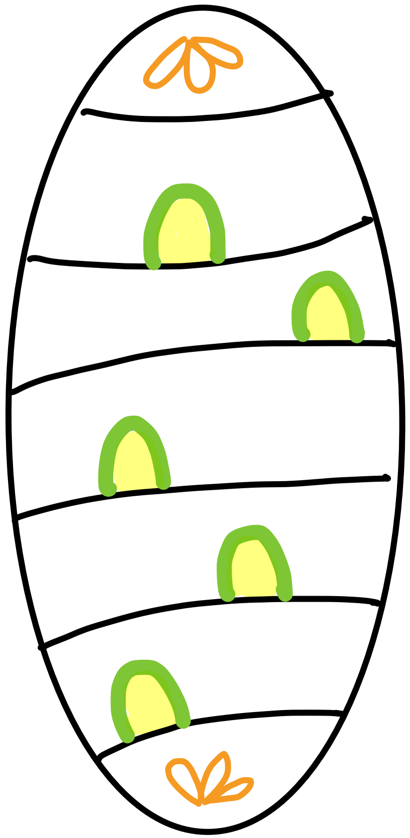
SRNI - LECTURE 1

INTRODUCTION

SUBMANIFOLDS OF CONTACT STRUCTURES

- CONTACT STRUCTURES
- (1D) LEGENDRIAN & TRANSVERSE KNOTS, LEGENDRIAN GRAPHS
- (2D) CONVEX SURFACES
- NEIGHBOURHOOD THEOREMS
- TIGHT & OVERTWISTED CONTACT STRUCTURES

THE GIROUX CORRESPONDENCE VIA CONVEX SURFACES



VERA VÉRTESI

JOINT WORK WITH JOAN LICATA

UNIVERSITY OF VIENNA

INTRODUCTION

1973 VINKELNKEMPER: FIRST USED THE WORD "OPEN BOOK DECOMPOSITION"

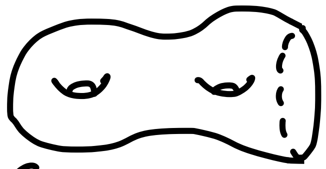
BUT IT WAS ALREADY KNOWN & STUDIED UNDER DIFFERENT NAMES:

- GLOBAL POINCARÉ-BIRKHOFF SECTION
- RELATIVE MAPPING TORUS
- LEFSHETZ/MILNOR FIBRATION
- FIBERED LINKS
- SPINNABLE STRUCTURES

DEF: OPEN BOOK: (S, φ)

$\varphi: S \circlearrowleft$ HOMEOMORPHISM : MONODROMY

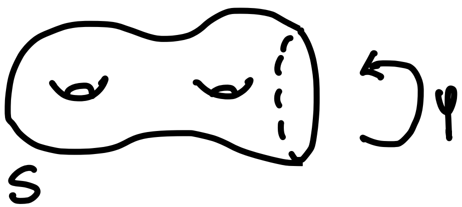
SURFACE WITH BOUNDARY



S

INTRODUCTION

OPEN BOOKS — **CONTACT STRUCTURES**



$$\xrightarrow[\text{WINKELNKEMPER}]{\text{THURSTON}}$$

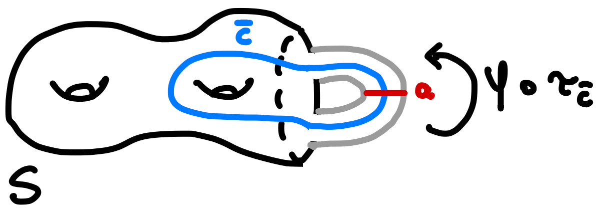
(M^3, ξ)

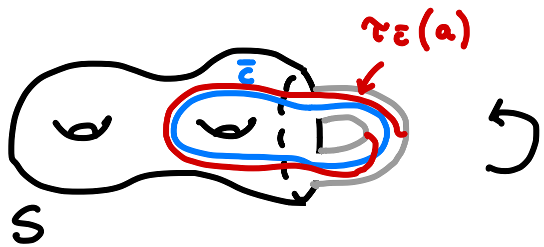
INTRODUCTION

OPEN BOOKS $\xrightarrow[\text{VINKENKEMPLER}]{\text{THURSTON}}$ **CONTACT STRUCTURES**

(M^3, ξ)

POSITIVE STABILISATION



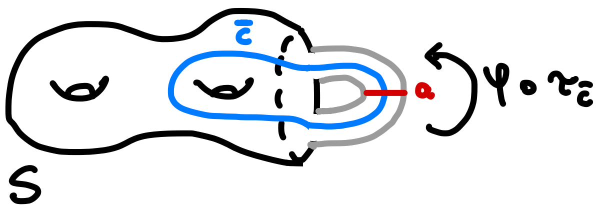


INTRODUCTION

OPEN BOOKS $\qquad\qquad$ **CONTACT STRUCTURES**

$$\xrightarrow[\text{VINKENKEMPLER}]{\text{THURSTON}} \quad (M^3, \xi)$$

$\downarrow$ POSITIVE STABILISATION



$$(S, h) \xleftarrow{\text{GIROUX}} (M^3, \xi)$$

GIROUX CORRESPONDENCE

$$\text{OPEN BOOKS} \,/\, \text{POSITIVE STABILISATION} \longleftrightarrow \text{CONTACT STRUCTURES} \,/\, \text{CONTACTOMORPHISM}$$

INTRODUCTION

GIROUX CORRESPONDENCE

OPEN BOOKS / POSITIVE STABILISATION $\longleftrightarrow$ CONTACT STRUCTURES / CONTACTOMORPHISM

GC HAS BEEN EXTENSIVELY USED TO PROVE THNS ABOUT CTCT 3-MFDS

2000 THE ORIGINAL PROOF OF GIROUX WAS INCOMPLETE

(MASSOT WROTE DOWN A COMPLETE PROOF BUT DIDN'T PUBLISH IT)

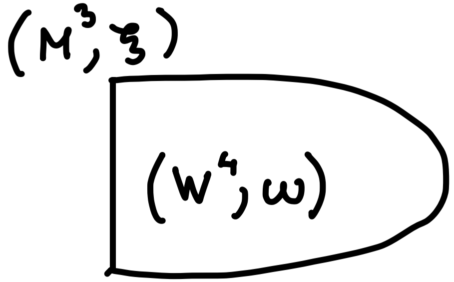
2023 BREEN-HONDA-HUANG: PROOF OF THE GIROUX CORRESPONDENCE FOR CONTACT STRUCTURES IN ANY ODD DIMENSIONS

2023 LICATA-V.: PROOF OF THE GIROUX CORRESPONDENCE FOR TIGHT CONTACT 3-MANIFOLDS (INDEPENDENT)

2024 LICATA-V.: EXTENDED OUR PROOF TO WORK FOR ANY CONTACT 3-MANIFOLD

APPLICATIONS

IN CONTACT TOPOLOGY



→ FILLABILITY

GIROUX : TOPOLOGICAL DESCRIPTION OF STEIN-FILLABLE CONTACT 3-MANIFOLDS

ELIASHBERG, ETNYRE : ANY WEAK SYMPLECTIC FILLING OF A CONTACT 3-MANIFOLD CAN BE EMBEDDED INTO A CLOSED SYMPLECTIC MANIFOLD

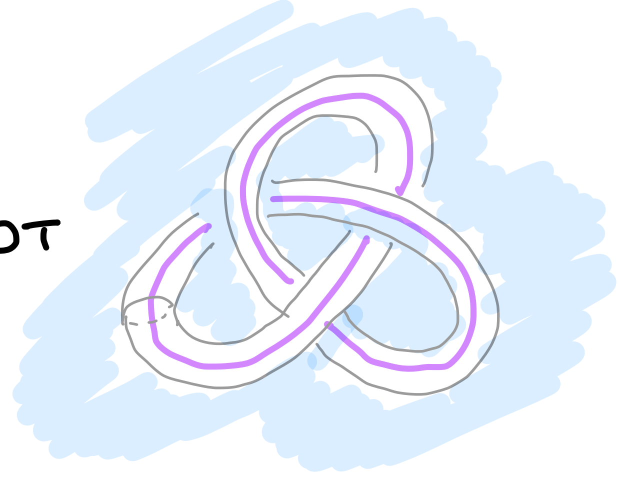
→ CONTACT SURGERY

WAND: CONTACT SURGERY PRESERVES TIGHTNESS

KEGEL- STENHENDE - V - ZUDDAS : CLASSIFICATION OF LEGENDRIAN SURGERY DIAGRAMS DESCRIBING THE SAME CONTACT MANIFOLD

APPLICATIONS

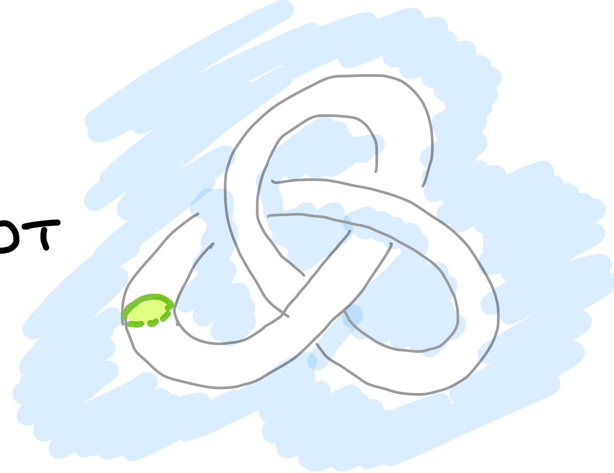
SURGERY: REMOVE NEIGHBOURHOOD OF A KNOT

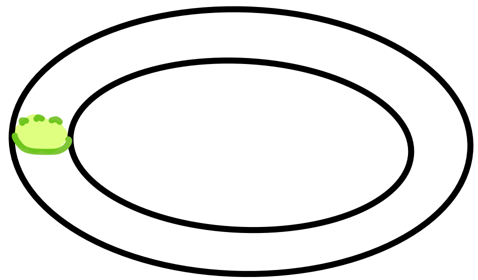


APPLICATIONS

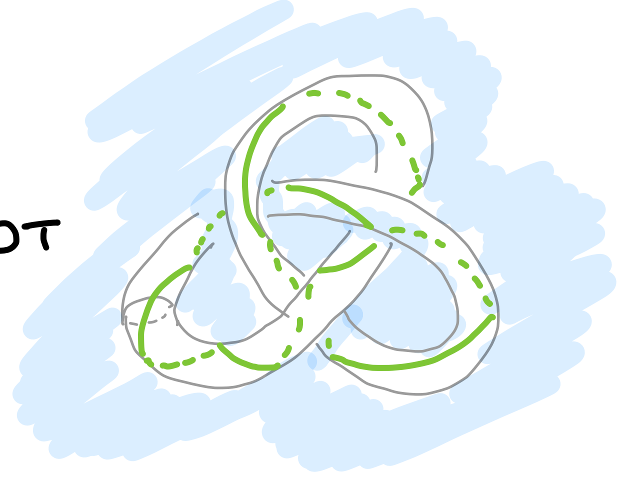
SURGERY: REMOVE NEIGHBOURHOOD OF A KNOT

& GLUE BACK A $D^2 \times S^1$ DIFFERENTLY





APPLICATIONS



SURGERY: REMOVE NEIGHBOURHOOD OF A KNOT

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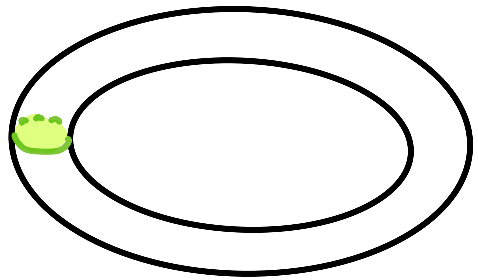

TOPOLOGY

KRONHEIMER - MROWKA: EVERY NONTRIVIAL KNOT HAS PROPERTY P

OZSVÁTH - SZABÓ: THE UNKNOT, TREFOIL & FIGURE-EIGHT KNOT ARE CHARACTERISED BY THEIR SURGERIES

OZSVÁTH - SZABÓ: THE THURSTON NORM IS DETERMINED BY HEEGAARD FLOER HOMOLOGY

GIROUX - GOODMAN: INDUCTIVE CONSTRUCTION OF FIBERED KNOTS IN S^3

Plan of Talks

- LECTURE 1: SUBMANIFOLDS OF CONTACT STRUCTURES
- LECTURE 2: DESCRIBING CONTACT STRUCTURES
- LECTURE 3: PROOF OF GIROUX CORRESPONDENCE

PLAN OF TALKS

- LECTURE 1: SUBMANIFOLDS OF CONTACT STRUCTURES
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PLAN OF TALKS

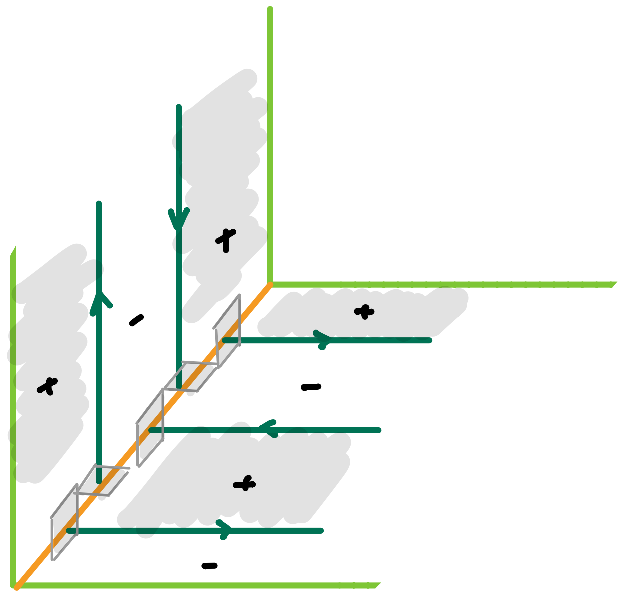
- LECTURE 1: SUBMANIFOLDS OF CONTACT STRUCTURES
- LECTURE 2: DESCRIBING CONTACT STRUCTURES
  - CONTACT CELL DECOMPOSITIONS
  - CONVEX SURFACE THEORY - BYPASSES
  - CONTACT HEEGAARD SPLITTINGS (PROOF OF EXISTENCE)
  - OPEN BOOK DECOMPOSITIONS
  - OPEN BOOK DECOMPOSITIONS & CONTACT HEEGAARD SPLITTINGS
- LECTURE 3: PROOF OF GIROUX CORRESPONDENCE

PLAN OF TALKS

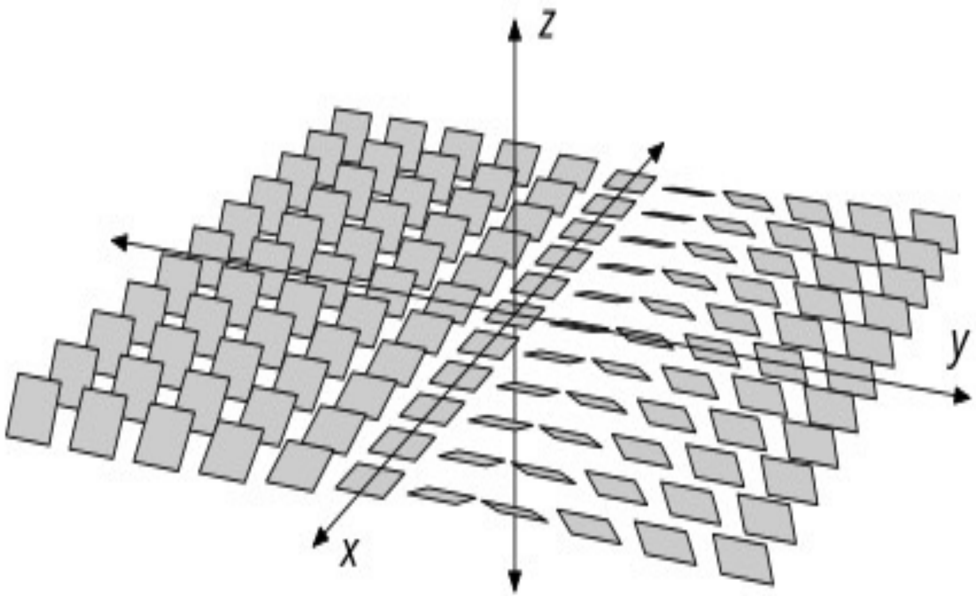
- LECTURE 1: SUBMANIFOLDS OF CONTACT STRUCTURES
- LECTURE 2: DESCRIBING CONTACT STRUCTURES
- LECTURE 3: PROOF OF GIROUX CORRESPONDENCE
  - STABILISATION
  - STATEMENT OF GIROUX CORRESPONDENCE
  - IDEA OF PROOF
  - FURTHER DIRECTIONS

LECTURE 1

SUBMANIFOLDS IN CONTACT STRUCTURES



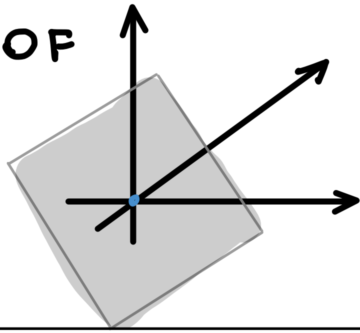
CONTACT STRUCTURES



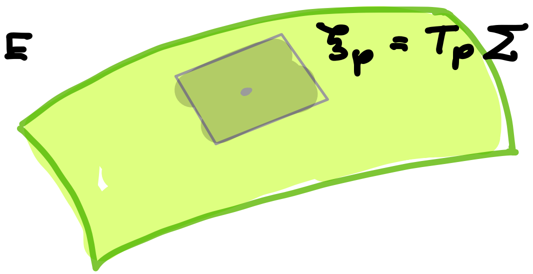
Contact Structures

DEF: A **CONTACT STRUCTURE** ON A CLOSED, ORIENTED SMOOTH 3-MANIFOLD M^3 IS A TOTALLY NONINTEGRABLE 2-PLANE-DISTRIBUTION $\xi \subset TM$

SMOOTH CHOICE OF $\xi_p^2 \subset T_pM \quad \forall p$



NOT TANGENT TO ANY SURFACE $\Sigma \subseteq M$



$\Updownarrow$ FROBENIUS

$\alpha \wedge d\alpha > 0$

LOCALLY: $\xi = \ker \alpha \qquad \alpha \in \Omega^1(M)$

COORIENTED CONTACT STRUCTURE: GLOBAL α

CONTACT STRUCTURES

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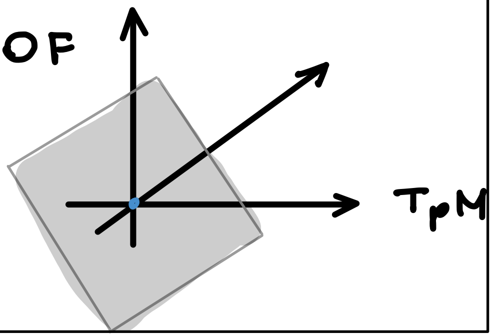

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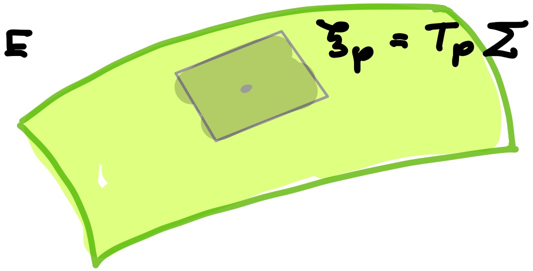


$\Updownarrow$ FROBENIUS

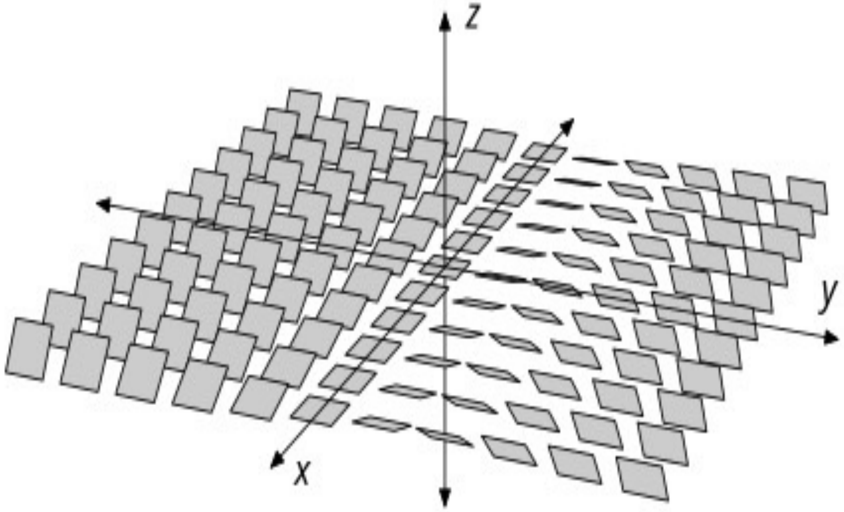
$\alpha \wedge d\alpha > 0$

LOCALLY: $\xi = \ker \alpha \qquad \alpha \in \Omega^1(M)$

COORIENTED CONTACT STRUCTURE: GLOBAL α

DARBOUX THM: LOCALLY ANY CONTACT STRUCTURE IS CONTACTOMORPHIC TO $\mathbb{R}^3$, $\xi_{st} = \ker(dz - y\,dx)$

DIFFEOMORPHISM THAT CARRIES ξ TO ξ'



EQUIVALENCE OF CONTACT STRUCTURES

(M,ξ) & (M',ξ') CONTACT STRUCTURES

- **CONTACTOMORPHISM**: $(M,\xi) \cong (M',\xi')$ IF $\exists$ DIFFEOMORPHISM $\phi : M \to M'$ THAT CARRIES ξ TO ξ' : $\phi_* \xi = \xi'$

WHEN $\boxed{M = M'}$

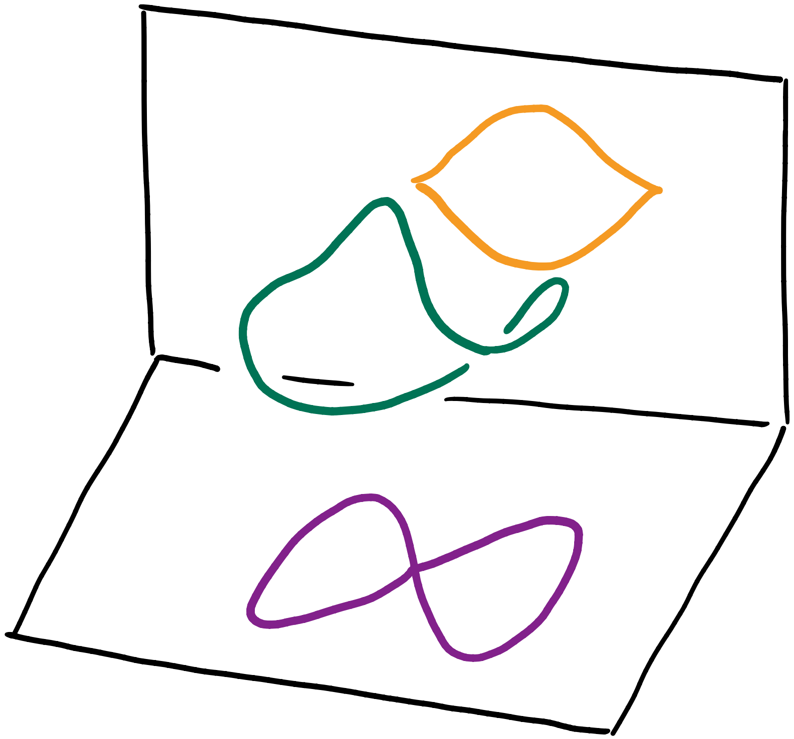
- **HOMOTOPY**: $\xi \simeq \xi'$ IF $\exists$ 1-PARAMETER FAMILY OF CONTACT STRUCTURES $(\xi_t)_{t\in[0,1]}$ ON M WITH $\xi = \xi_0$ & $\xi' = \xi_1$

- **ISOTOPY**: $\xi \approx \xi'$ IF $\exists$ 1-PARAMETER FAMILY OF SELF-DIFFEOMORPHISM $(\phi_t)_{t\in[0,1]}$ OF M WITH
  - $\phi_0 = \mathrm{Id}$ &
  - $\xi' = (\phi_1)_* \xi$

THM (GRAY STABILITY): "HOMOTOPY = ISOTOPY"

ANY HOMOTOPY $(\xi_t)_{t\in[0,1]}$ OF CONTACT STRUCTURES IS INDUCED BY AN ISOTOPY $(\phi_t)_{t\in[0,1]}$:
- $\phi_0 = \mathrm{Id}$ &
- $\xi_t = (\phi_t)_* \xi_0$

