible with the filtration, which fulfill the following conditions:

1. **(extend ev_0)** $\mathcal{E}$ is the ordinary evaluation ev_0 at zero on holom. germs;
2. **(partial multiplicativity)** $\mathcal{E}(f_1 \cdot f_2) = \mathcal{E}(f_1) \cdot \mathcal{E}(f_2)$ if f_1 and f_2 depend on different sets (we call them independent) of variables z_i;
3. $\mathcal{E}$ is invariant under permutations of the variables $\mathcal{E}_k \circ \sigma^* = \mathcal{E}_k$ for any $\sigma \in \Sigma_k$, with $\sigma^* f(z_1, \cdots, z_k) := f(z_{\sigma(1)}, \cdots, z_{\sigma(k)})$;
4. (continuity) If $f_n(\vec{z}_k) \cdot L_1^{s_1} \cdots L_m^{s_m} \xrightarrow[n\to\infty]{\text{uniformly}} g(\vec{z}_k)$ as holomorphic germs, then $\mathcal{E}_k(f_n) \underset{n\to\infty}{\longrightarrow} \mathcal{E}_k(\lim_{n\to\infty} f_n)$ (investigated in [Dahmen, Schmeding, S.P. 2023] in the context of Silva spaces).

Drawback: Speer's approach depends on the choice of coordinates $z_1, \cdots, z_k, \cdots$.

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Locality on meromorphic germs with linear poles

Meromorphic germs with linear poles

- $\mathcal{M}_0(\mathbb{C}^k) \ni f = \frac{h(\ell_1, \cdots, \ell_n)}{L_1^{s_1} \cdots L_n^{s_n}}$, h holomorphic germ, $s_i \in \mathbb{Z}_{\geq 0}$,
- $\ell_i : \mathbb{C}^k \to \mathbb{C}$, $L_j : \mathbb{C}^k \to \mathbb{C}$ linear forms.
- Dependence space $\mathrm{Dep}(f) := \langle \ell_1, \cdots, \ell_m, L_1, \cdots, L_n \rangle$.

Locality on meromorphic germs

On $\mathcal{M}_0(\mathbb{C}^\infty) = \bigcup_{k \in \mathbb{N}} \mathcal{M}_0(\mathbb{C}^k)$; $\quad f_1 \perp^Q f_2 :\Longleftrightarrow \mathrm{Dep}(f_1) \perp^Q \mathrm{Dep}(f_2)$.

$\mathcal{M}_0^-(\mathbb{C}^k)$ is the set of polar germs $f = \frac{h}{g}$ with $h \perp^Q g$.

Reminder: Decomposition of meromorphich germs

$\perp^Q$ induces a splitting:

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Meromorphic germs with prescribed types of poles

Data

- $(\mathcal{M}^\bullet(\mathbb{C}^k), \perp^Q)$ an (locality) algebra of meromorphic germs at zero with a prescribed type of poles (e.g. Chen $\subset$ Speer $\subset$ Feynman) in $\mathcal{S}^\bullet$;
- $\mathcal{M}_0^+(\mathbb{C}^k) \subset \mathcal{M}^\bullet(\mathbb{C}^k)$ the algebra of holomorphic germs at zero;
- the evaluation at zero: $\mathrm{ev}_0 : \mathcal{M}_0^+(\mathbb{C}^k) \to \mathbb{C}$, $h \mapsto h(0)$;
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Functions with a prescribed set of poles

A function f in $\mathcal{M}_0^\bullet(\mathbb{C}^k)$ with poles in $\mathcal{S}^\bullet$ decomposes uniquely

$$f = \underbrace{h_0}_{\in \mathcal{M}_0^+(\mathbb{C}^k)} + \underbrace{\sum_{S \in \mathcal{S}^\bullet} \frac{h_S}{S}}_{\in \mathcal{M}_0^-(\mathbb{C}^k)}, \quad h_S \perp^Q S.$$

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Principle of locality: factorisation on independent events

$$\underbrace{a \text{ and } b}_{\in \mathcal{A}} \textit{ independent } \underset{\text{factorisation}}{\Longrightarrow} \text{Meas} \underbrace{(a \vee b)}_{\text{concatenation}} = \text{Meas}(a) \cdot \text{Meas}(b).$$

We consider $\mathcal{M}^\bullet := \mathcal{M}_0(\mathbb{C}^\infty) := \cup_{k=1}^\infty \mathcal{M}_0^\bullet(\mathbb{C}^k)$ consisting of meromorphic functions/germs $f : \mathbb{C}^k \to \mathbb{C}$ with linear poles at zero,

$$f = \frac{h(\vec{z})}{L_1^{s_1}(\vec{z}) \cdots L_m^{s_m}(\vec{z})}, \quad L_i \text{ linear in } \vec{z} := (z_1, \cdots, z_k), \quad h \text{ holom. at zero.}$$

Aim: evaluate meromorphic germs at poles according to the principle of locality: "*two events separated in space can be measured independently.*"

Generalised evaluators

We want to build locality linear forms:

$$\mathcal{E} : \left(\mathcal{M}^\bullet, \perp^Q\right) \longrightarrow \mathbb{C}, \qquad f \perp^Q g \Longrightarrow \mathcal{E}(f \cdot g) = \mathcal{E}(f) \cdot \mathcal{E}(g).$$

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Back to the locality principle in QFT

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A locality Galois type group

Where we stand

- $(\mathcal{M}^\bullet, \perp^Q)$ an (locality) algebra of meromorphic germs at zero with a prescribed type of poles (e.g. Chen $\subset$ Speer $\subset$ Feynman);
- $\mathcal{M}_+ \subset \mathcal{M}^\bullet$ the algebra of holomorphic germs at zero;
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- $\mathcal{M}_-^{\bullet Q}$ is generated by polar germs $f = \frac{h}{g}$ with $h \perp^Q g$.

Locality projection

$\perp^Q$ induces a locality projection onto the holomorphic part:

$$\mathcal{M}^\bullet = \mathcal{M}_+ \oplus^Q \mathcal{M}_-^{\bullet Q} \Longrightarrow \pi_+{}^Q : \mathcal{M}^\bullet \longrightarrow \mathcal{M}_+ \quad \text{is a locality projection.}$$

Definition

$\mathrm{Gal}^Q(\mathcal{M}^\bullet/\mathcal{M}_+)$ is the **Galois geoup** *of (locality) isomorphisms of $(\mathcal{M}^\bullet, \perp^Q)$ that leave holomorphic germs invariant.*

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Classification of locality evaluators

Theorem [Guo, S.P., Zhang, CMP 2024]

Definition

A locality evaluator at zero $\mathcal{E}:\mathcal{M}^\bullet\longrightarrow\mathbb{C}$ is a linear form, which i) extends the ordinary evaluation ev_0 at zero, and ii) factorises on independent germs (i.e., it is a locality character):

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An emblematic evaluator   Minimal subtraction scheme

$\mathcal{E}^{\mathrm{MS}}:\quad \mathcal{M}^\bullet\overset{\pi_+{}^Q}{\longrightarrow}\mathcal{M}_+\overset{\mathrm{ev}_0}{\longrightarrow}\mathbb{C}$ is a locality evaluator.

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Main theorem: A classification of locality evaluators

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THANK YOU FOR YOUR ATTENTION!

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