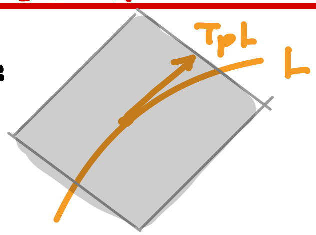
1-DIM: KNOTS

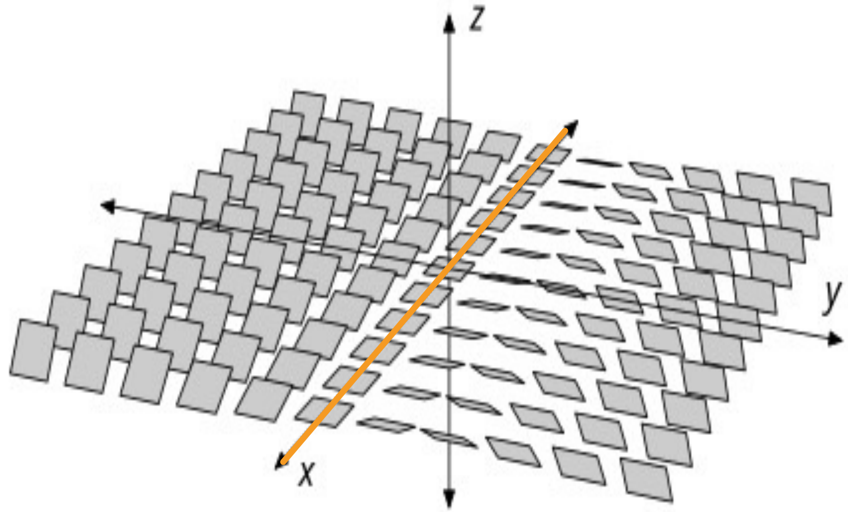


Knots in Contact Structures

DEF: $L^1 \subset M$ is a **LEGENDRIAN KNOT**

IF $T_pL \subset \xi_p \quad \forall p$:





$L = \{y = z = 0\}$

Knots in Contact Structures

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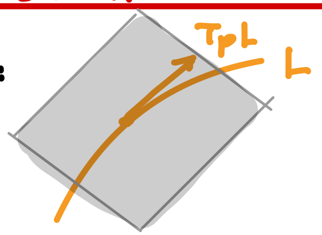
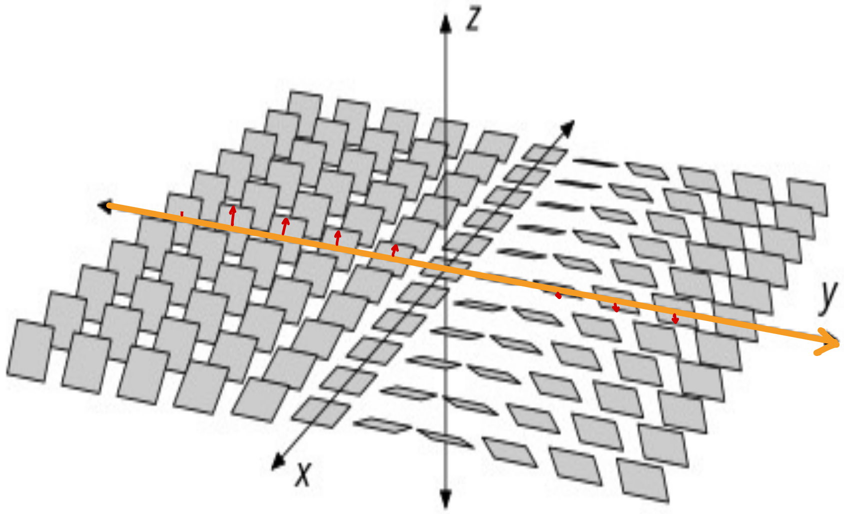
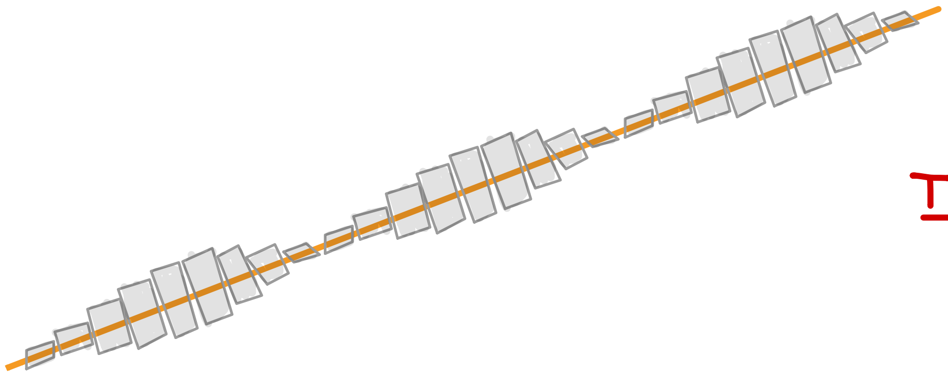




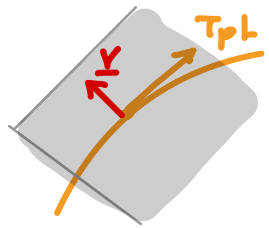
$L = \{x = z = 0\}$

MOTTO: THE CONTACT STRUCTURE ALWAYS "ROTATES" ALONG LEGENDRIANS



THURSTON-BENNEQUIN FRAMING:

PUSH L IN THE DIRECTION OF $\underline{v}$ WHERE $\underline{v}_p \perp T_pL$ & $\underline{v}_p \in \xi_p$



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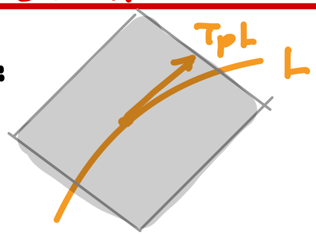

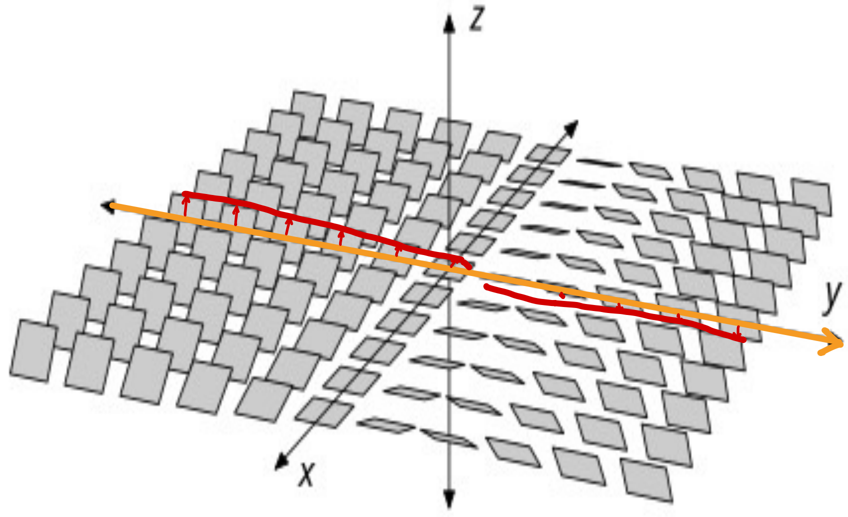

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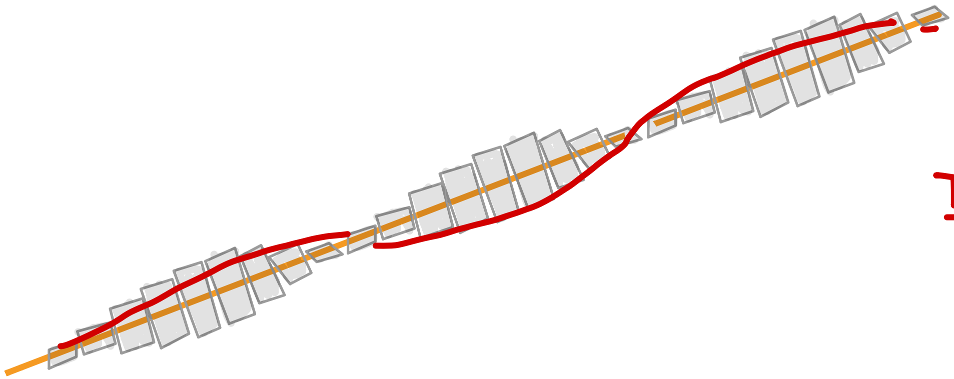

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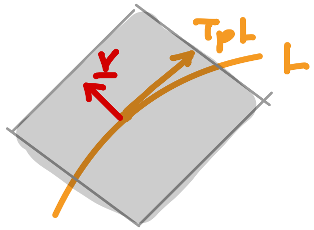
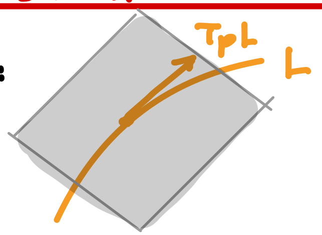
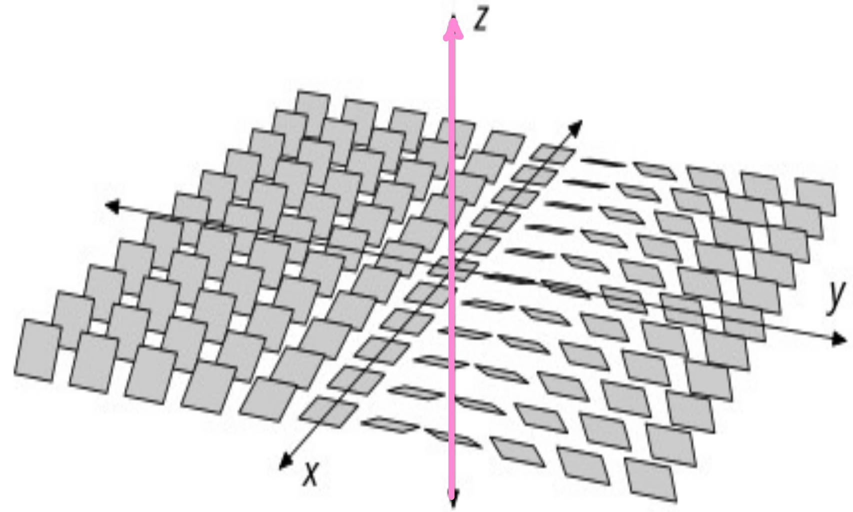


Knots in Contact Structures

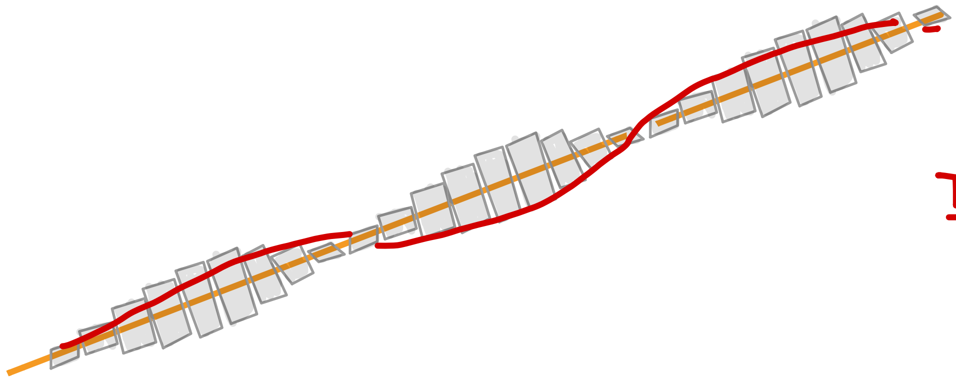
DEF: $L^1 \subset M$ is a **LEGENDRIAN KNOT** if $T_pL \subset \xi_p \ \ \forall p$:





$T = \{x = y = 0\}$

MOTTO: THE CONTACT STRUCTURE ALWAYS "ROTATES" ALONG LEGENDRIANS



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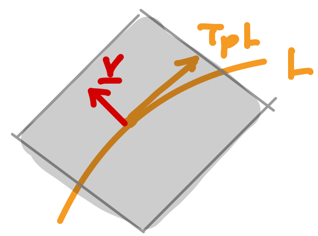
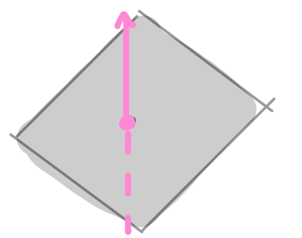


DEF: $T^1 \subset M$ is a **TRANSVERSE KNOT** if $T_pT \pitchfork \xi_p \ \ \forall p$:
$\alpha(T_pT) > 0$

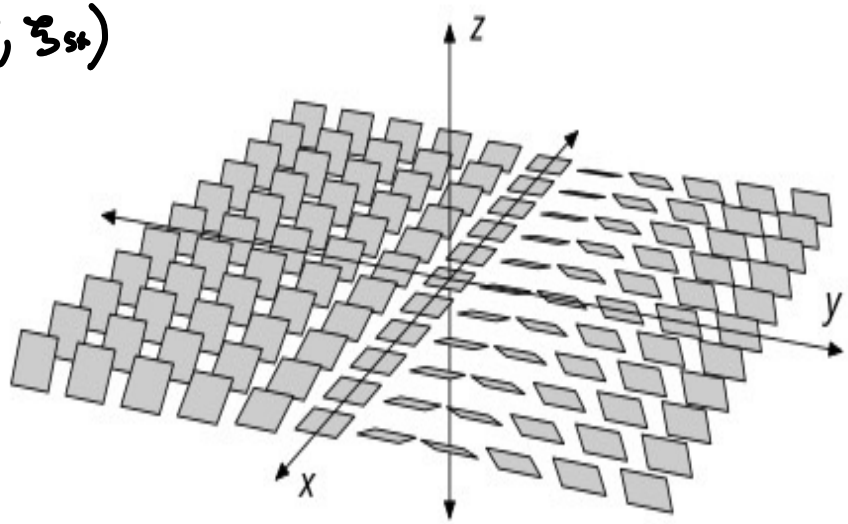


LEGENDRIAN APPROXIMATION

THM: ANY KNOT $K \hookrightarrow (M, \xi)$ CAN BE C^0-APPROXIMATED BY A LEGENDRIAN KNOT

IDEA OF PROOF: ENOUGH TO APPROXIMATE LOCALLY & BY DARBOUX THM WE CAN WORK IN $(\mathbb{R}^3, \xi_{st})$

$$\xi_{st} = \ker(dz - y\,dx)$$



Legendrian Approximation

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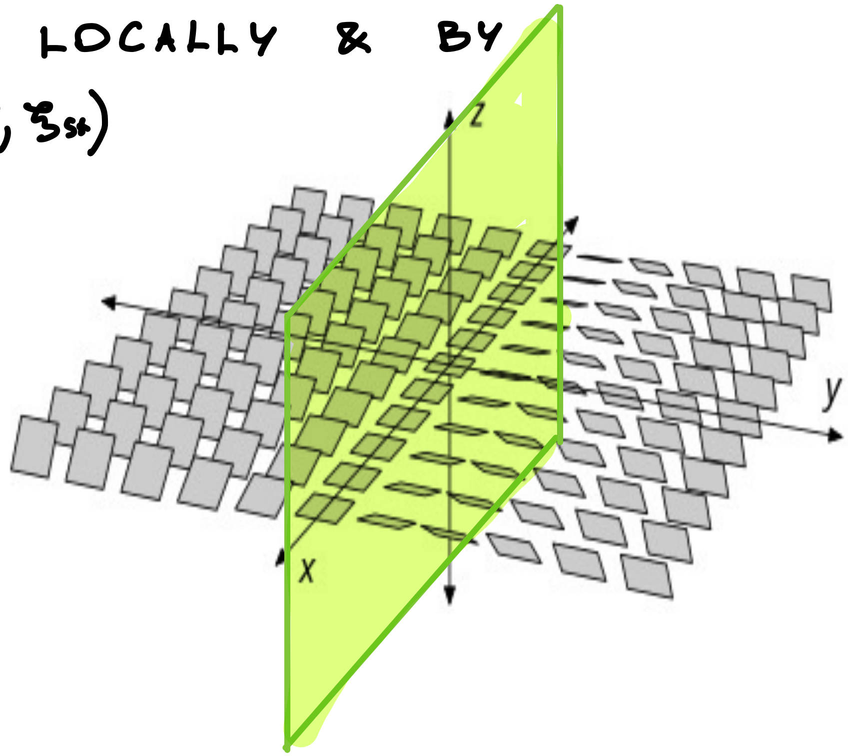
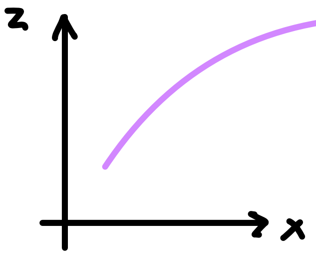


$$\xi_{st} = \ker(dz - y\,dx) \longleftrightarrow \text{"}\, y = \frac{dz}{dx} \text{"}$$

$\boxed{\text{WE CAN READ OFF } y\text{-COORDINATE FROM THE PROJECTION TO } (x,z)\text{-PLANE}}$

- PROJECT K TO THE (x,z)-PLANE

$(x(t), z(t), y(t))$



Legendrian Approximation

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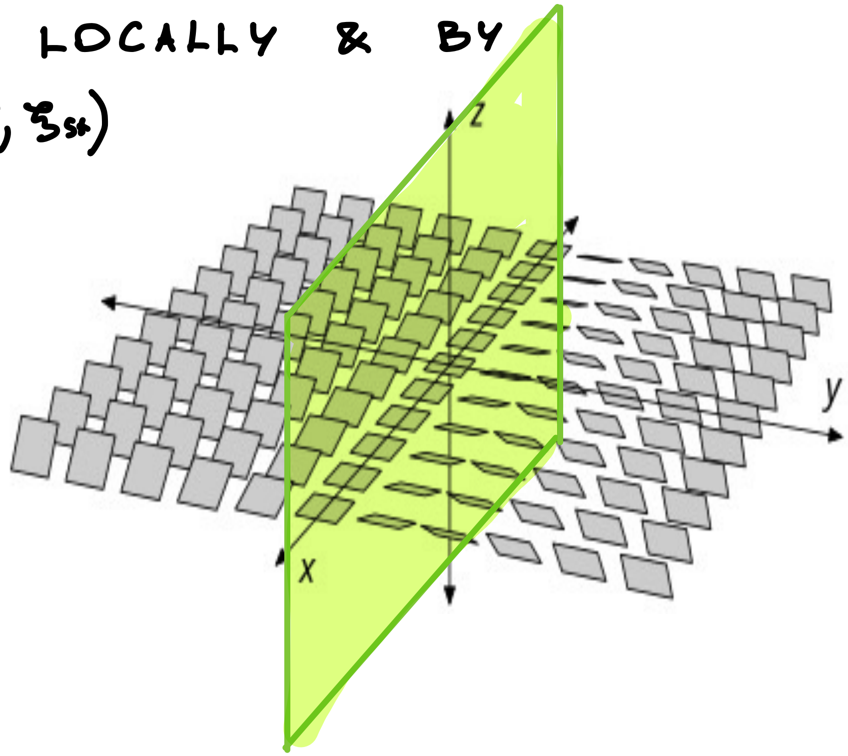

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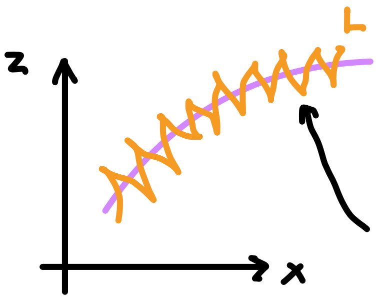
- PROJECT K TO THE (x,z)-PLANE



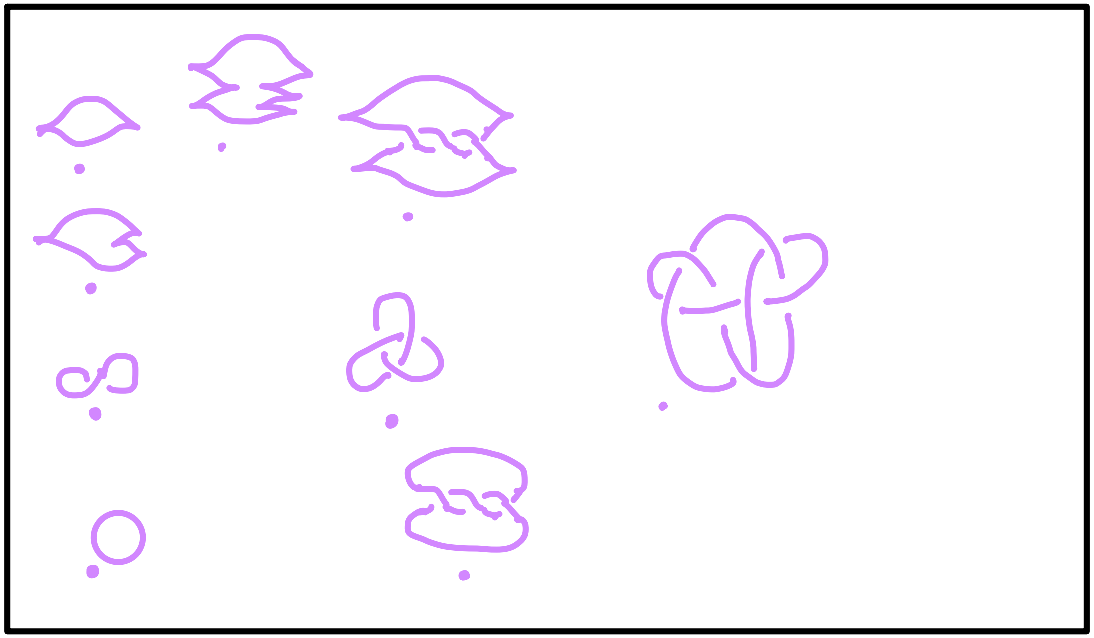
$(x(t), z(t), y(t))$

SLOPE $\frac{dz}{dx}$ "CLOSE" TO $y(t)$ $\Rightarrow$ L IS C^0-CLOSE TO K

COR: ANY SMOOTH KNOT CAN BE REPRESENTED BY A LEGENDRIAN KNOT

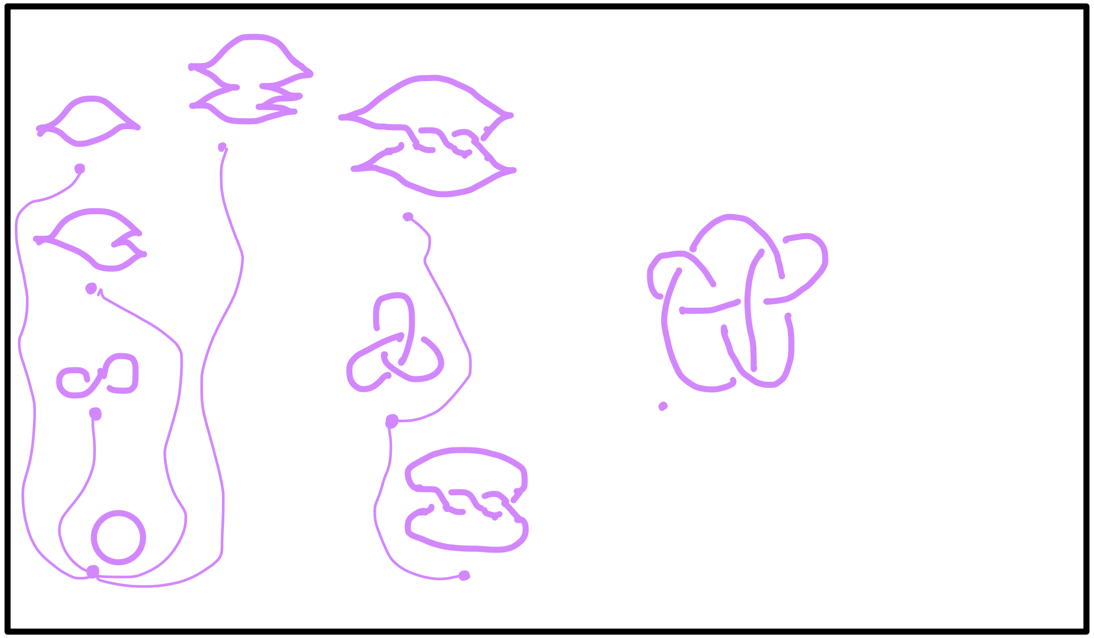
Isotopies of Knots

SMOOTH KNOTS



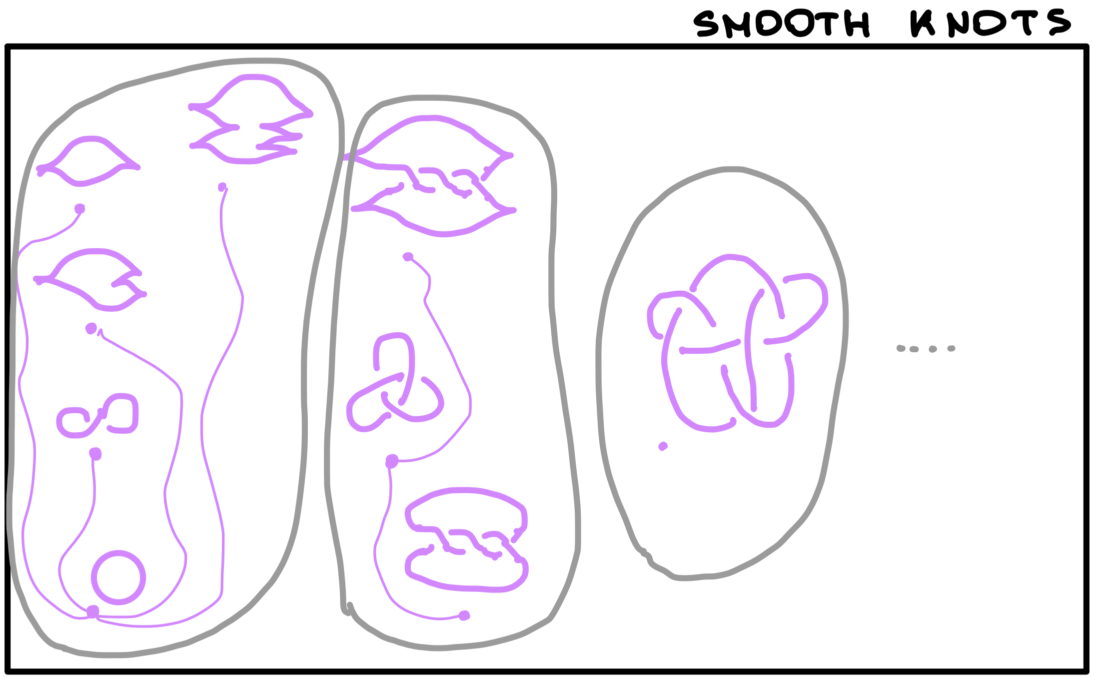
ISOTOPIES OF KNOTS

SMOOTH KNOTS



ISOTOPY: PATH IN THE SPACE OF KNOTS

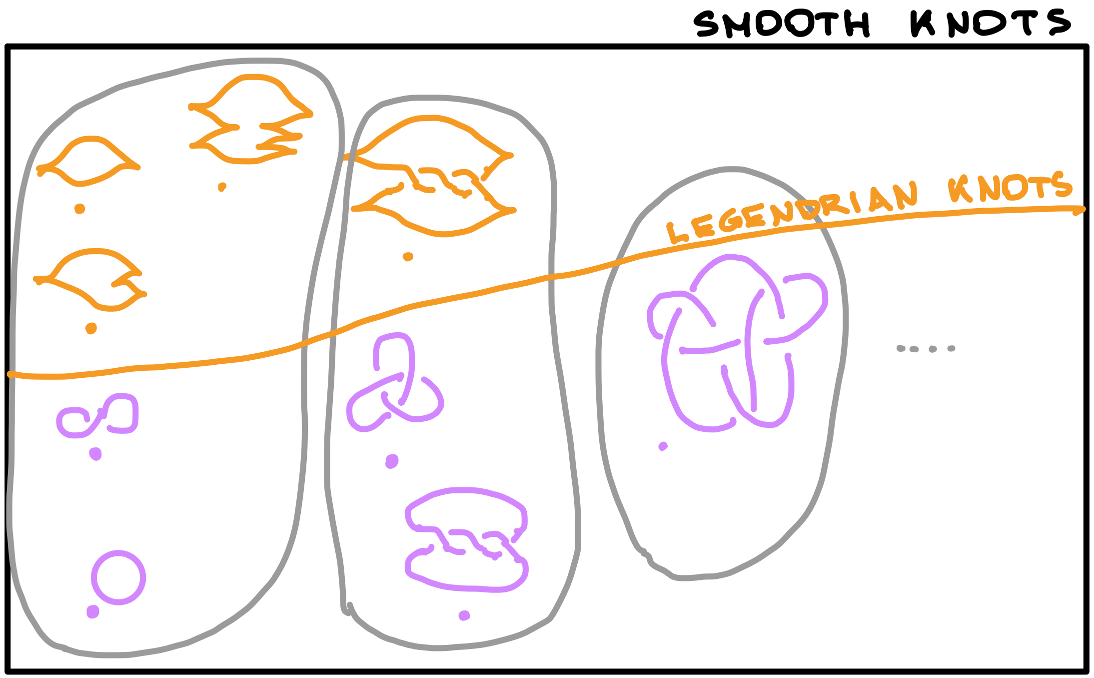
ISOTOPIES OF KNOTS



ISOTOPY: PATH IN THE SPACE OF KNOTS

ISOTOPY CLASS: CONNECTED COMPONENT

ISOTOPIES OF KNOTS



ISOTOPY: PATH IN THE SPACE OF KNOTS

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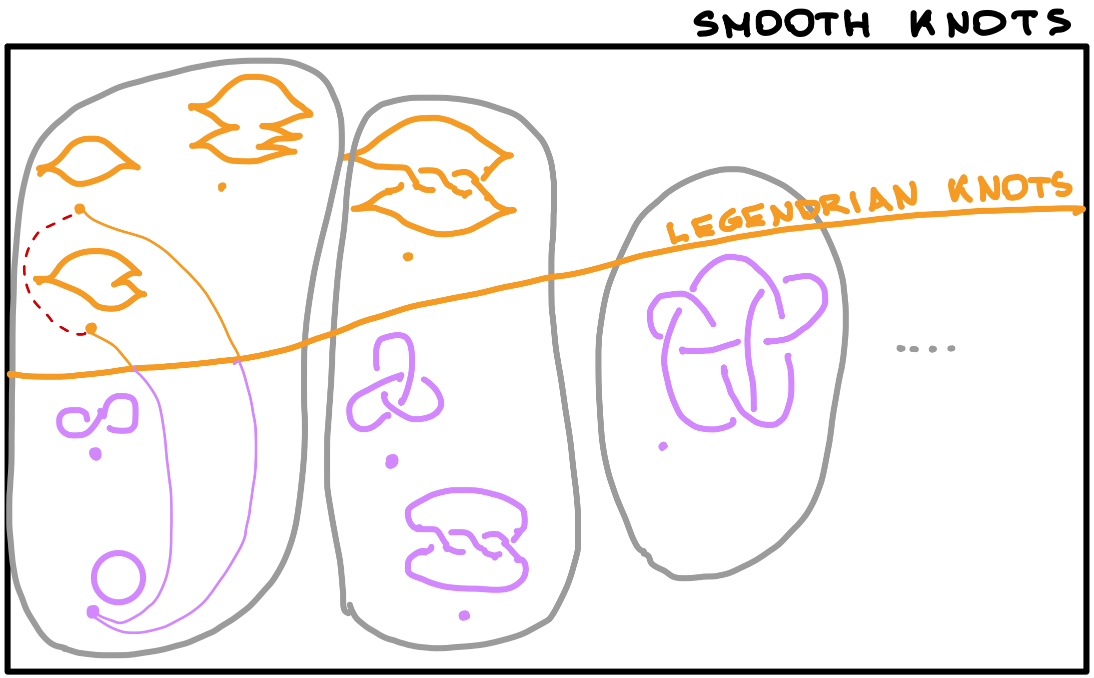
ISOTOPIES OF KNOTS



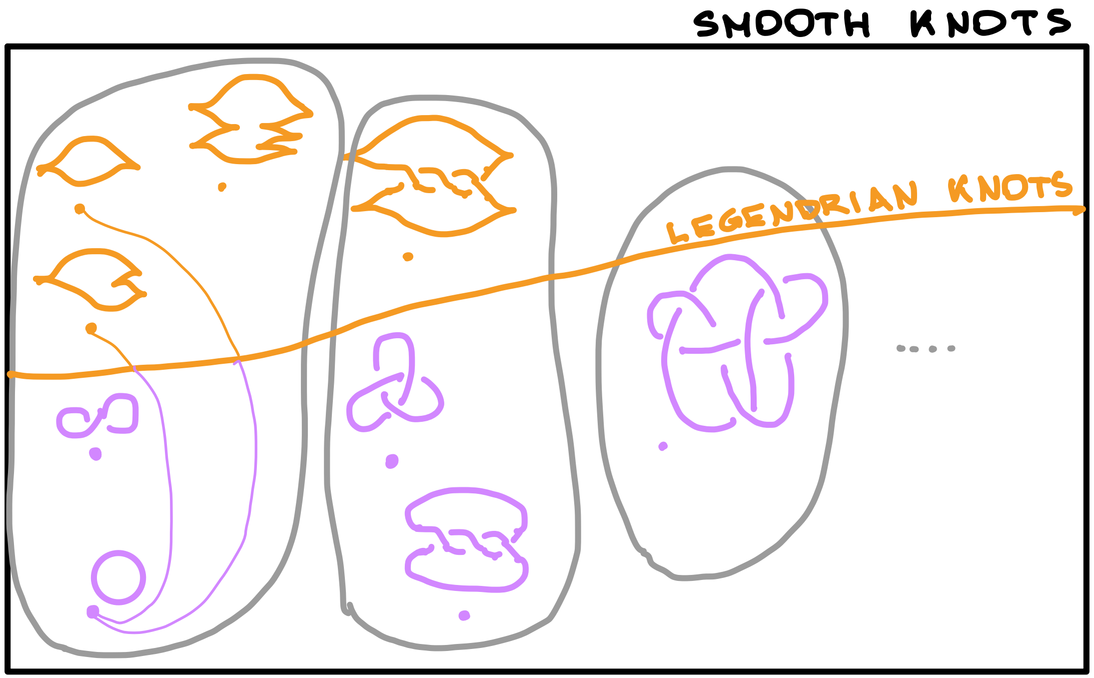
ISOTOPY: PATH IN THE SPACE OF KNOTS

ISOTOPY CLASS: CONNECTED COMPONENT

LEGENDRIAN ISOTOPY: PATH IN THE SPACE OF LEGENDRIAN KNOTS

IF L ISOTOPIC TO L' IMPLIES L LEGEDRIAN ISOTOPIC TO L'?

Isotopies of Knots



ISOTOPY: PATH IN THE SPACE OF KNOTS

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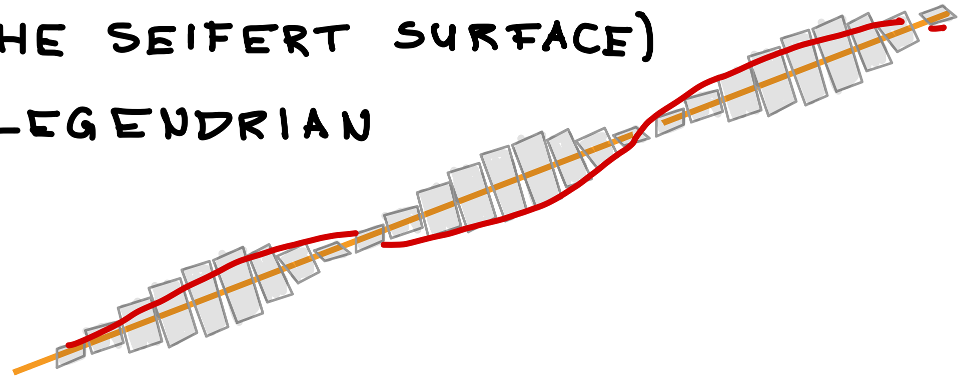
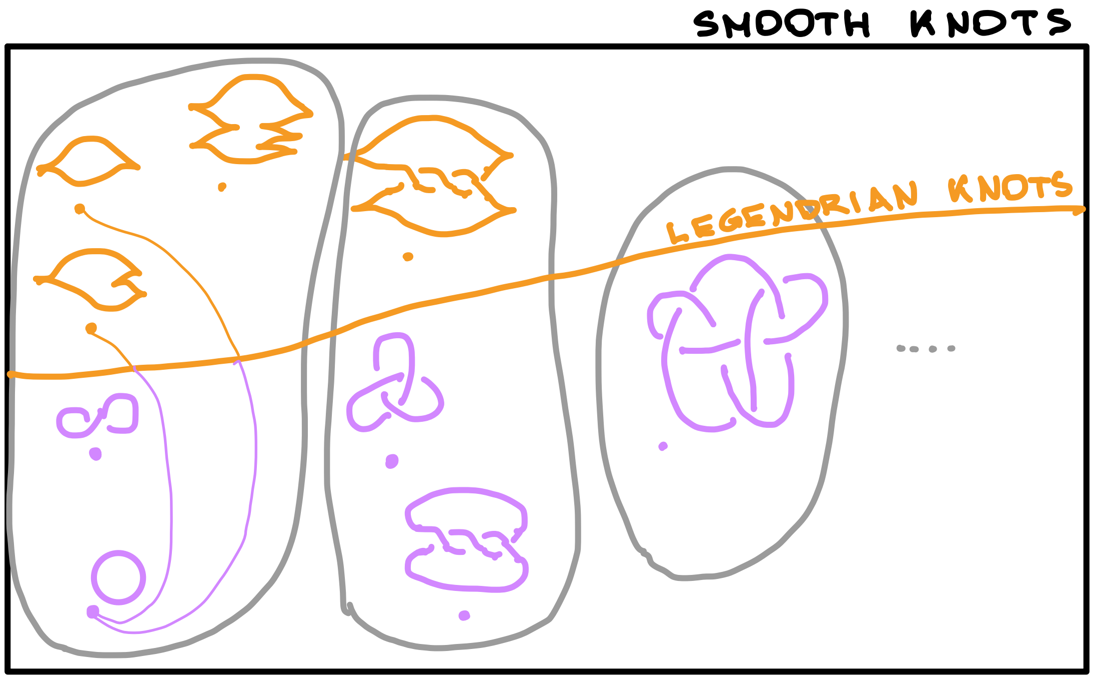
LEGENDRIAN ISOTOPY: PATH IN THE SPACE OF LEGENDRIAN KNOTS

IF L ISOTOPIC TO L' IMPLIES L LEGEDRIAN ISOTOPIC TO L'?

NO! - THE TWISTING OF ξ (W.R.T THE SEIFERT SURFACE) DOESN'T CHANGE DURING LEGENDRIAN ISOTOPY



ISOTOPIES OF KNOTS



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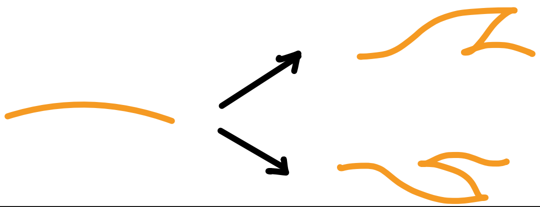
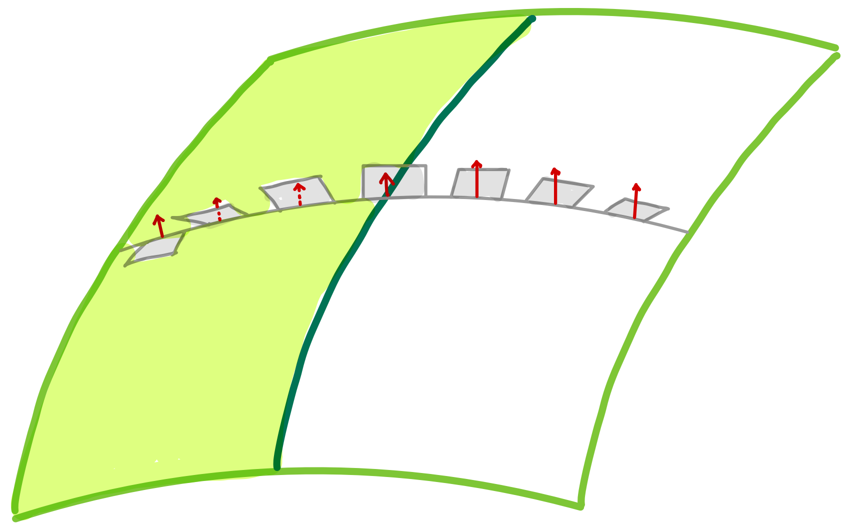
STABILISATION:



CHANGES TWISTING

THM (FUCHS-TABACHNIKOV): L IS ISOTOPIC TO L' $\iff$ AFTER SOME STABILISATIONS $St(L)$ IS LEGEDRIAN ISOTOPIC TO $St(L')$

2-DIM: SURFACES

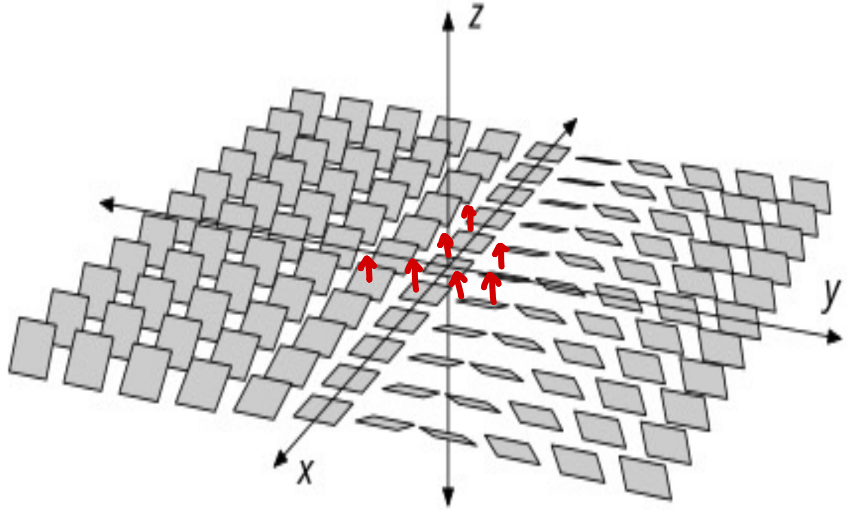


SURFACES IN CONTACT STRUCTURES

DEF: A CONTACT VECTORFIELD $X \in \mathfrak{X}(M)$ IS A VECTORFIELD WHOSE FLOW PRESERVS ξ

$$\Updownarrow$$

$\mathcal{L}_X \alpha = g\alpha$ FOR SOME $g : M \to \mathbb{R}$



$X = \frac{\partial}{\partial z}$

Surfaces in Contact Structures

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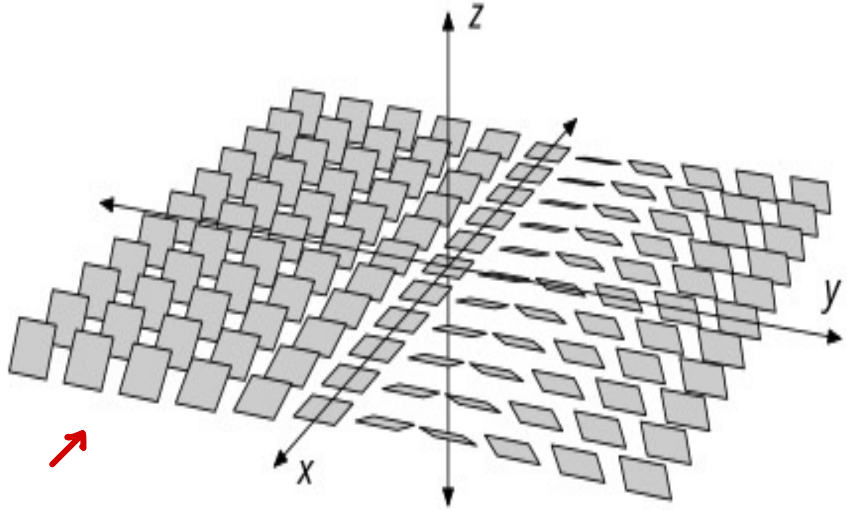


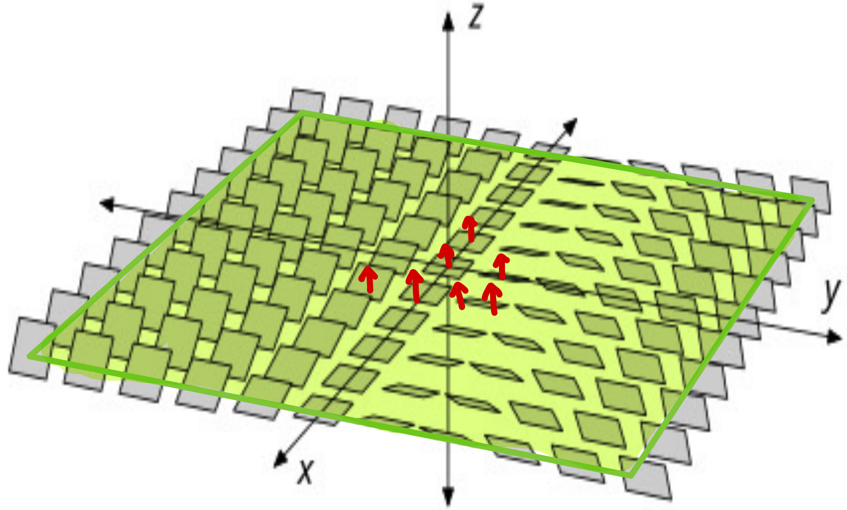
$X = \frac{\partial}{\partial x}$

Surfaces in Contact Structures

DEF: A <u>CONTACT VECTORFIELD</u> $X \in \mathfrak{X}(M)$ IS A VECTORFIELD WHOSE FLOW PRESERVES ξ

$$\Updownarrow$$

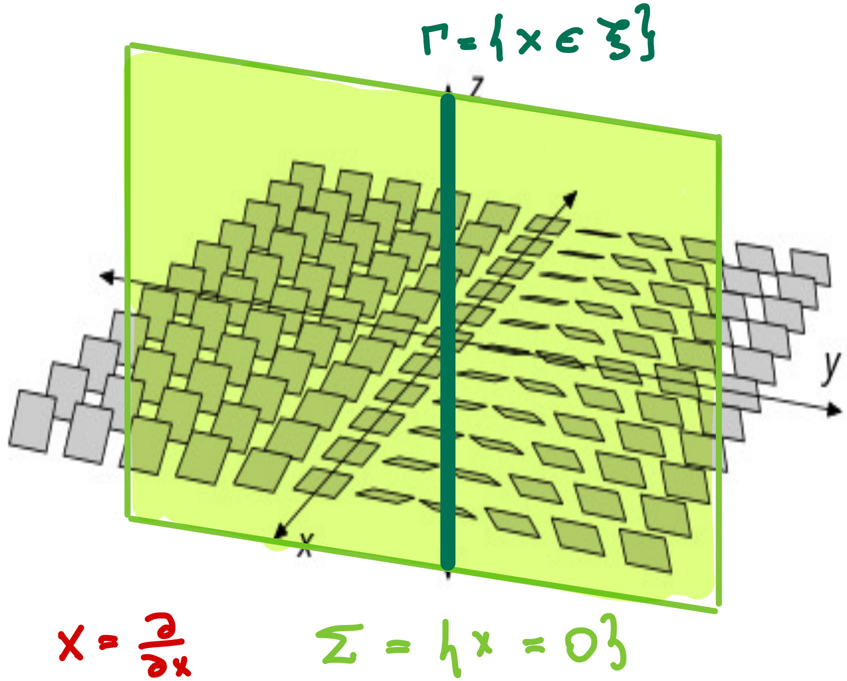
$\mathcal{L}_X \alpha = g\alpha$ FOR SOME $g : M \to \mathbb{R}$



$X = \frac{\partial}{\partial z}$ $\Sigma = \{z = 0\}$

DEF: $\Sigma \hookrightarrow M$ IS <u>CONVEX</u> IF $\exists$ X CONTACT VECTORFIELD $X \pitchfork \Sigma$

SURFACES IN CONTACT STRUCTURES



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$$\Updownarrow$$

$\mathcal{L}_X \alpha = g\alpha$ FOR SOME $g : M \to \mathbb{R}$

DEF: $\Sigma \hookrightarrow M$ IS CONVEX IF $\exists$ X CONTACT VECTORFIELD $X \pitchfork \Sigma$

EQUIVALENTLY: Σ HAS A NEIGHBOURHOOD $N(\Sigma) \cong \Sigma \times \underset{t}{I}$ WITH I-INVARIANT CONTACT STRUCTURE ξ

$$\Updownarrow$$

$\alpha = \beta + g\,dt$ WHERE $\beta \in \Omega^1(\Sigma)$ & $g : M \to \mathbb{R}$

FACT (GIROUX): TO UNDERSTAND ξ ON $N(\Sigma)$ ONE ONLY NEEDS TO KNOW $\Gamma = \{g = 0\} = \{\frac{\partial}{\partial t} \in \xi\} = \{X \in \xi\}$

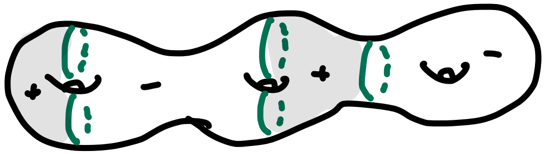
CONVEX SURFACES (GIROUX)

DEF: $\Sigma \hookrightarrow M$ IS **CONVEX** IF $\exists$ CONTACT VECTORFIELD X : $\Sigma \pitchfork X$

DEF: $\Gamma = \{x \in \xi\} = \{\alpha(X) = 0\} \subset \Sigma$ IS THE **DIVIDING CURVE**

PROP: - THE ISOTOPY CLASS OF Γ IS INDEPENDENT OF THE CHOICE OF X

- Γ DIVIDES Σ INTO TWO PIECES: $\Sigma_+ = \{\alpha(X) > 0\}$
  $\Sigma_- = \{\alpha(X) < 0\}$



THM (THE DIVIDING CURVE DETERMINES ξ NEAR Σ)

Σ, Σ' CONVEX SURFACES W/ ISOTOPIC DIVIDING CURVES

$\Longrightarrow \exists$ $N(\Sigma), N(\Sigma')$ NEIGHBOURHOODS THAT ARE CONTACTOMORPHIC

THM (CONVEX SURFACES ARE C^∞-GENERIC): ANY SURFACE Σ CAN BE C^∞-SMALL ISOTOPED TO BE CONVEX

CONTACT MANIFOLDS WITH BOUNDARY

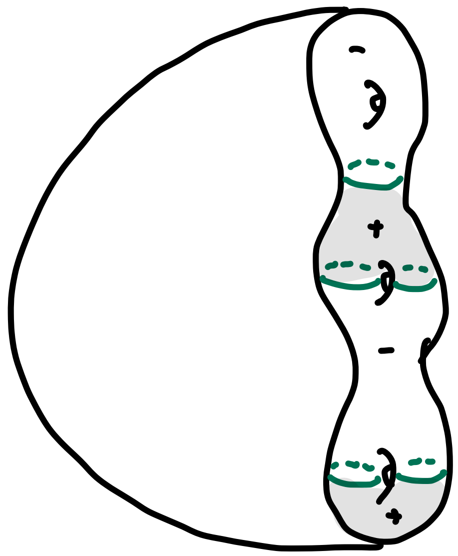
DEF: $(\Sigma_t)_{t\in[0,1]}$ IS A CONVEX ISOTOPY IF Σ_t IS CONVEX $(\forall t \in [0,1])$

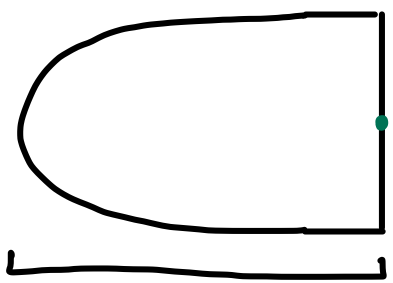
WE WILL WORK WITH

M^3 3-MANIFOLD WITH BOUNDARY,

ξ CONTACT STRUCTURE ON M, S.T

∂M IS CONVEX





(M, ξ)

Contact Manifolds with Boundary

DEF: $(\Sigma_t)_{t\in[0,1]}$ is a **convex isotopy** if Σ_t is convex $(\forall t \in [0,1])$

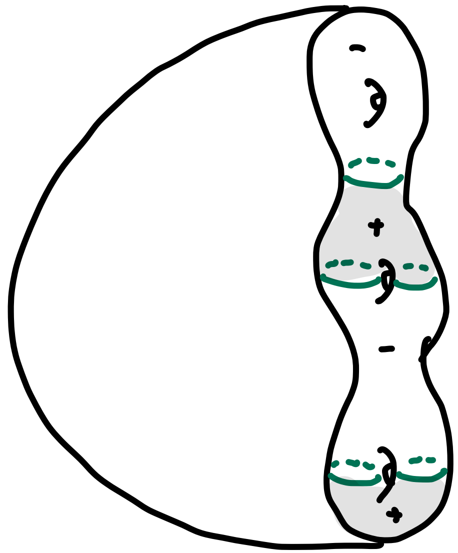
We will work with

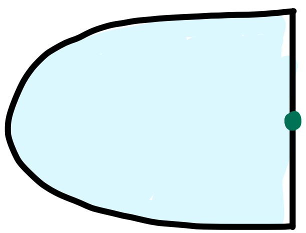
M^3 3-manifold with boundary,

ξ contact structure on M, s.t.

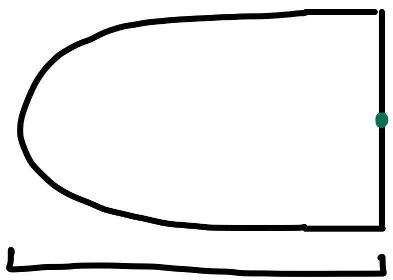
∂M is convex

• same for (M', ξ')





(M', ξ')



(M, ξ)

Contact Manifolds with Boundary

DEF: $(\Sigma_t)_{t\in[0,1]}$ IS A **CONVEX ISOTOPY** IF Σ_t IS CONVEX $(\forall t \in [0,1])$

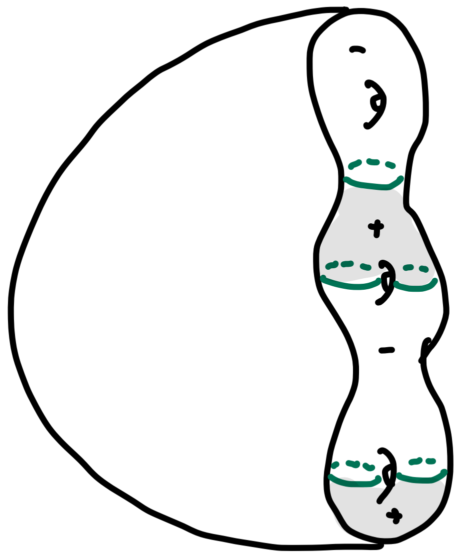
WE WILL WORK WITH

M^3 3-MANIFOLD WITH BOUNDARY,

ξ CONTACT STRUCTURE ON M, S.T

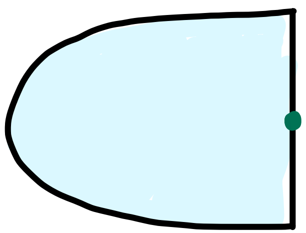
∂M IS CONVEX

- SAME FOR (M', ξ')

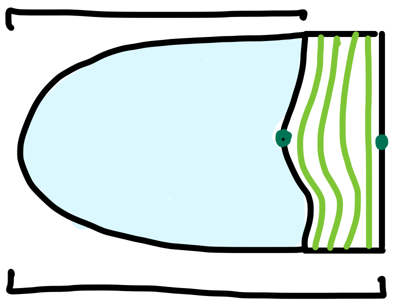


DEF: **WEAKLY CONTACTOMORPHIC:** $\exists$ EMBEDDING $(M', \xi') \xhookrightarrow{\iota} (M, \xi)$

SUCH THAT $\iota(\partial M')$ IS CONVEX ISOTOPIC TO ∂M



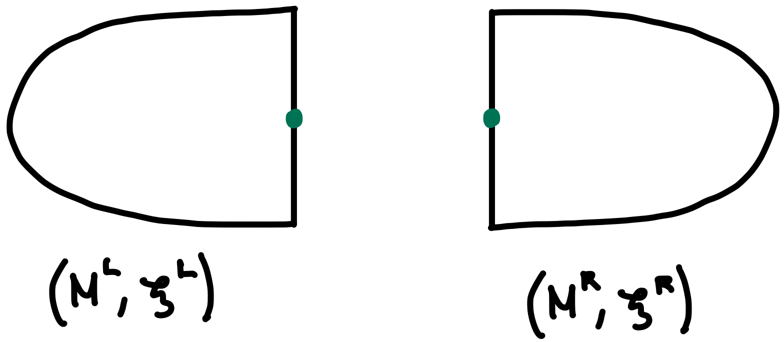
(M', ξ')



Gluing Contact Structures

WE CAN GLUE CONTACT STRUCTURES ALONG SURFACES WITH MATCHING DIVIDING CURVES

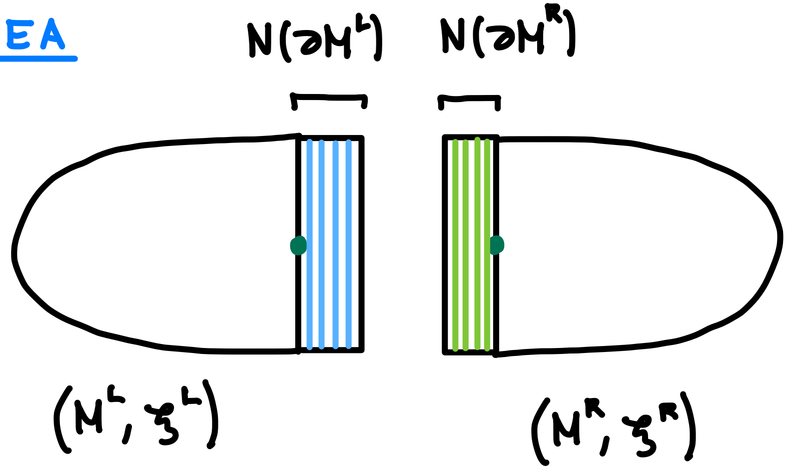
IDEA



Gluing Contact Structures

We can glue contact structures along surfaces with matching dividing curves

IDEA

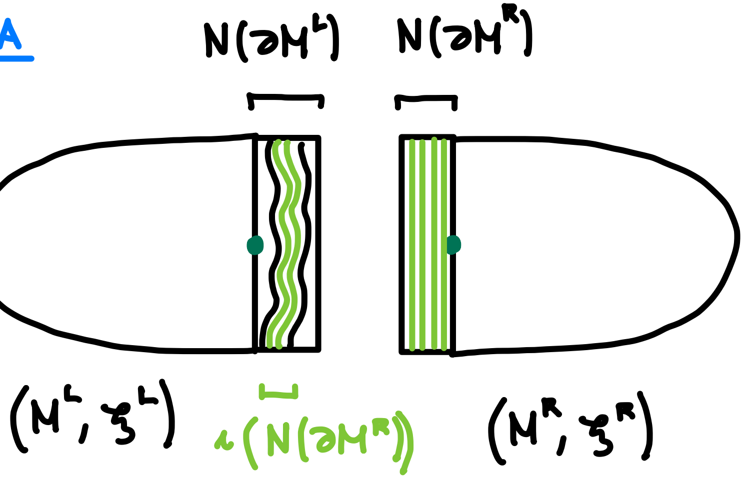


STEP 1: ADD I-INVARIANT PART TO EACH

GLUING CONTACT STRUCTURES

WE CAN GLUE CONTACT STRUCTURES ALONG SURFACES WITH MATCHING DIVIDING CURVES

IDEA



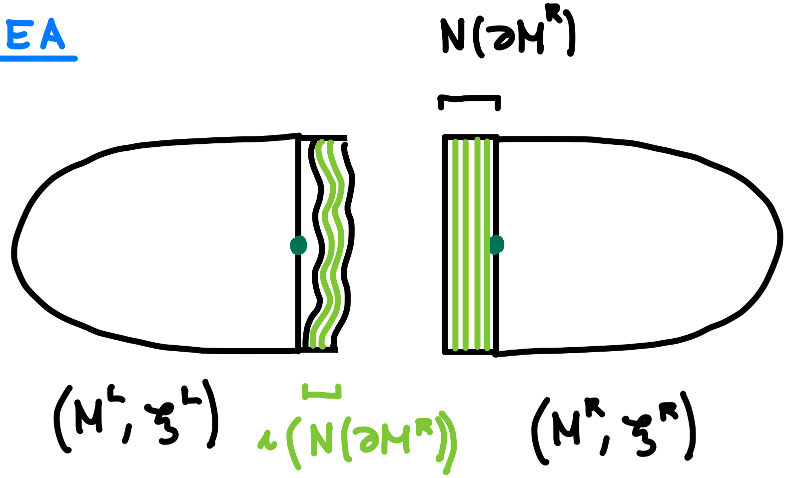
STEP 1: ADD I-INVARIANT PART TO EACH

STEP 2: FIND $N(\partial M^R)$ IN $N(\partial M^L)$

Gluing Contact Structures

WE CAN GLUE CONTACT STRUCTURES ALONG SURFACES WITH MATCHING DIVIDING CURVES

IDEA



STEP 1: ADD I-INVARIANT PART TO EACH

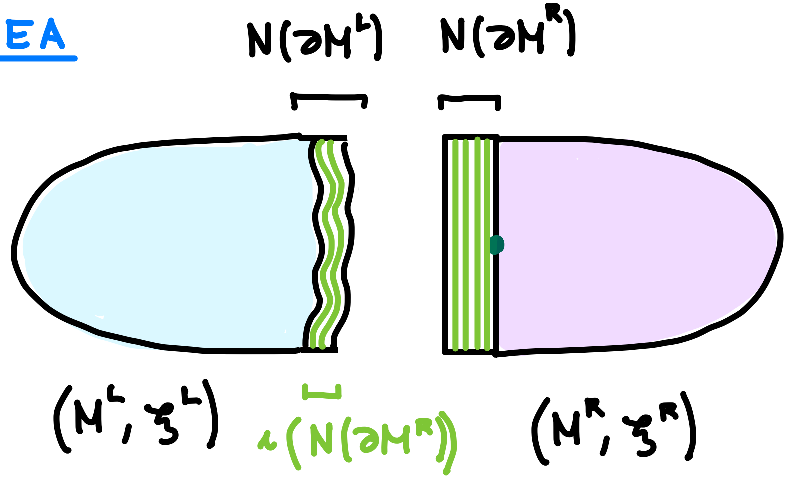
STEP 2: FIND $N(\partial M^R)$ IN $N(\partial M^L)$

STEP 3: TRUNCATE M^L AT $\iota(\partial M^R \times \{1\})$

GLUING CONTACT STRUCTURES

WE CAN GLUE CONTACT STRUCTURES ALONG SURFACES WITH MATCHING DIVIDING CURVES

IDEA

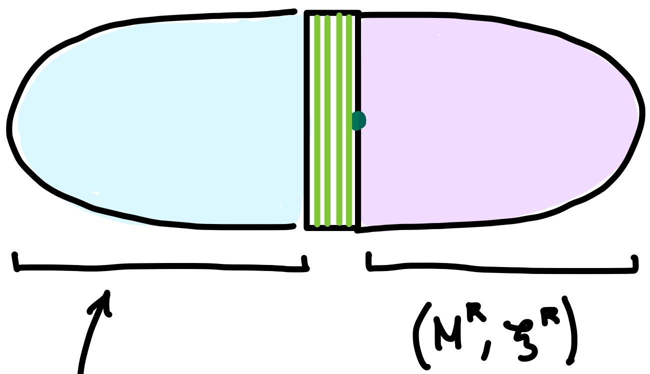


STEP 1: ADD I-INVARIANT PART TO EACH

STEP 2: FIND $N(\partial M^R)$ IN $N(\partial M^L)$

STEP 3: TRUNCATE M^L AT $\iota(\partial M^R \times \{1\})$

STEP 4: OVERLAP $\iota(N(\partial M^R))$ WITH $N(\partial M^R)$



THE OBTAINED CONTACT MANIFOLD IS $(M^L, \xi^L) \cup (M^R, \xi^R)$

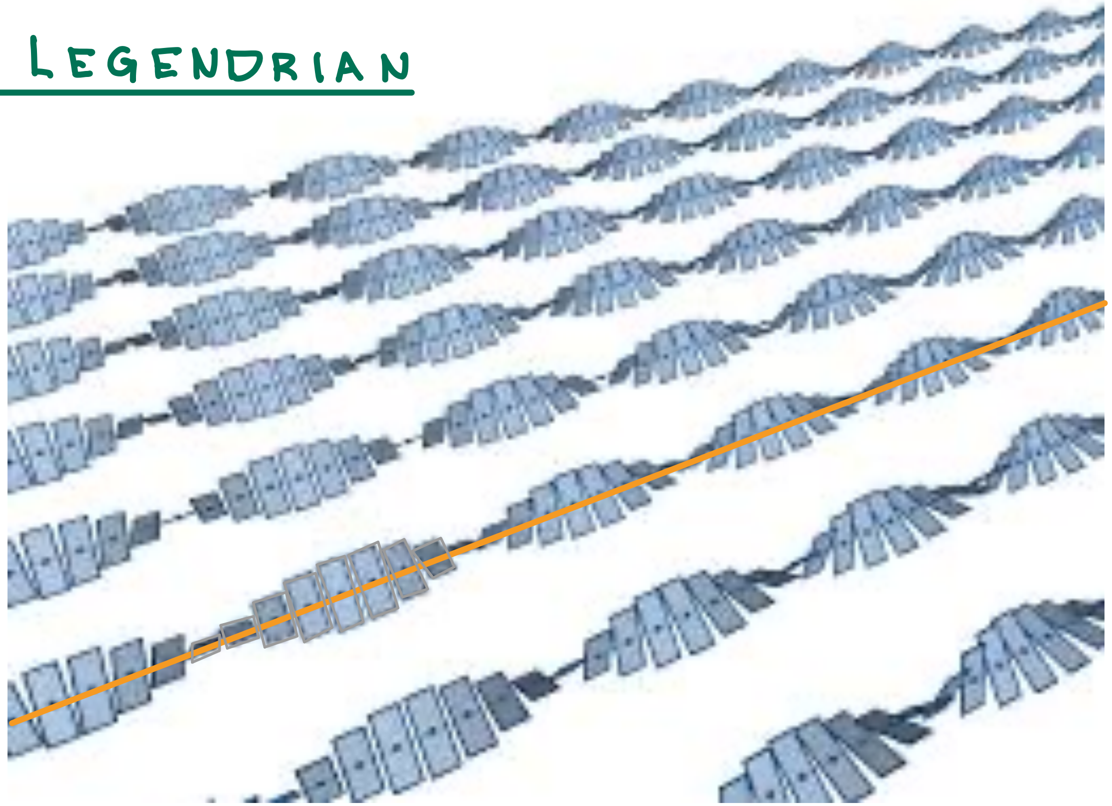
& IT IS WELL DEFINED UP TO CONTACTOMORPHISM

WEAKLY CONTACT ISOTOPIC TO (M^L, ξ^R)

STANDARD NEIGHBOURHOOD OF A LEGENDRIAN

E.G: $\xi = \ker(\cos(z)\,dx - \sin(z)\,dy)$

(ISOTOPIC TO ξ_{st})



STANDARD NEIGHBOURHOOD OF A LEGENDRIAN

E.G: $\xi = \ker\left(\cos(z)\,dx - \sin(z)\,dy\right)$

(ISOTOPIC TO ξ_{st})

IDENTIFY $(x,y,z) \sim (x,y,z+2\pi n)$

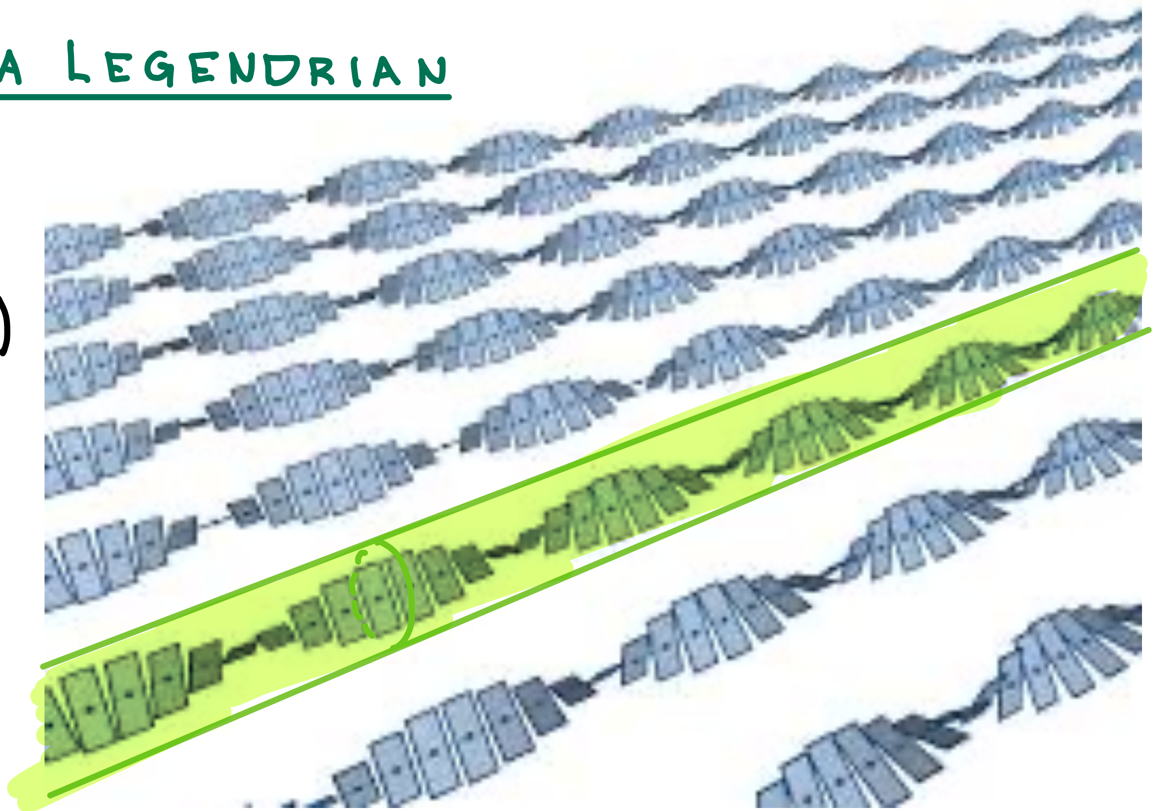
$\rightsquigarrow$ CONTACT STRUCTURE

ON $\mathbb{R}^2 \times S^1$

WITH LEGENDRIAN KNOT

$L_0 = (0,0) \times S^1 \subset \mathbb{R}^2 \times S^1$

NEIGHBOURHOOD $N(L_0) = D^2 \times S^1$



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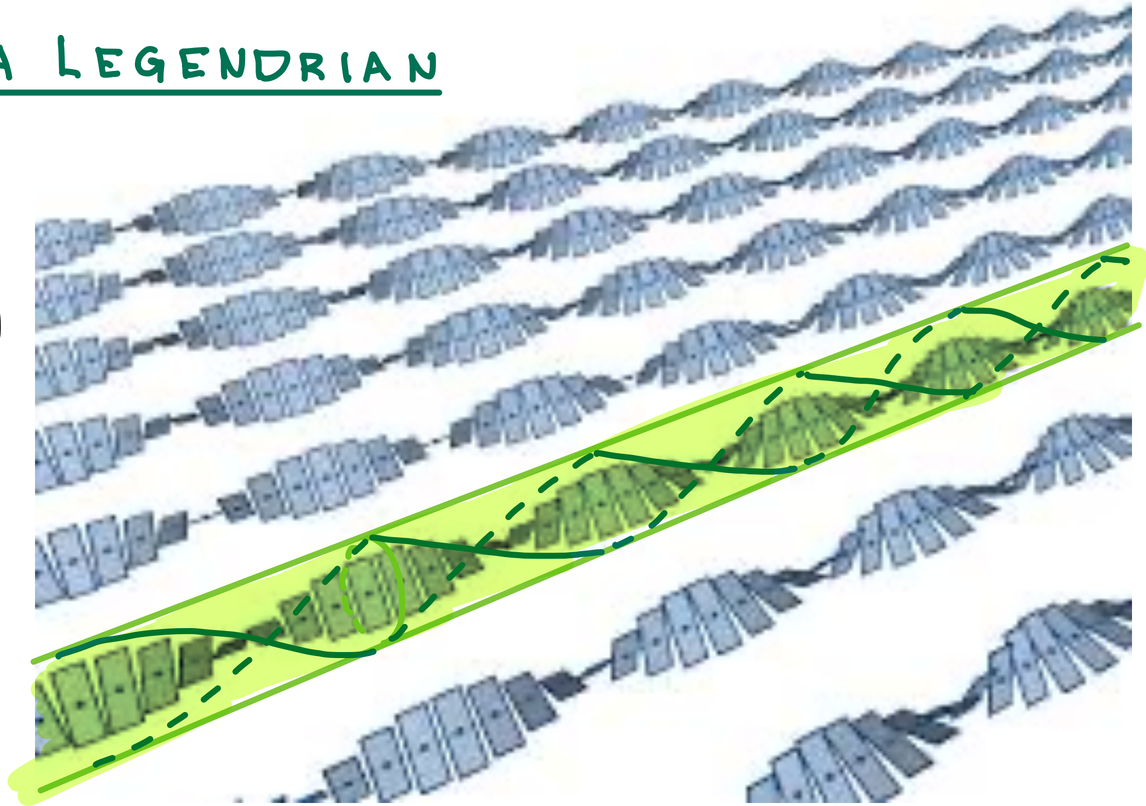
ON $\mathbb{R}^2 \times S^1$

WITH LEGENDRIAN KNOT

$L_0 = (0,0) \times S^1 \subset \mathbb{R}^2 \times S^1$



NEIGHBOURHOOD $N(L_0) = D^2 \times S^1$

RMK: $\partial N(L_0)$ IS NOT CONVEX, BUT BY A C^∞-SMALL ISOTOPY IT CAN BE MADE CONVEX WITH A TWO COMPONENT DIVIDING CURVE PARALLEL TO $\left(\sin(z), \cos(z), z\right)/\sim$

THIS GIVES THE THURSTON-BENNEQUIN FRAMING

STANDARD NEIGHBOURHOOD OF A LEGENDRIAN

E.G: $\xi = \ker\left(\cos(z)\,dx - \sin(z)\,dy\right)$

(ISOTOPIC TO ξ_{st})

IDENTIFY $(x,y,z) \sim (x,y,z+2\pi n)$

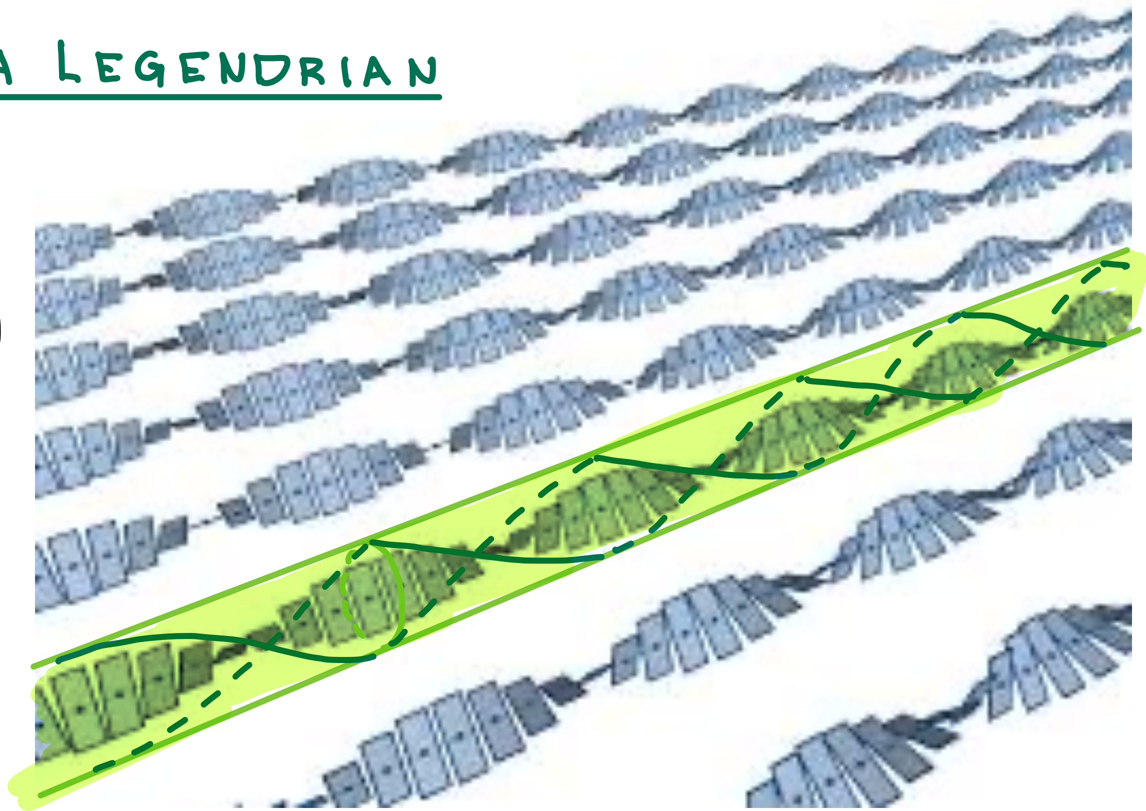
$\rightsquigarrow$ CONTACT STRUCTURE

ON $\mathbb{R}^2 \times S^1$

WITH LEGENDRIAN KNOT

$L_0 = (0,0) \times S^1 \subset \mathbb{R}^2 \times S^1$

NEIGHBOURHOOD $N(L_0) = D^2 \times S^1$



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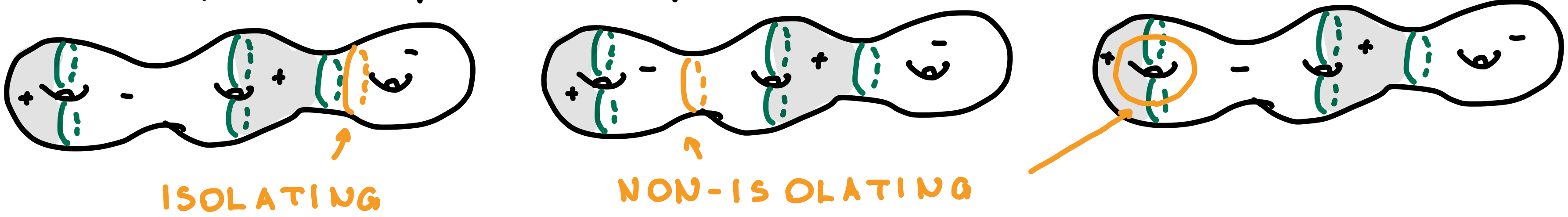
THIS GIVES THE THURSTON-BENNEQUIN FRAMING

$\boxed{\text{THM: ANY LEGENDRIAN KNOT } L \subset (M,\xi) \text{ HAS A NEIGHBOURHOOD } N(L) \text{ CONTACTOMORPHIC TO } N(L_0)}$

LEGENDRIANS ON CONVEX SURFACES

DEF: $C \subset (\Sigma, \Gamma)$ IS AN ISOLATING CURVE, IF SOME COMPONENT OF $\Sigma \setminus C$ IS DISJOINT FROM Γ:



THM (LEGENDRIAN REALISATION PRINCIPLE)

(Σ, Γ) CONVEX SURFACE, $C \subset \Sigma$ NON-ISOLATING CURVE

$\Longrightarrow$ Σ CAN BE ISOTOPED THROUGH CONVEX SURFACES $\psi_t(\Sigma)$ S.T. AFTER THE ISOTOPY $\psi_1(C) \subset \psi_1(\Sigma)$ IS LEGENDRIAN

LEGENDRIANS ON CONVEX SURFACES

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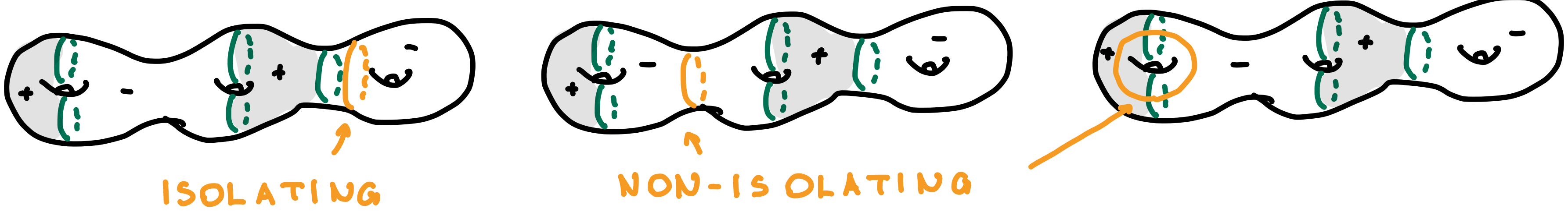

THM (LEGENDRIAN REALISATION PRINCIPLE)

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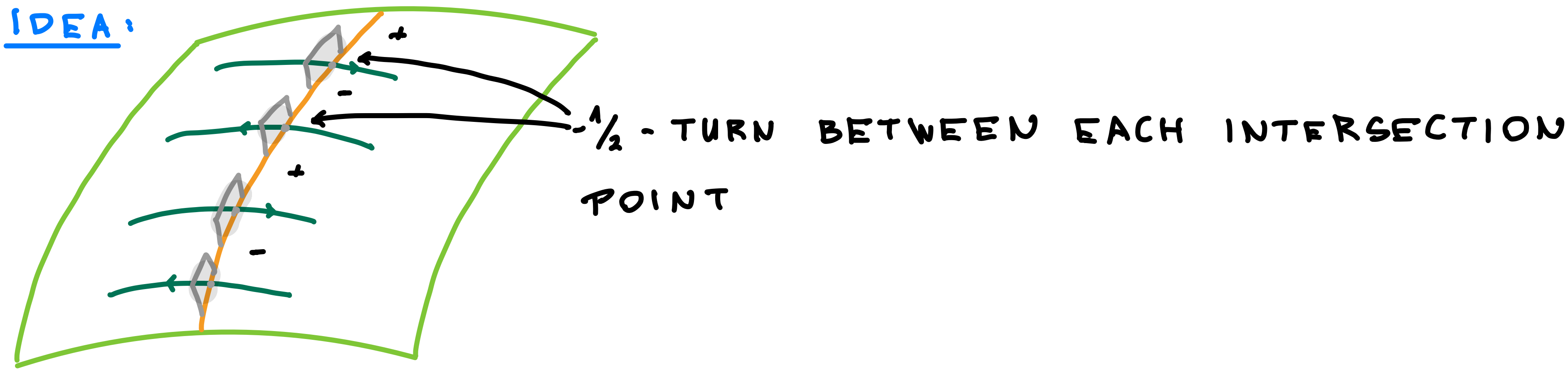
RMK: THE TWISTING OF ξ W.R.T. $T\Sigma$ ALONG C $= -\frac{1}{2}|C \cap \Gamma|$

IDEA:



CONVEX SURFACES WITH LEGENDRIAN BOUNDARY

THM (KANDA): Σ SURFACE WITH LEGENDRIAN BOUNDARY L CAN BE ISOTOPED REL ∂ TO BE CONVEX

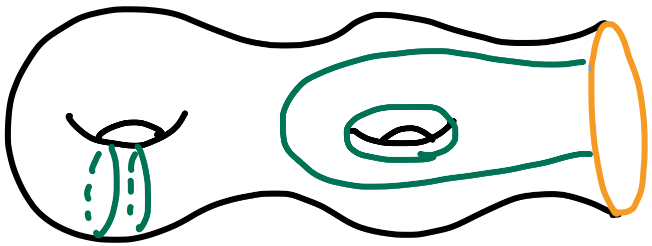
$$\Updownarrow$$

TWISTING OF ξ W.R.T. Σ ALONG L IS ≤ 0

- THE ISOTOPY CAN BE ASSUMED TO BE C^0-SMALL

(C$^\infty$-SMALL IF Σ IS ALREADY CONVEX NEAR $\partial\Sigma$)

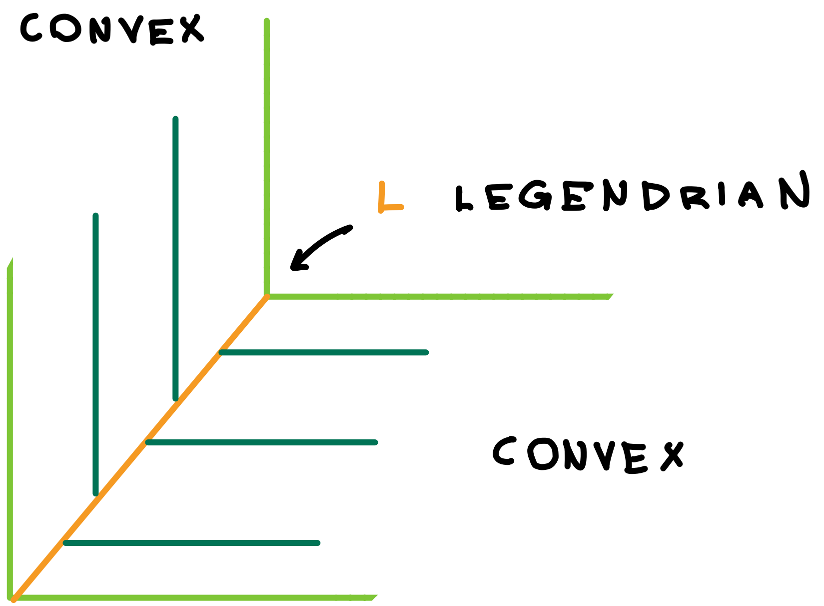
RMK: AFTER THE ISOTOPY



TWISTING OF ξ W.R.T. Σ ALONG L $= -\frac{1}{2}|\Gamma \cap L|$

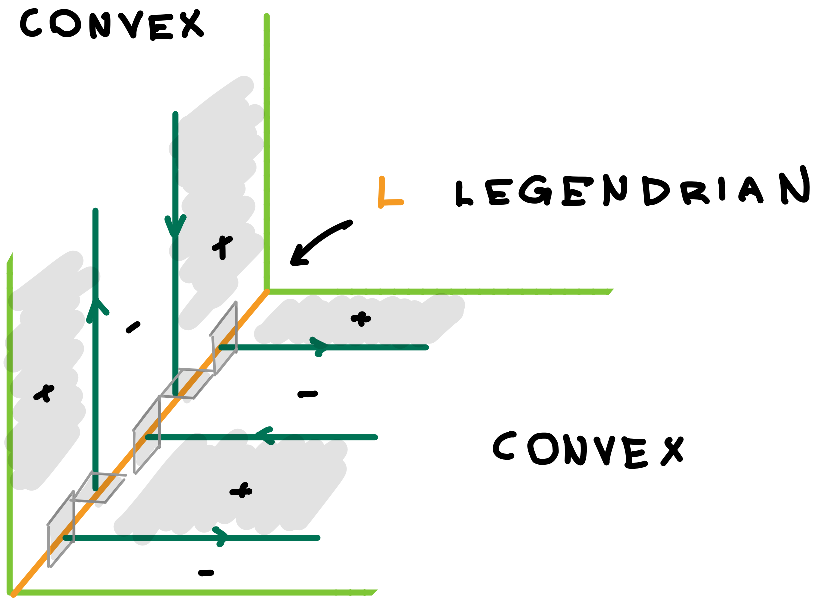
Rounding Edges

Σ_1 & Σ_2 convex surfaces with common Legendrian boundary L



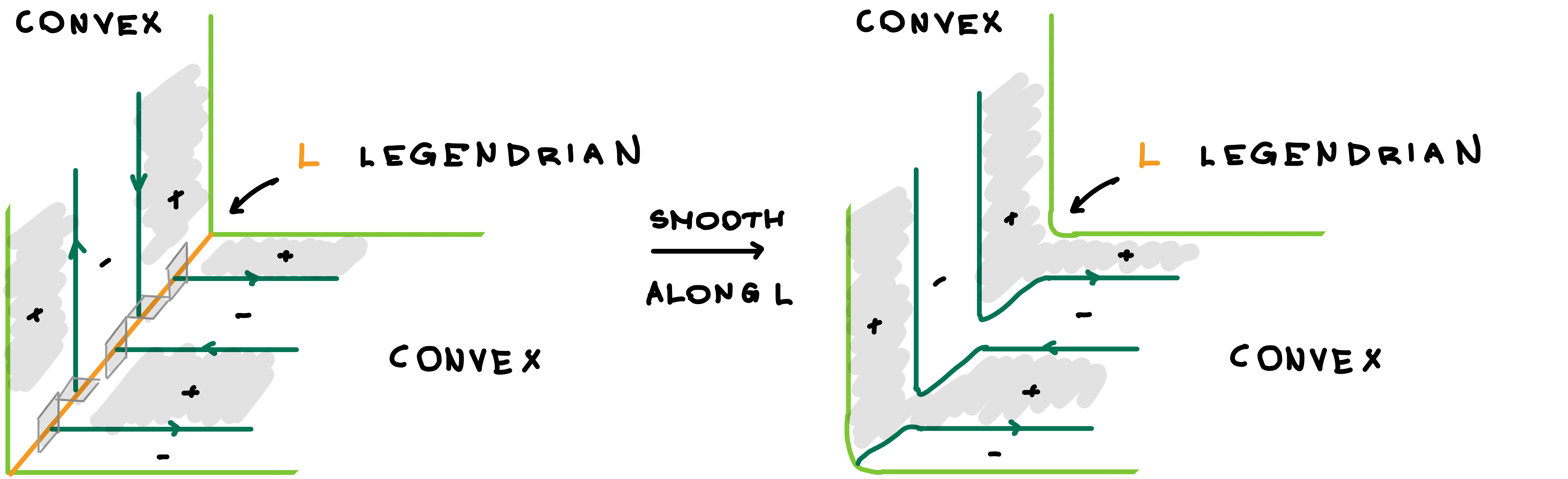
ROUNDING EDGES

Σ_1 & Σ_2 CONVEX SURFACES WITH COMMON LEGENDRIAN BOUNDARY L



ROUNDING EDGES

Σ_1 & Σ_2 CONVEX SURFACES WITH COMMON LEGENDRIAN BOUNDARY L THEN THE EDGE L CAN BE ROUNDED & WE GET A NEW SMOOTH CONVEX SURFACE Σ WITH DIVIDING CURVE AS BELOW

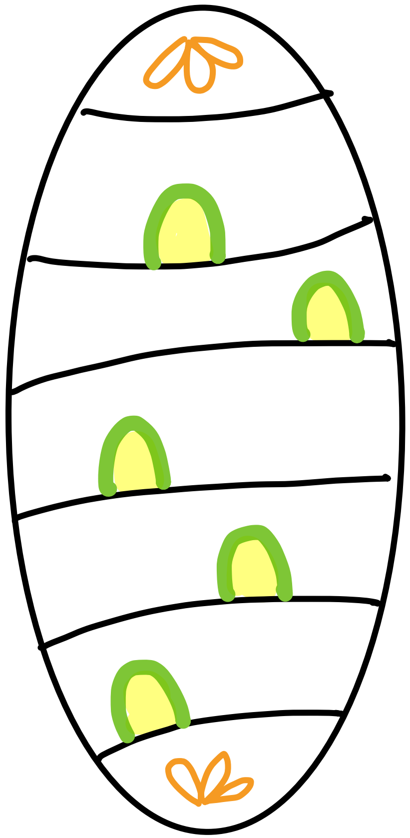


SRNI - LECTURE 2

DESCRIBING CONTACT STRUCTURES

- TIGHT & OVERTWISTED CONTACT STRUCTURES
- CONTACT CELL DECOMPOSITIONS
- CONVEX SURFACE THEORY - BYPASSES
- CONTACT HEEGAARD SPLITTINGS (PROOF OF EXISTENCE)
- OPEN BOOK DECOMPOSITIONS
- OPEN BOOK DECOMPOSITIONS & CONTACT HEEGAARD SPLITTINGS

THE GIROUX CORRESPONDENCE VIA CONVEX SURFACES

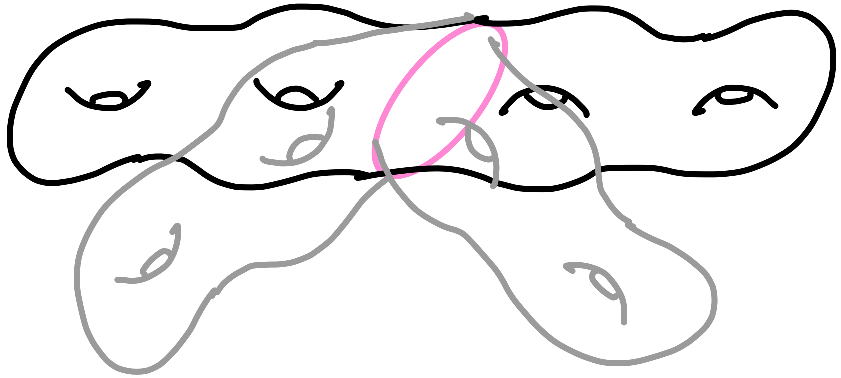


VERA VÉRTESI

JOINT WORK WITH JOAN LICATA

UNIVERSITY OF VIENNA

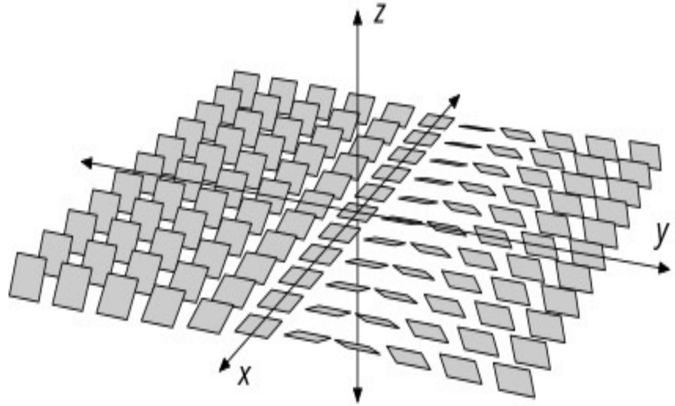
LECTURE 2



DESCRIBING CONTACT STRUCTURES

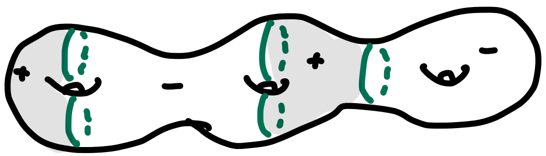
- TIGHT & OVERTWISTED CONTACT STRUCTURES
- CONTACT CELL DECOMPOSITIONS
- CONVEX SURFACE THEORY - BYPASSES
- CONTACT HEEGAARD SPLITTINGS (PROOF OF EXISTENCE)
- OPEN BOOK DECOMPOSITIONS
- OPEN BOOK DECOMPOSITIONS & CONTACT HEEGAARD SPLITTINGS

LAST TIME (GIROUX)



- CONTACT STRUCTURE ON CLOSED, SMOOTH M^3
  LOCALLY $\xi = \ker(dz - y\,dx)$

- $\Sigma \hookrightarrow M$ IS CONVEX IF $\exists$ CONTACT VECTORFIELD X : $\Sigma \pitchfork X$



$\Sigma = \Sigma_+ \cup_\Gamma \Sigma_-$

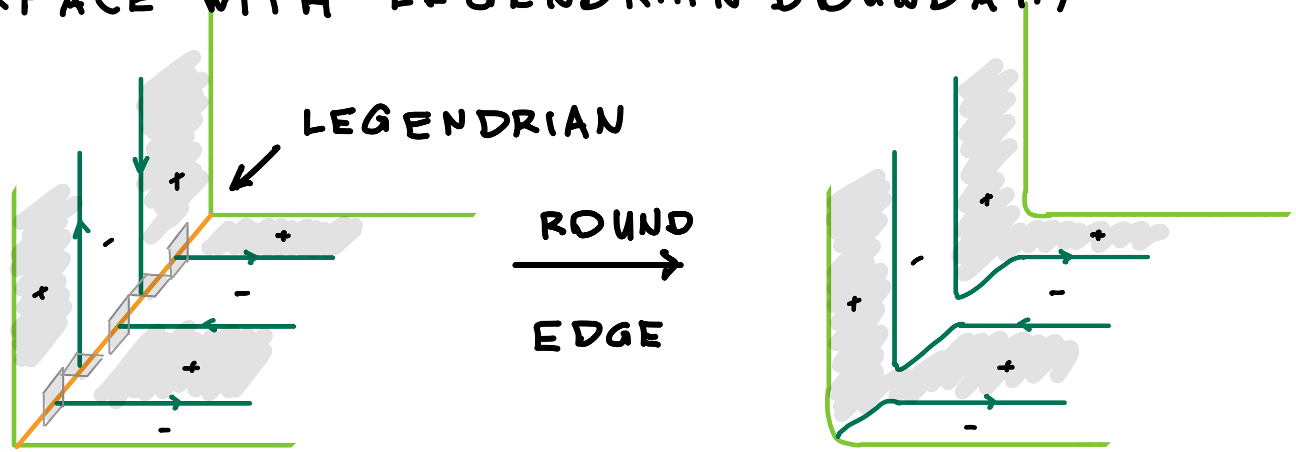
$\alpha(X) > 0$ $\alpha(X) = 0$ $\alpha(X) < 0$

→ GENERIC $\Sigma \hookrightarrow M$ IS CONVEX

→ $\xi|_{N(\Sigma)}$ IS DETERMINED BY $\Gamma \subset \Sigma$

- Σ CONVEX SURFACE WITH LEGENDRIAN BOUNDARY

EDGE ROUNDING

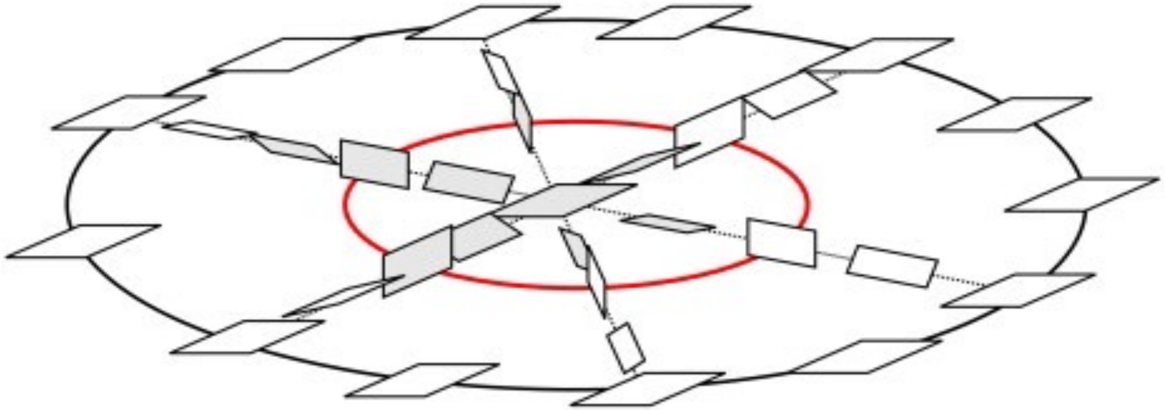


CONV: ALL CPCT M HAS CONVEX ∂ $\rightsquigarrow$ CAN GLUE ξ ON THEM

TIGHT

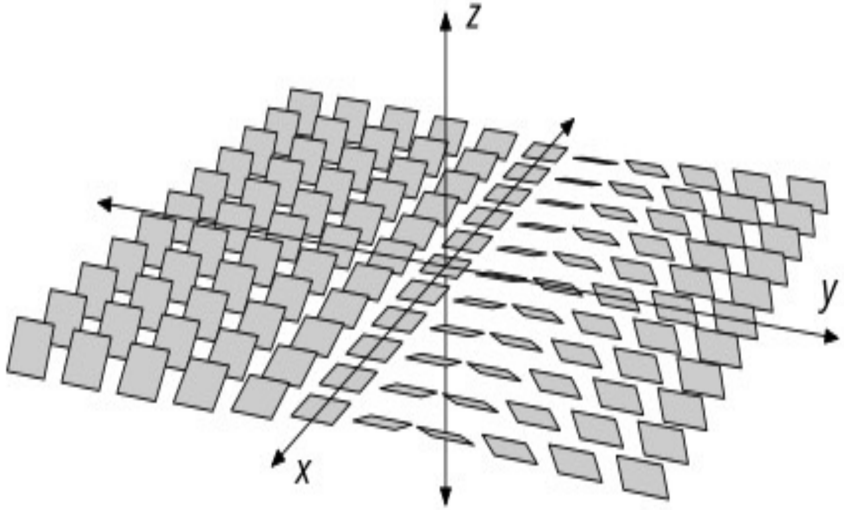
&

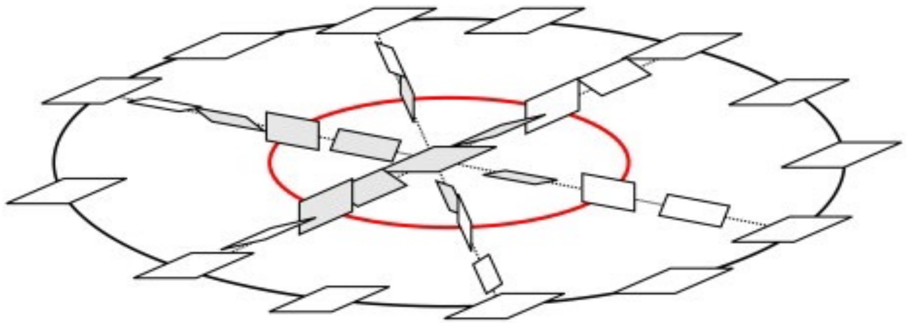
OVERTWISTED



Two Contact Structures

STANDARD CONTACT STRUCTURE | **OVERTWISTED CONTACT STRUCTURE**





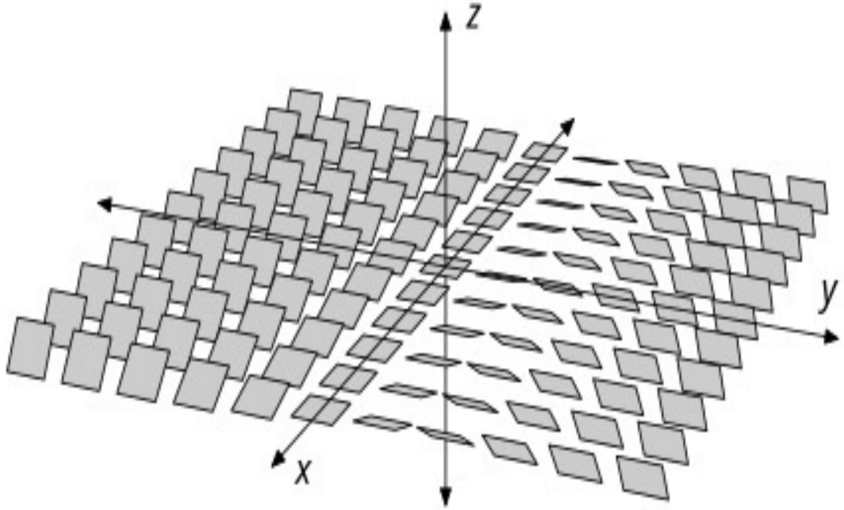
$$\xi_{st} = \ker(dz - y\,dx)$$

$$\xi_{OT} = \ker(\cos(r)\,dz + r\sin(r)\,d\vartheta)$$

ARE ξ_{st} & ξ_{OT} ISOTOPIC/CONTACTOMORPHIC ?

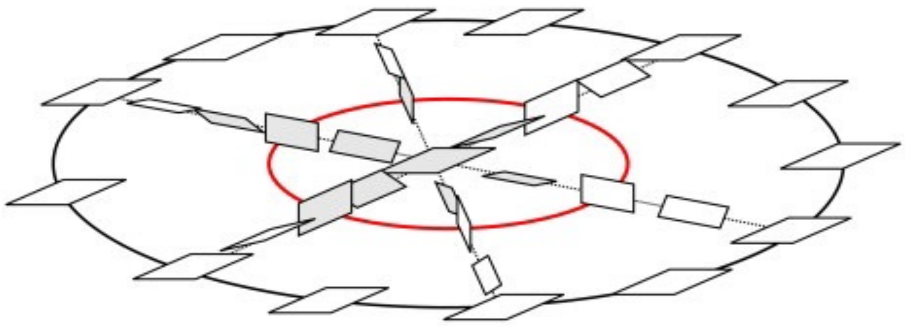
Two Contact Structures

STANDARD CONTACT STRUCTURE



$$\xi_{st} = \ker(dz - y\,dx)$$

OVERTWISTED CONTACT STRUCTURE



$$\xi_{OT} = \ker(\cos(r)\,dz + r\sin(r)\,d\vartheta)$$

ARE ξ_{st} & ξ_{OT} ISOTOPIC/CONTACTOMORPHIC ?

BENNEQUIN (1982): NO! CONSIDER $D = \{r \leq 2\pi\} \times \{0\}$

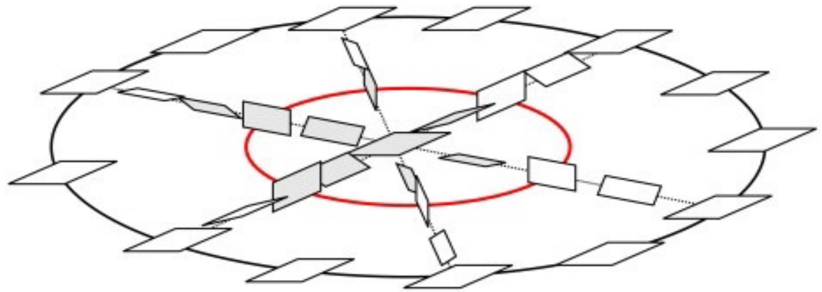
DEF: $D \hookrightarrow (M,\xi)$ IS AN OVERTWISTED DISK IF $TD|_{\partial D} = \xi|_{\partial D}$

RMK: ENOUGH TWISTING OF ξ ALONG ∂D W.R.T D IS 0

DEF: • ξ IS OVERTWISTED IF ξ CONTAINS AN OVERTWISTED DISK

• ξ IS TIGHT IF IT IS NOT OVERTWISTED

OVERTWISTED CONTACT STRUCTURES



THM (ELIASHBERG) ξ, ξ' OVERTWISTED CONTACT STRUCTURES & $\xi \simeq \xi'$ AS PLANEFIELDS $\Longrightarrow \xi \approx \xi'$

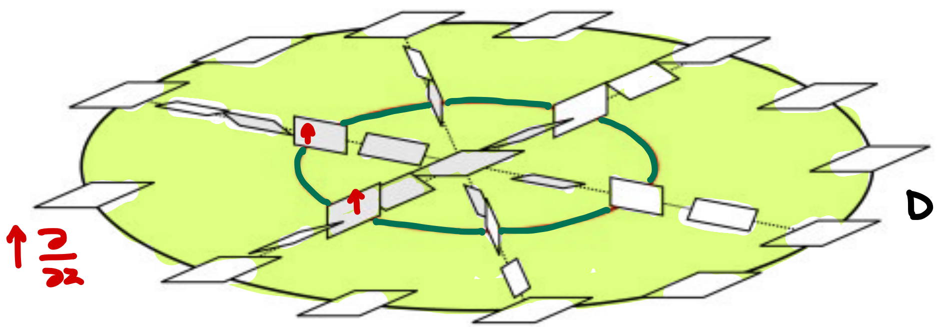
THM (LUTZ-MARTINEZ) ANY HOMOTOPY CLASS OF PLANEFIELDS IS REPRESENTED BY A(N OVERTWISTED) CONTACT STRUCTURE

OVERTWISTED CONTACT STRUCTURES CAN BE UNDERSTOOD THROUGH ALGEBRAIC TOPOLOGICAL INVARIANTS : d_2, d_3

RMK: TIGH CONTACT STRUCTURES ARE HARDER TO CLASSIFY

- ELIASHBERG: S^3 ADMITS A UNIQUE TIGHT CONTACT STRUCTURE
- GIROUX: $\exists$ ∞-LY MANY 3-MANIFOLDS WITH ∞-LY MANY TIGHT CONTACT STRUCTURES
- ETNYRE: $-\Sigma(2,3,5)$ ADMITS NO TIGHT CONTACT STRUCTURES
- ...

RECOGNISING OVERTWISTED CONTACT STRUCTURES



D IS CONVEX : $X = \frac{\partial}{\partial z}$ IS A CONTACT VECTORFIELD, $X \pitchfork D$

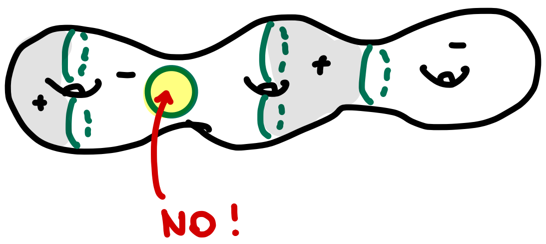
$\Gamma = \{\frac{\partial}{\partial z} \in \xi\} = \{r = \pi\} \times \{0\}$

$\xi_{OT} = \ker(\cos(r)\,dz + r\sin(r)\,d\vartheta)$

THM (GIROUX'S CRITERION): $\Sigma \hookrightarrow (M, \xi)$ CONVEX SURFACE ADMITS A TIGHT NEIGHBOURHOOD $(N(\Sigma), \xi|_{N(\Sigma)})$ IFF

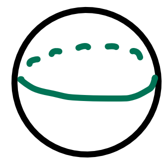
- $\Sigma = S^2$ & $|\Gamma| = 1$
- $\Sigma \neq S^2$ & NO COMPONENT OF Γ BOUNDS A DISC

OR



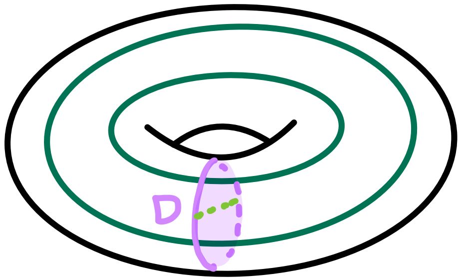
Classification of Tight Contact Structures

THM (Eliashberg): D^3 admits a unique tight contact structure with connected dividing curve on S^2:



This result allows us to prove other uniqness results

E.G.: $M = D^2 \times S^1$ given any contact structurf ξ on M



- **STEP 1:** By the Legendrian realisation principle we can assume ∂D is Legendrian

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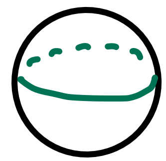

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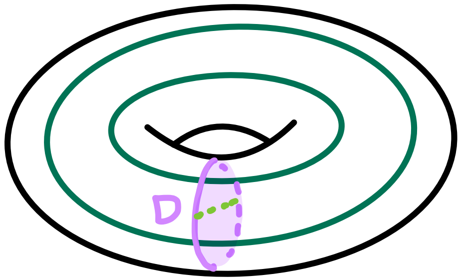
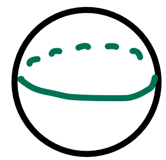


- **STEP 1:** By the Legendrian realisation principle we can assume ∂D is Legendrian
- **STEP 2:** Isotope D rel. ∂ to be convex: the dividing curve on D is a single arc

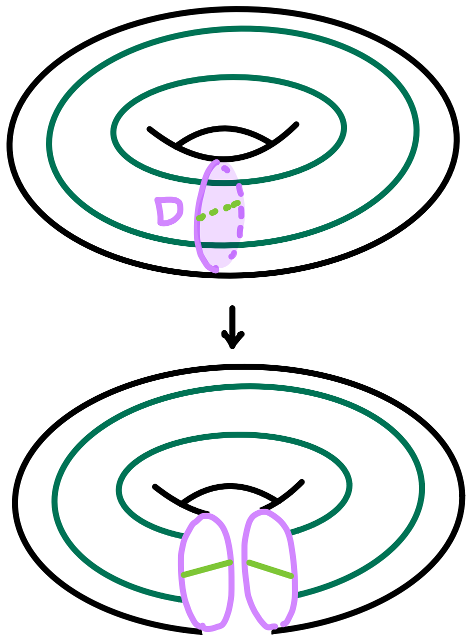
CLASSIFICATION OF TIGHT CONTACT STRUCTURES

THM (ELIASHBERG): D^3 ADMITS A UNIQUE TIGHT CONTACT STRUCTURE WITH CONNECTED DIVIDING CURVE ON S^2:



THIS RESULT ALLOWS US TO PROVE OTHER UNIQNESS RESULTS

E.G.: $M = D^2 \times S^1$ GIVEN ANY CONTACT STRUCTURF ξ ON M



- STEP 1: BY THE LEGENDRIAN REALISATION PRINCIPLE WE CAN ASSUME ∂D IS LEGENDRIAN
- STEP 2: ISOTOPE D REL. ∂ TO BE CONVEX: THE DIVIDING CURVE ON D IS A SINGLE ARC
- STEP 3: CUT M ALONG D

CLASSIFICATION OF TIGHT CONTACT STRUCTURES

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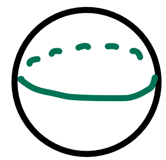

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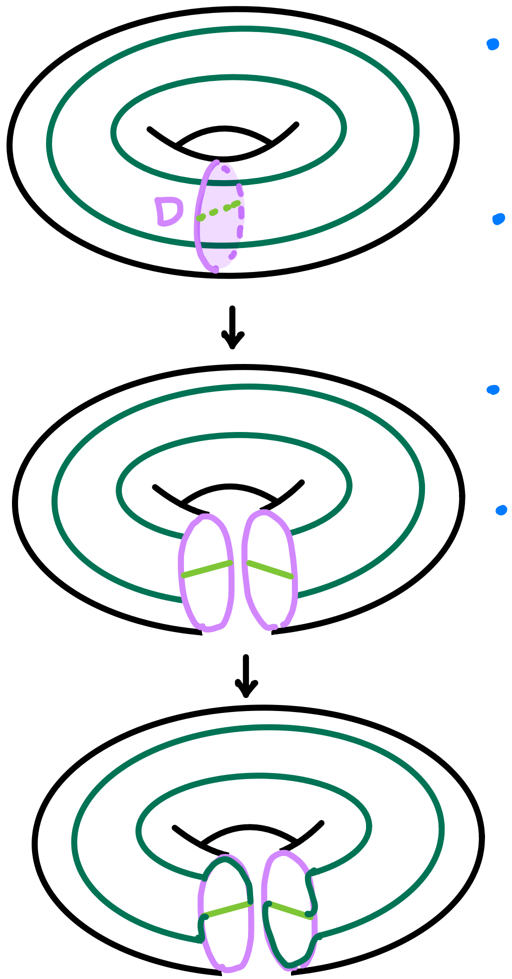
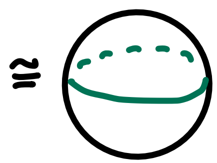


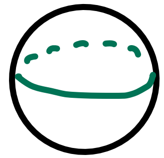
- **STEP 1:** BY THE LEGENDRIAN REALISATION PRINCIPLE WE CAN ASSUME ∂D IS LEGENDRIAN
- **STEP 2:** ISOTOPE D REL. ∂ TO BE CONVEX: THE DIVIDING CURVE ON D IS A SINGLE ARC
- **STEP 3:** CUT M ALONG D
- **STEP 4:** ROUND THE EDGES: WE GET A D^3 WHICH HAS A UNIQUE CONTACT STRUCTURE ξ_0



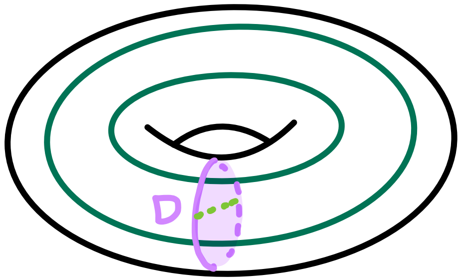
$\Longrightarrow$ ANY ξ CAN BE OBTAINED FROM ξ_0 BY GLUEING $\Longrightarrow$ ξ IS UNIQUE TOO ∎

CLASSIFICATION OF TIGHT CONTACT STRUCTURES

THM (ELIASHBERG): D^3 ADMITS A UNIQUE TIGHT CONTACT STRUCTURE WITH CONNECTED DIVIDING CURVE ON S^2:



E.G.:

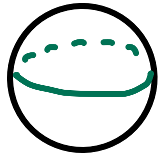


ADMITS A UNIQUE TIGHT CONTACT STRUCTURE

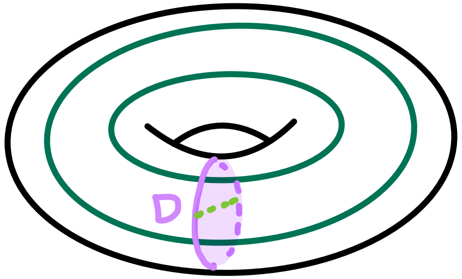
- WHAT DID WE USE IN THE PROOF?

Classification of Tight Contact Structures

THM (Eliashberg): D^3 admits a unique tight contact structure with connected dividing curve on S^2:



E.G.:

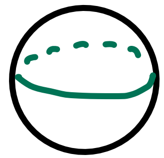


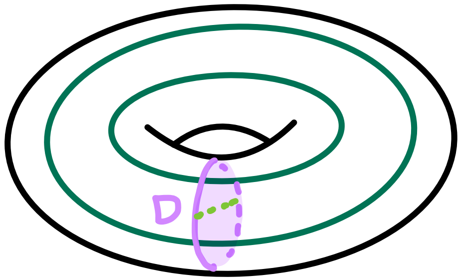
admits a unique tight contact structure

- What did we use in the proof?
- That $|\partial D \cap \Gamma| = 2$ thus the dividing curve on D was well defined

Classification of Tight Contact Structures

> **THM (ELIASHBERG):** D^3 admits a unique tight contact structure with connected dividing curve on S^2:



E.G.: 

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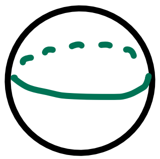
- What did we use in the proof?
- That $|\partial D \cap \Gamma| = 2$ thus the dividing curve on D was well defined
- & that we got  after cutting and rounding

CLASSIFICATION OF TIGHT CONTACT STRUCTURES

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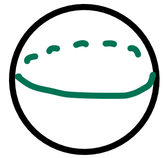

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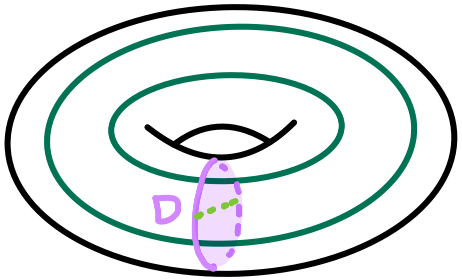
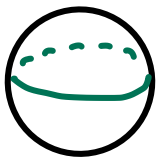
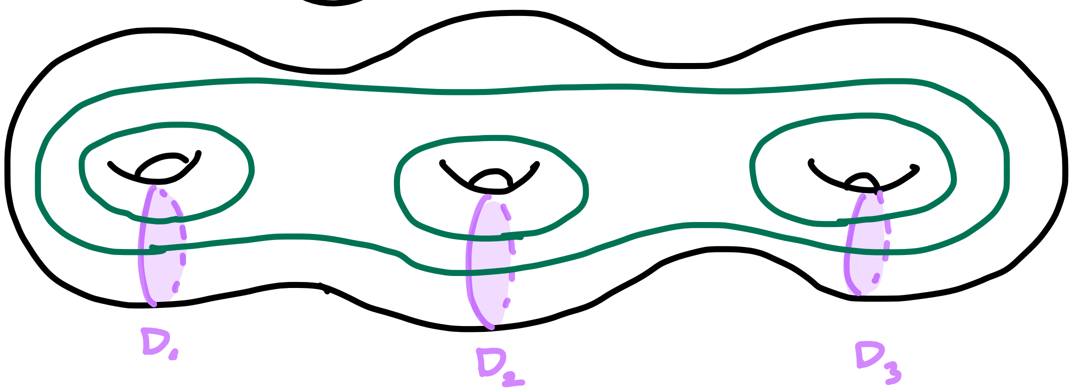


ADMITS A UNIQUE TIGHT CONTACT STRUCTURE

- WHAT DID WE USE IN THE PROOF?
- THAT $|\partial D \cap \Gamma| = 2$ THUS THE DIVIDING CURVE ON D WAS WELL DEFINED
- & THAT WE GOT  AFTER CUTTING AND ROUNDING

SIMILARLY:

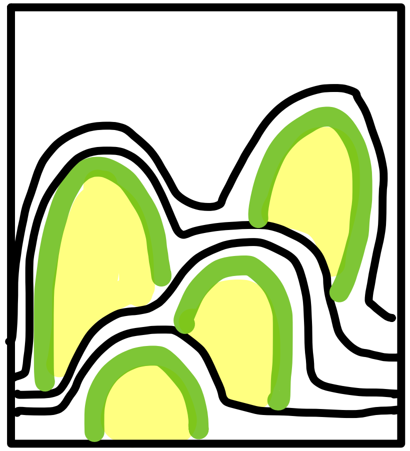


ADMITS A UNIQUE TIGHT CONTACT STRUCTURE

DEF: PRODUCT DISC DECOMPOSABLE

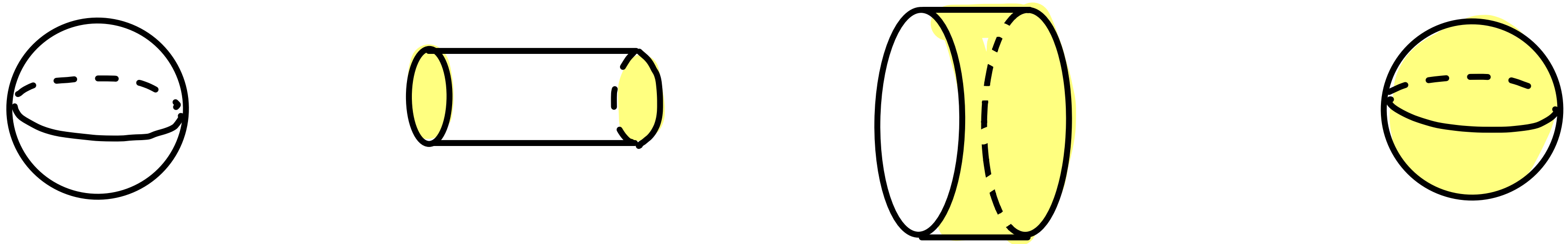
CONTACT CELL-

-DECOMPOSITIONS



Handle Decompositions of Smooth 3-Manifolds

0-HANDLE	1-HANDLE	2-HANDLE	3-HANDLE
$h^0 = D^0 \times D^3$	$h^1 = D^1 \times D^2$	$h^2 = D^2 \times D^1$	$h^3 = D^3 \times D^0$



THM (MORSE): ANY 3-MANIFOLD CAN BE OBTAINED VIA SUCCESIVELY ATTACHING HANDLES ONTO EACH OTHER ALONG $\partial D^i \times D^{3-i}$ (HANDLE DECOMPOSITION)

Handle Decompositions of Smooth 3-Manifolds

0-Handle

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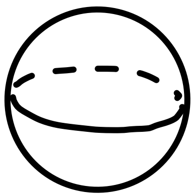
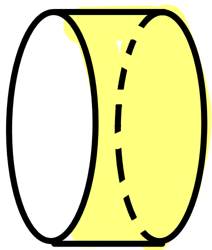


1-Handle

$h^1 = D^1 \times D^2$

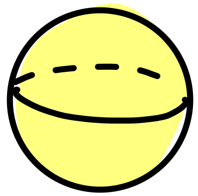
2-Handle

$h^2 = D^2 \times D^1$



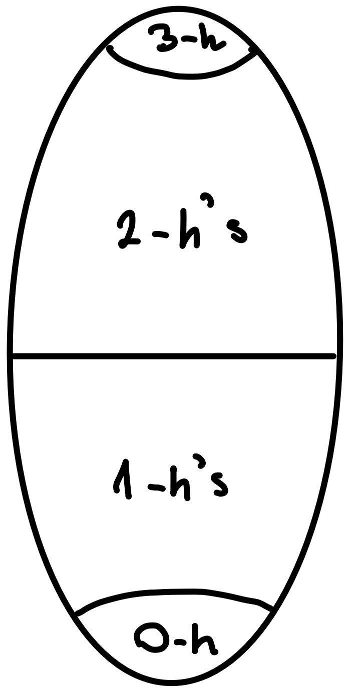
3-Handle

$h^3 = D^3 \times D^0$



THM (Morse): Any 3-manifold can be obtained via succesively attaching handles onto each other along $\partial D^i \times D^{3-i}$ (**Handle Decomposition**)

Moreover the indices of handles can be assumed to be in increasing order



HANDLE DECOMPOSITIONS OF SMOOTH 3-MANIFOLDS

0-HANDLE

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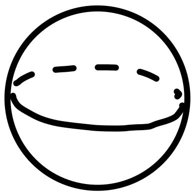

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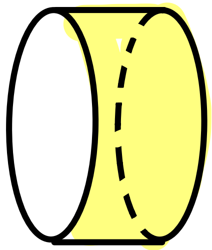

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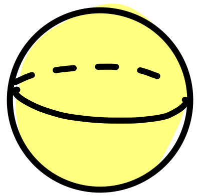
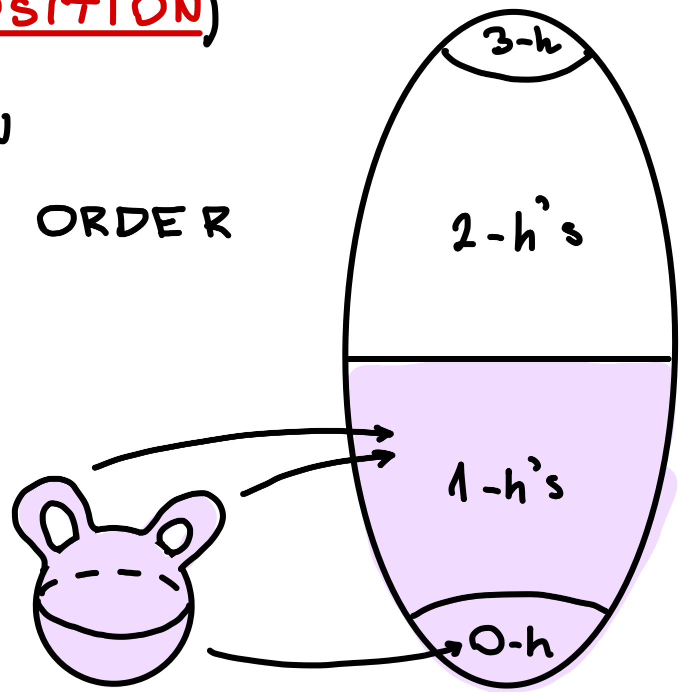


THM (MORSE): ANY 3-MANIFOLD CAN BE OBTAINED VIA SUCCESIVELY ATTACHING HANDLES ONTO EACH OTHER ALONG $\partial D^i \times D^{3-i}$ (HANDLE DECOMPOSITION)

MOREOVER THE INDICES OF HANDLES CAN BE ASSUMED TO BE IN INCREASING ORDER

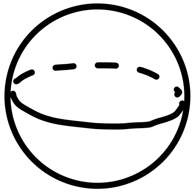


HANDLEBODY

HANDLE DECOMPOSITIONS OF SMOOTH 3-MANIFOLDS

0-HANDLE

$h^0 = D^0 \times D^3$



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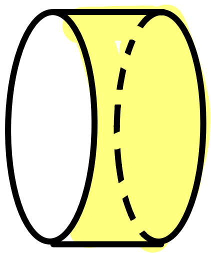

3-HANDLE

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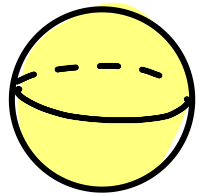

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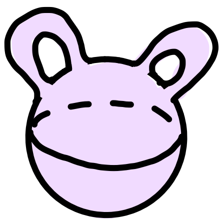
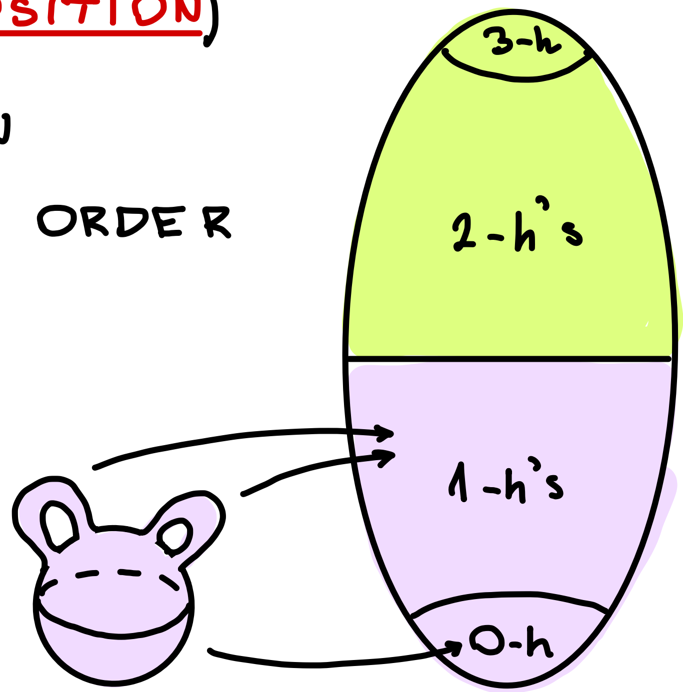
MOREOVER THE INDICES OF HANDLES CAN BE ASSUMED TO BE IN INCREASING ORDER

COR: $M =$  $\cup_\Sigma$ 



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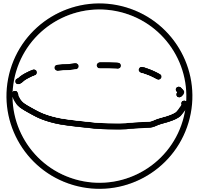
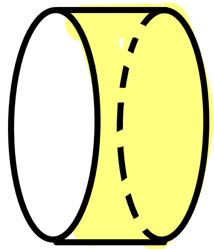
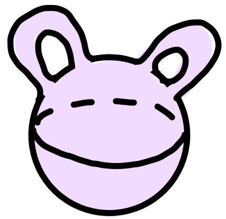
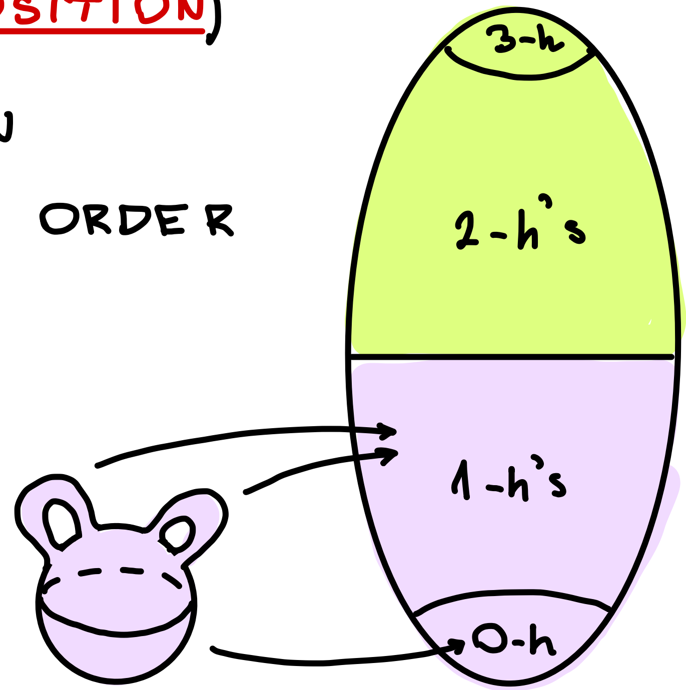
0-HANDLE	1-HANDLE	2-HANDLE	3-HANDLE
$h^0 = D^0 \times D^3$	$h^1 = D^1 \times D^2$	$h^2 = D^2 \times D^1$	$h^3 = D^3 \times D^0$
			

THM (MORSE): ANY (3-)MANIFOLD CAN BE OBTAINED VIA SUCCESIVELY ATTACHING HANDLES ONTO EACH OTHER ALONG $\partial D^i \times D^{3-i}$ (HANDLE DECOMPOSITION)

MOREOVER THE INDICES OF HANDLES CAN BE ASSUMED TO BE IN INCREASING ORDER

COR: M =  $\cup_\Sigma$ 



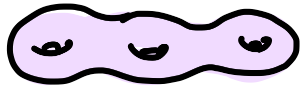
HANDLEBODY

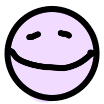
> EVERY 3-MANIFOLD ADMITS A HEEGAARD DECOMPOSITION

HEEGAARD DECOMPOSITIONS

THM (ALEXANDER) EVERY 3-MANIFOLD ADMITS A **HEEGAARD DECOMPOSITION**

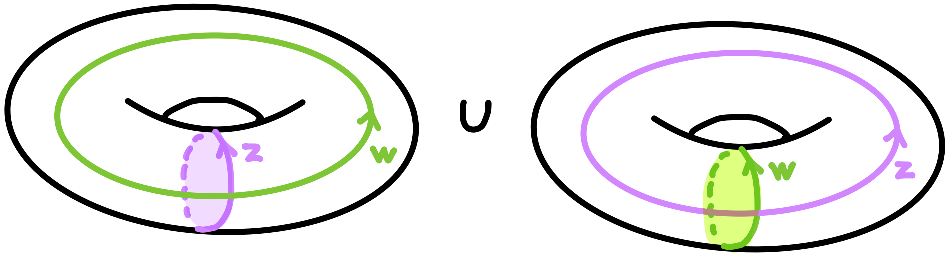
$M = U \cup V$

HANDLEBODIES: 

E.G.: • $S^3 =$  $\cup$  (SCHEMATIC )

• $S^3 = \{ |z|^2 + |w|^2 = 1 \} \subseteq \mathbb{C}^2$

$= \{ |z|^2 \leq 1/2 \} \cup \{ |w|^2 \leq 1/2 \}$, $\qquad \Sigma = \{ |z| = |w| = 1/2 \} \cong S^1 \times S^1$



CONTACT HANDLES

0-HANDLE

$D^0 \times D^3$

h^0

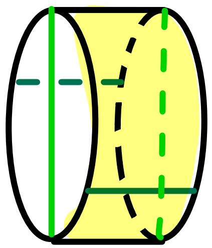
1-HANDLE

$D^1 \times D^2$

h^1

2-HANDLE

$D^2 \times D^1$



h^2

3-HANDLE

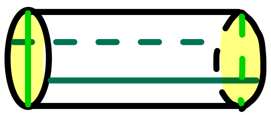
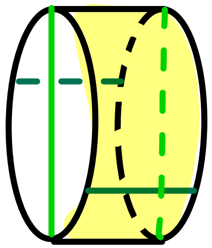
$D^3 \times D^0$

h^3

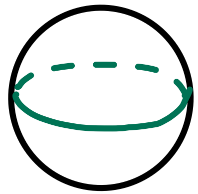
AFTER ROUNDING THE EDGES EACH BECOMES 

& BY ELIASHBERG'S THM THIS ADMITS A UNIQE TIGHT CONTACT STRUCTURE

CONTACT HANDLES

0-HANDLE	1-HANDLE	2-HANDLE	3-HANDLE
$D^0 \times D^3$	$D^1 \times D^2$	$D^2 \times D^1$	$D^3 \times D^0$
			
(h^0, ξ^0)	(h^1, ξ^1)	(h^2, ξ^2)	(h^3, ξ^3)

AFTER ROUNDING THE EDGES EACH BECOMES



& BY ELIASHBERG'S THM THIS ADMITS A UNIQE TIGHT CONTACT STRUCTURE

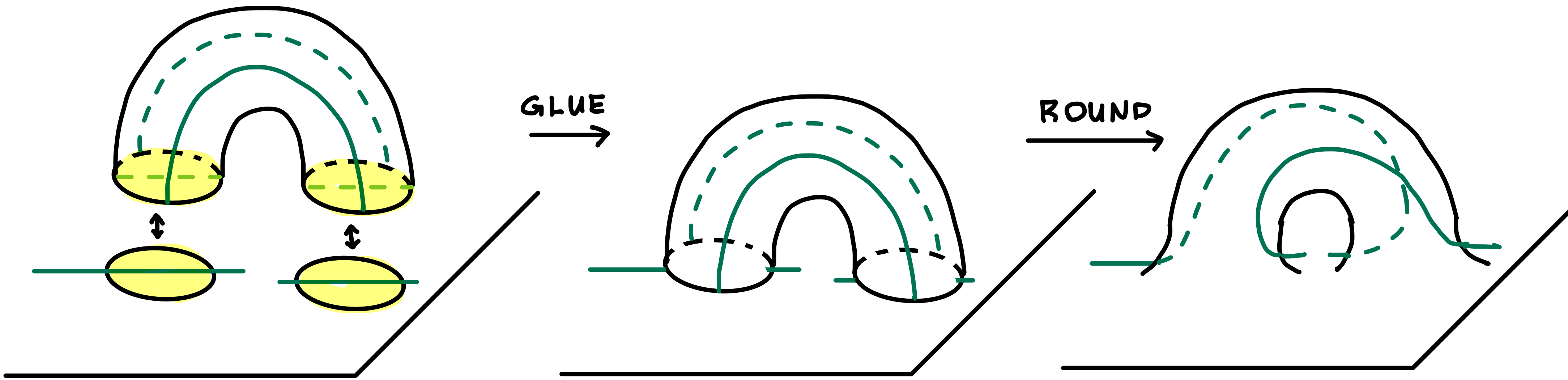
$\rightsquigarrow$ UP TO ISOTOPY WE GET WELL DEFINED CONTACT STRUCTURES ON THE h^i

ATTACHING CONTACT HANDLES

i-HANDLE $h^i = D^i \times D^{3-i}$

$$\boxed{\partial h^i = \partial D^i \times D^{3-i} \cup D^i \times \partial D^{-i}}$$

WE CAN CONSTRUCT CONTACT MANIFOLDS BY SUCCESIVELY GLUING HANDLES ALONG $\partial D^i \times D^{3-i}$:



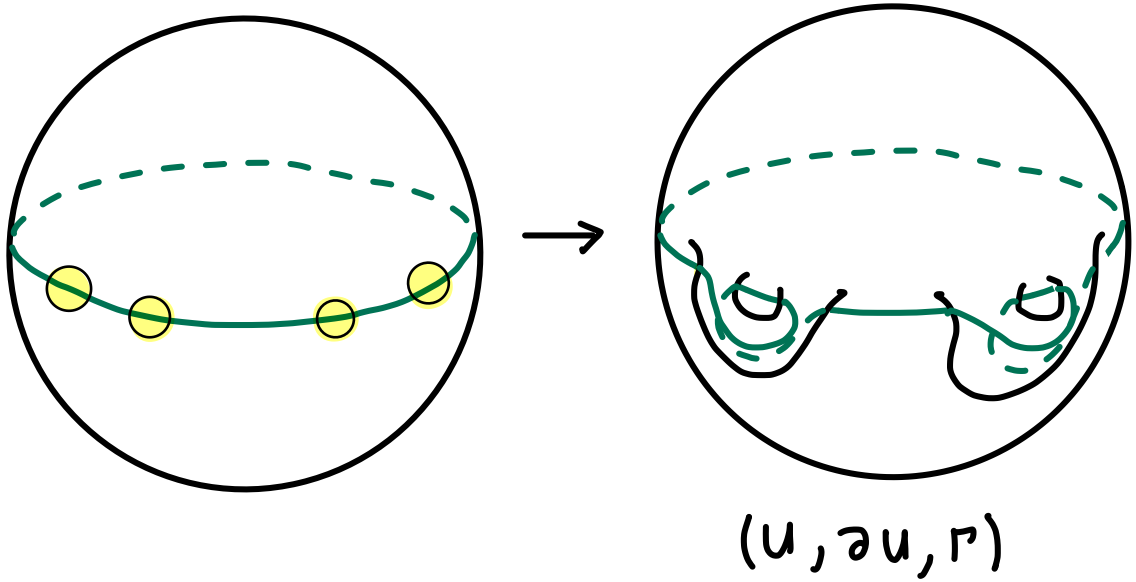
CONTACT HANDLEDECOMPOSITION

THM (GIROUX) ANY CONTACT 3-MANIFOLD ADMITS A CONTACT HANDLEDECOMPOSITION

Contact Handlebody

0-handle ∪ some 1-handles

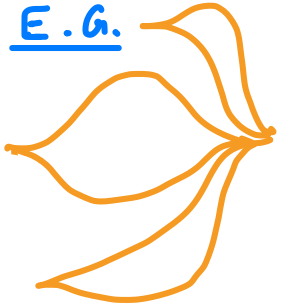
E.G.



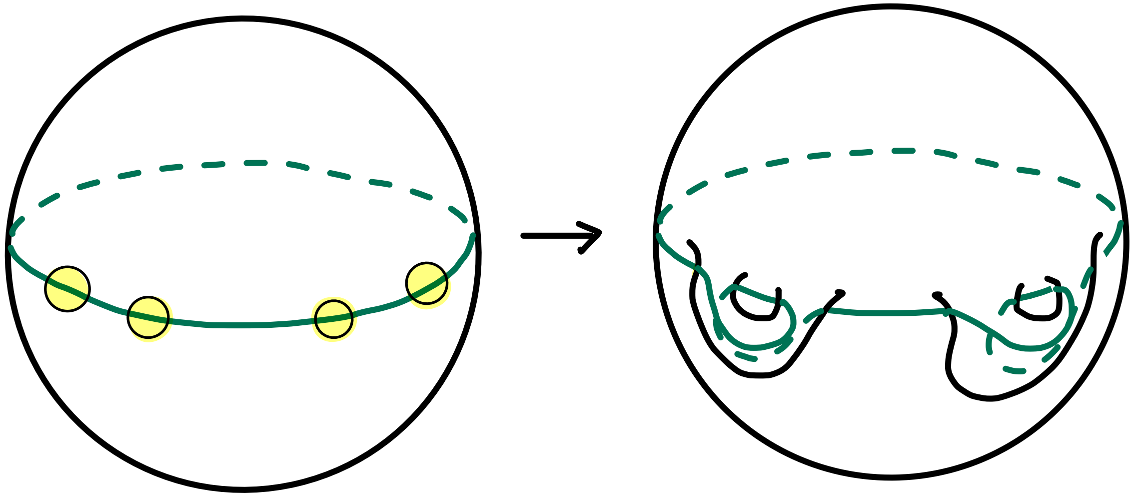
Contact Handlebody

0-HANDLE U SOME 1-HANDLES

E.G.



NEIGHBOURHOOD OF A LEGENDRIAN GRAPH:

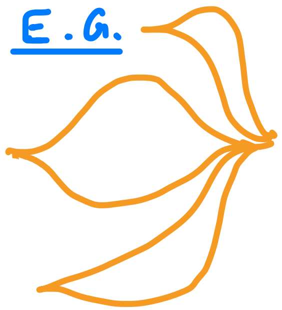


$(U, \partial U, \Gamma)$

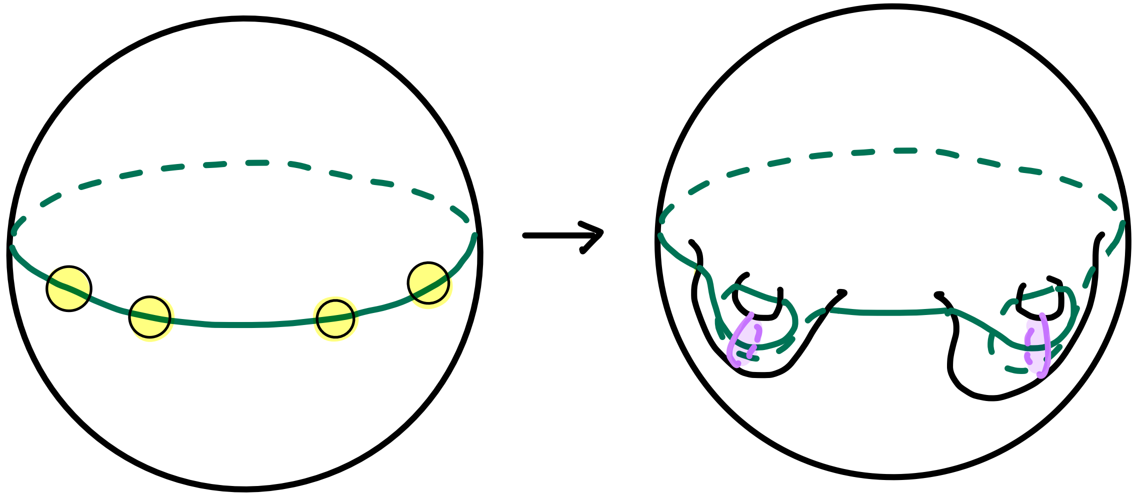
CONTACT HANDLEBODY

0-HANDLE $\cup$ SOME 1-HANDLES

E.G.



NEIGHBOURHOOD OF A LEGENDRIAN GRAPH:



$(\mathcal{U}, \partial\mathcal{U}, \Gamma)$

NOTE: CONTACT HANDLEBODIES ARE PRODUCT DISC DECOMPOSABLE

RECALL: A PRODUCT DISC DECOMPOSABLE HANDLEBODY $\mathcal{U}$ ADMITS A UNIQUE TIGHT CONTACT STRUCTURE WITH DIVIDING CURVE Γ ON $\partial\mathcal{U}$.

THUS

THM: A PRODUCT DISC DECOMPOSABLE HANDLEBODY $\mathcal{U}$ WITH TIGHT CONTACT STRUCTURE ξ IS A CONTACT HANDLEBODY

Rearranging Contact Handles

∃ UST AS IN THE SMOOTH CASE

IN A CONTACT HANDLE DECOMPOSITION ONE CAN ASSUME THAT:

- CONTACT 0-h's ARE ATTACHED FIRST
- CONTACT 3-h's ARE ATTACHED LAST

Rearranging Contact Handles

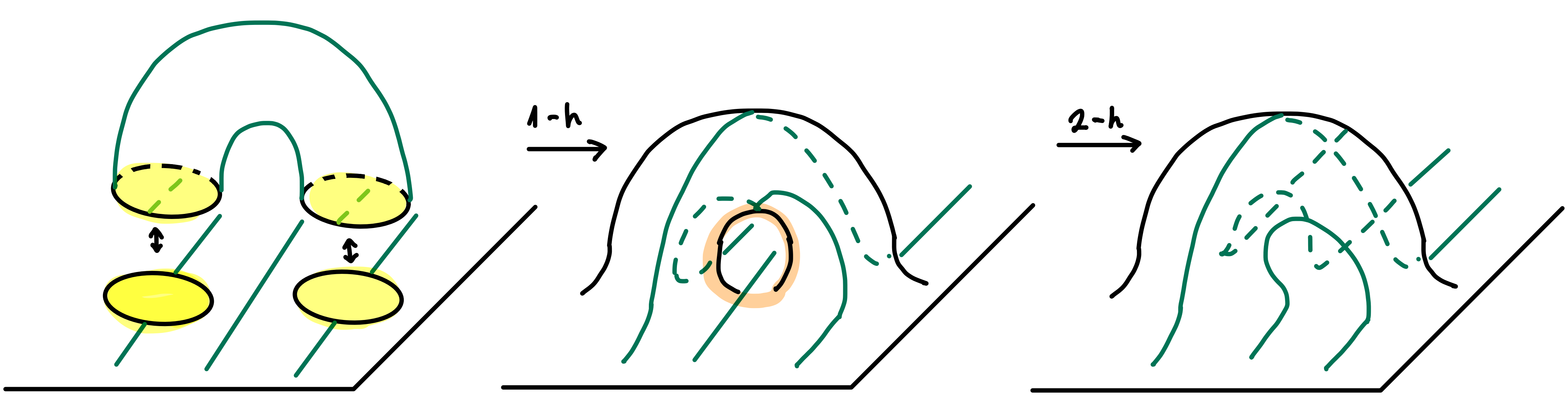
JUST AS IN THE SMOOTH CASE

IN A CONTACT HANDLE DECOMPOSITION ONE CAN ASSUME THAT:

- CONTACT 0-h's ARE ATTACHED FIRST
- CONTACT 3-h's ARE ATTACHED LAST

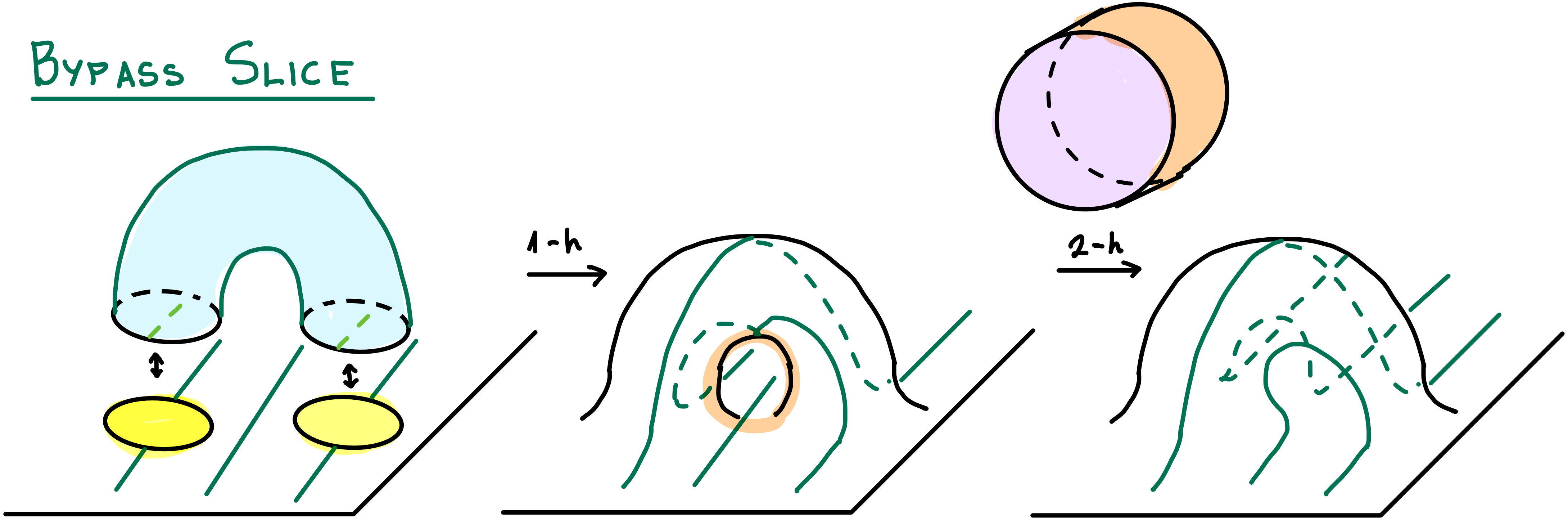
BUT! CONTACT 1-h's CANNOT ALWAYS BE ATTACHED BEFORE CONTACT 2-h's

Bypass Slice



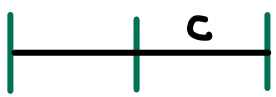
FROM THE TOP

BYPASS SLICE



FROM THE TOP

THE ABOVE PAIR OF CONTACT 1- & 2-h CAN BE ATTACHED TO ANY CONVEX SURFACE (Σ, Γ) ALONG ANY ARC c INTERSECTING Γ AS



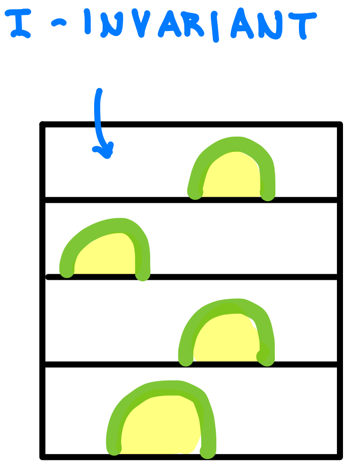
WILL SEE: BYPASSES ARE BASIC BUILDING BLOCKS OF CONTACT STRUCTURES ON $\Sigma \times I$

Convex Surface Theory

0-PARAMETER

RECALL (GIROUX): CONVEX SURFACES ARE C^∞-GENERIC

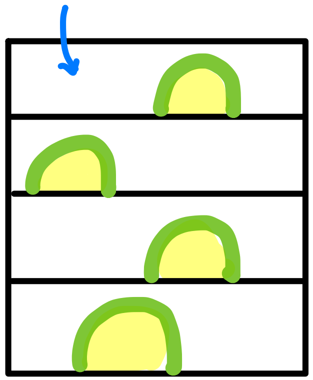
THM (GIROUX REPHRASED BY HONDA) ANY CONTACT STRUCTURE ON $\Sigma \times I$ IS CONTACTOMORPHIC TO A STACK OF BYPASS SLICES



CONVEX SURFACE THEORY

0-PARAMETER

RECALL (GIROUX): CONVEX SURFACES ARE C^∞-GENERIC

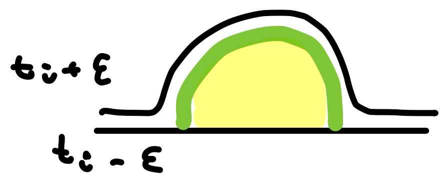


THM (GIROUX REPHRASED BY HONDA) ANY CONTACT STRUCTURE ON $\Sigma \times I$ IS CONTACTOMORPHIC TO A STACK OF BYPASS SLICES

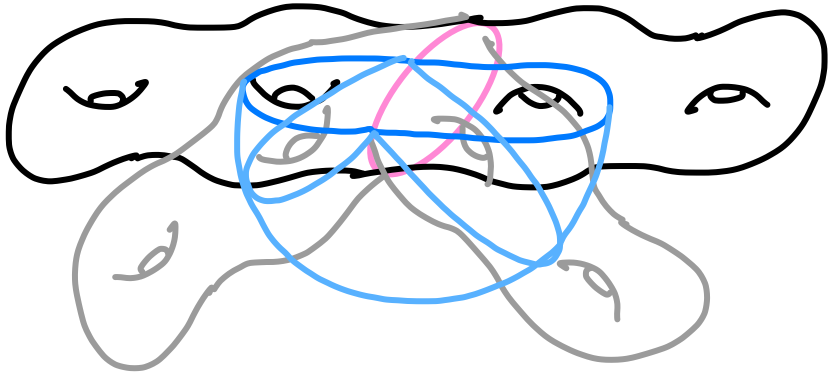
MORE GENERALLY WE CAN THINK OF THIS AS GENERICITY STATEMENT FOR 1-PARAMETER FAMILIES:

THM (GIROUX, COLIN, REPHRASED BY HONDA) ANY 1-PARAMETER FAMILY OF SURFACES $(\Sigma_t)_{t\in[0,1]}$ WITH Σ_0, Σ_1 CONVEX CAN BE ISOTOPED TO $(\Sigma'_t)_{t\in[0,1]}$ SO THAT

- $\Sigma_t = \Sigma'_t$ NEAR $t = 0$ & 1
- Σ'_t IS CONVEX EXCEPT AT DISCRETE TIMES $\{t_1, .., t_k\} \subseteq [0,1]$
- $\Sigma'_{t_i - \varepsilon}$ & $\Sigma'_{t_i + \varepsilon}$ COBOUND A BYPASS SLICE $(i = 1..k)$



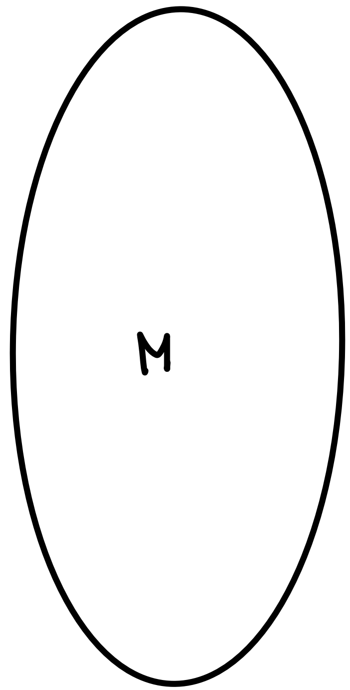
EXISTENCE OF CONTACT HEEGAARD DECOMPOSITIONS



Existence of Contact Heegaard Decompositions

THM (Giroux): Any contact 3-manifold (M,ξ) admits a contact Heegaard decomposition

PROOF (Licata - V)

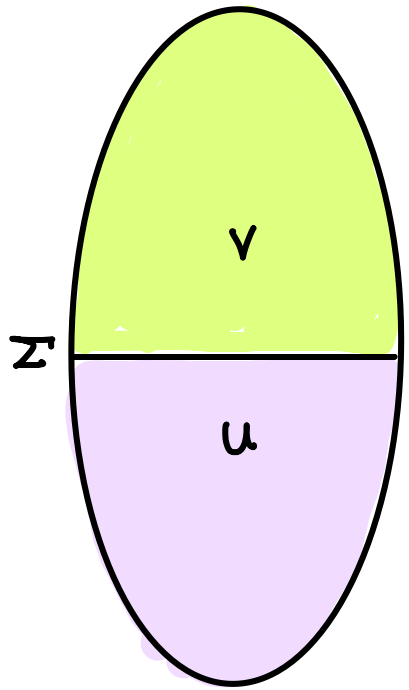


Existence of Contact Heegaard Decompositions

THM (GIROUX): ANY CONTACT 3-MANIFOLD (M,ξ) ADMITS A CONTACT HEEGAARD DECOMPOSITION

PROOF (LICATA - V)

STEP 1: TAKE ANY SMOOTH HEEGAARD DECOMPOSITION OF M: $M = U \cup_\Sigma V$



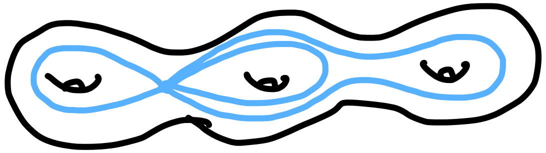
EXISTENCE OF CONTACT HEEGAARD DECOMPOSITIONS

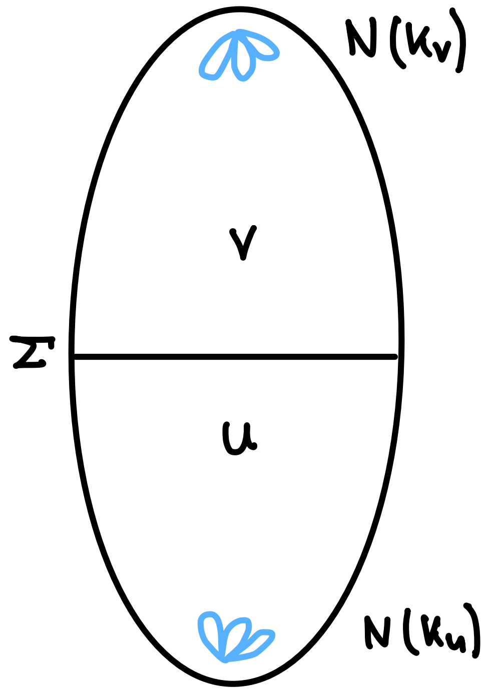
> **THM (GIROUX):** ANY CONTACT 3-MANIFOLD (M,ξ) ADMITS A CONTACT HEEGAARD DECOMPOSITION

PROOF (LICATA - V)

STEP 1: TAKE ANY SMOOTH HEEGAARD DECOMPOSITION OF M: $M = U \cup_{\Sigma} V$

STEP 2: TAKE SPINES $K_U \subset U$ & $K_V \subset V$





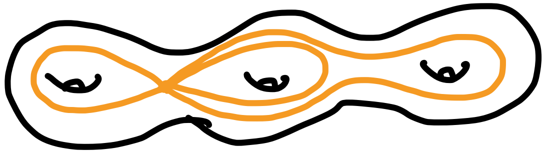
Existence of Contact Heegaard Decompositions

THM (GIROUX): ANY CONTACT 3-MANIFOLD (M,ξ) ADMITS A CONTACT HEEGAARD DECOMPOSITION

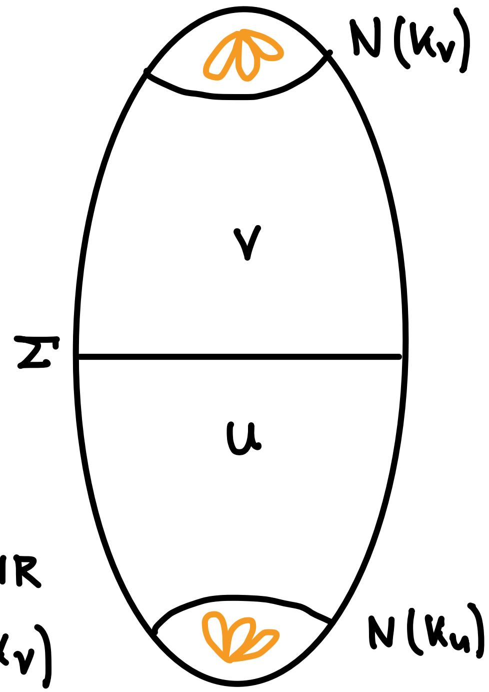
PROOF (LICATA - V)

STEP 1: TAKE ANY SMOOTH HEEGAARD DECOMPOSITION OF M : $M = U \cup_\Sigma V$

STEP 2: TAKE SPINES $K_U \subset U$ & $K_V \subset V$



STEP 3: LEGENDRIAN REALISE K_U & K_V & TAKE THEIR STANDARD CONTACT NEIGHBOURHOODS $N(K_U)$ & $N(K_V)$



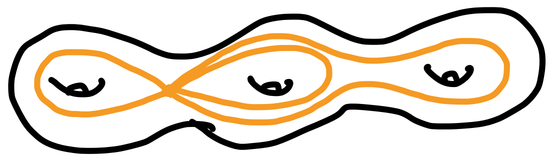
Existence of Contact Heegaard Decompositions

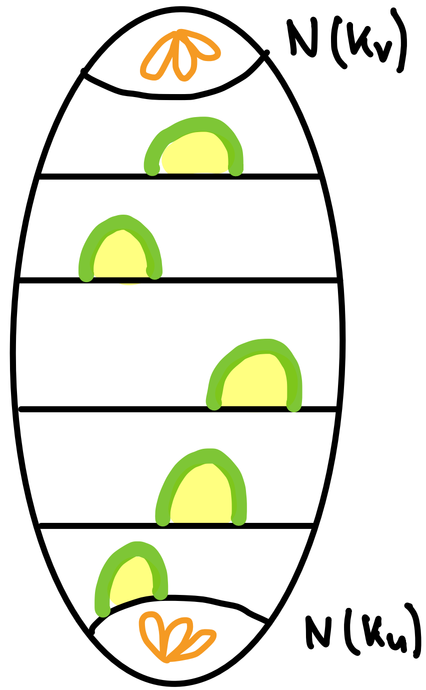
THM (Giroux): ANY CONTACT 3-MANIFOLD (M,ξ) ADMITS A CONTACT HEEGAARD DECOMPOSITION

PROOF (LICATA - V)

STEP 1: TAKE ANY SMOOTH HEEGAARD DECOMPOSITION OF M: $M = U \cup_{\Sigma} V$

STEP 2: TAKE SPINES $K_U \subset U$ & $K_V \subset V$





STEP 3: LEGENDRIAN REALISE K_U & K_V & TAKE THEIR STANDARD CONTACT NEIGHBOURHOODS $N(K_U)$ & $N(K_V)$

STEP 4: $X = M - (N(K_U) \cup N(K_V)) \overset{\text{DIFFEO}}{\cong} \Sigma \times I \Rightarrow \xi|_X$ CAN BE WRITTEN AS A STACK OF BYPASS-SLICES $\cong h^1 \cup h^2$

(AWAY FROM THE HANDLES ξ IS I-INVARIANT)

Existence of Contact Heegaard Decompositions

THM (GIROUX): ANY CONTACT 3-MANIFOLD (M,ξ) ADMITS A CONTACT HEEGAARD DECOMPOSITION

PROOF (LICATA - V)

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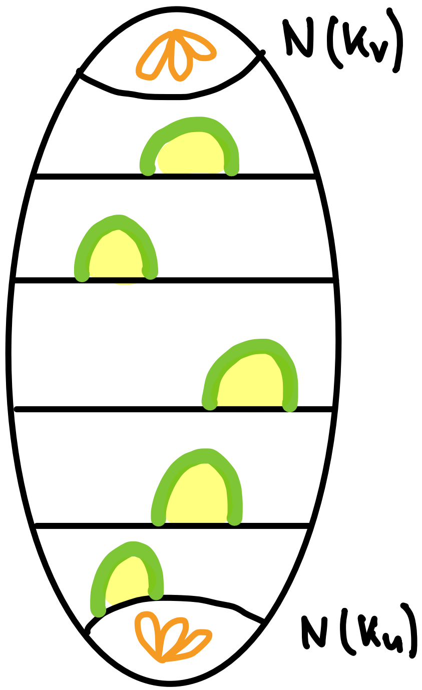
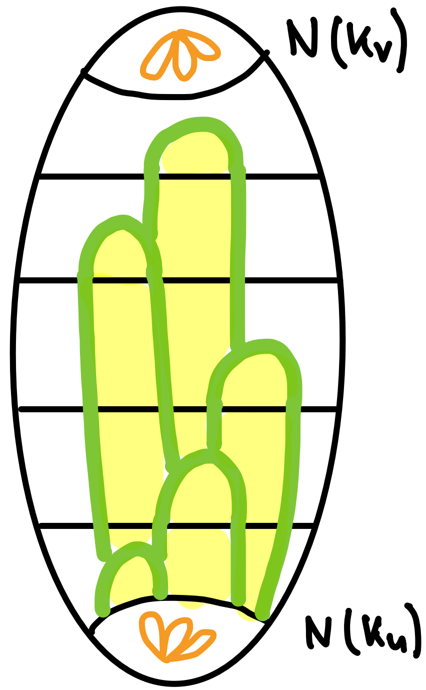


Existence of Contact Heegaard Decompositions

> **THM (GIROUX):** ANY CONTACT 3-MANIFOLD (M,ξ) ADMITS A CONTACT HEEGAARD DECOMPOSITION

PROOF (LICATA - V)

STEP 4: $X = M - (N(K_u) \cup N(K_v)) \overset{\text{DIFFEO}}{\cong} \Sigma \times I \Rightarrow \xi|_X$ CAN BE WRITTEN AS A STACK OF BYPASS-SLICES $\cong h^1 \cup h^2$ (AWAY FROM THE HANDLES ξ IS I-INVARIANT)



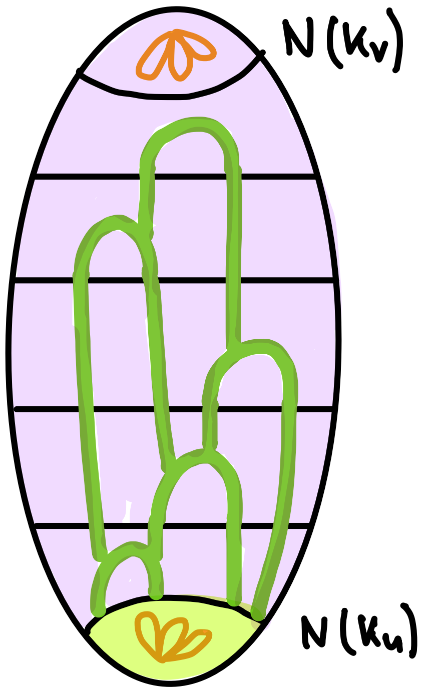
STEP 5: USE THIS FLOW TO EXTEND THE HANDLES

Existence of Contact Heegaard Decompositions

THM (GIROUX): ANY CONTACT 3-MANIFOLD (M,ξ) ADMITS A CONTACT HEEGAARD DECOMPOSITION

PROOF (LICATA - V)

STEP 4: $X = M - (N(K_u) \cup N(K_v)) \overset{\text{DIFFEO}}{\cong} \Sigma \times I \Rightarrow \xi|_X$ CAN BE WRITTEN AS A STACK OF BYPASS-SLICES $\cong h^1_i \cup h^2$ (AWAY FROM THE HANDLES ξ IS I-INVARIANT)



STEP 5: USE THIS FLOW TO EXTEND THE HANDLES

CONSIDER: $\hat{U} = N(K_u) \cup (\cup h^1_i)$

CONTACT HANDLEBODY

CONTACT 1-HANDLES

$\Longrightarrow \hat{U}$ IS A CONTACT HANDLEBODY

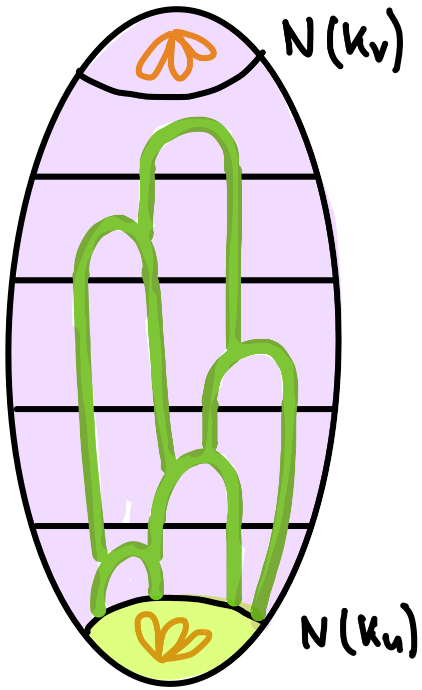
EXISTENCE OF CONTACT HEEGAARD DECOMPOSITIONS

THM (GIROUX): ANY CONTACT 3-MANIFOLD (M,ξ) ADMITS A CONTACT HEEGAARD DECOMPOSITION

PROOF (LICATA - V)

STEP 4: $X = M - (N(K_u) \cup N(K_v)) \overset{\text{DIFFEO}}{\cong} \Sigma \times I \Rightarrow \xi|_X$ CAN BE WRITTEN AS A STACK OF BYPASS-SLICES $\cong h^1_i \cup h^2$

(AWAY FROM THE HANDLES ξ IS I-INVARIANT)



STEP 5: USE THIS FLOW TO EXTEND THE HANDLES

CONSIDER: $\hat{U} = N(K_u) \cup (\cup h^1_i)$

CONTACT HANDLEBODY

CONTACT 1-HANDLES

$\Longrightarrow \hat{U}$ IS A CONTACT HANDLEBODY

UPSIDE DOWN: $\hat{V} = M \setminus \hat{U} \cong N(K_v) \cup (\cup h^2_i)$ IS A CONTACT HANDLEBODY

$\Longrightarrow M = \hat{U} \cup \hat{V}$ IS A CONTACT HEEGAARD DECOMPOSITION ∎

BRIDGING

$$\underset{\mathcal{H}}{M = U \cup V} \xrightarrow{\text{BRIDGING}} \underset{\hat{\mathcal{H}}}{M = \hat{U} \cup \hat{V}}$$

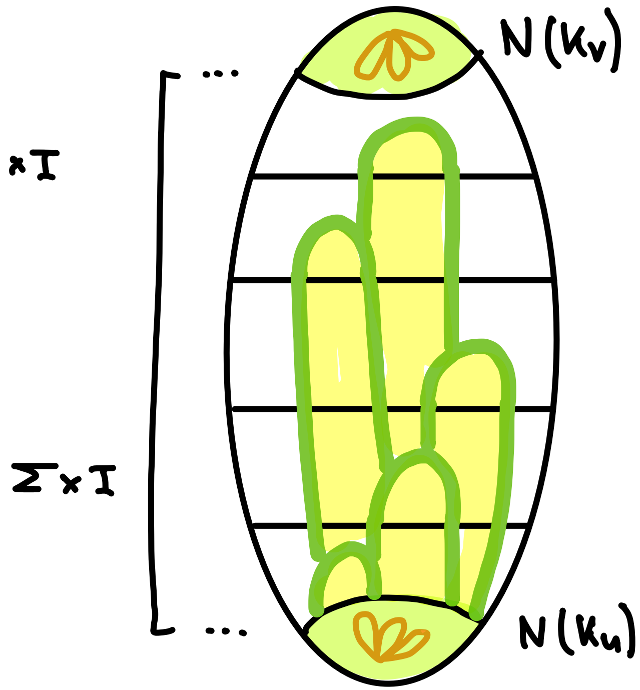
BRIDGE WHERE

- $\hat{U} = N(K_u) \cup (\cup h_i^1)$
- $\hat{V} = \overline{M \setminus \hat{U}}$

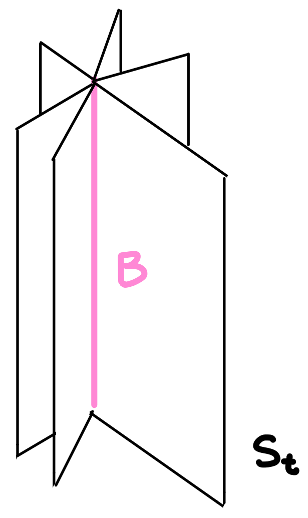
NOTE: THE BRIDGE DEPENDS ON

- $\mathcal{H}$
- K_u & K_v
- THE BYPASSES $\mathcal{B}$ BUILDING UP $\xi|_{\Sigma \times I}$

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}(K_u, K_v, \mathcal{B})$$



OPEN BOOK DECOMPOSITIONS

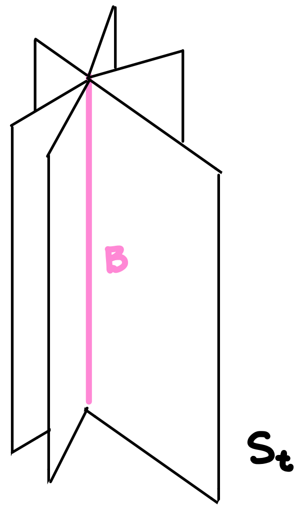


Open Book Decompositions

DEF: PAIR (B,π), WHERE

- $B^1 \hookrightarrow M$ EMBEDDED 1-MANIFOLD: **BINDING**
- $\pi: M - B \longrightarrow S^1$ FIBRATION SUCH THAT
  - $\to \forall\, t \in S^1$ $\quad S_t := \overline{\pi^{-1}(t)}$ IS A SEIFERT SURFACE FOR B
  - $\to$ & ON $N(B) \cong B \times D^2$ $\quad \pi =$ ANGLE

DEF: $S_t := \pi^{-1}(t)$ ARE THE **PAGES** OF (B,π)



Open Book Decompositions

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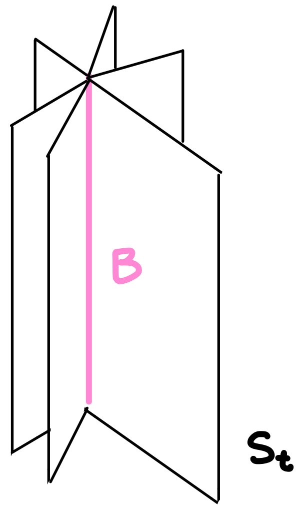


E.G.: $M = S^3 = \{|z|^2 + |w|^2 = 1\} \subseteq \mathbb{C}^2$

$B = \{|z| = 0\} \cong S^1$, $\quad \pi : S^3 \setminus B \longrightarrow S^1$

$$(z,w) \longmapsto \frac{z}{|z|}$$

Open Book Decompositions

DEF: PAIR (B,π), WHERE

- $B^1 \hookrightarrow M$ EMBEDDED 1-MANIFOLD: **BINDING**
- $\pi : M - B \longrightarrow S^1$ FIBRATION SUCH THAT
  - $\forall\, t \in S^1$ $\quad S_t := \overline{\pi^{-1}(t)}$ IS A SEIFERT SURFACE FOR B
  - & ON $N(B) \cong B \times D^2$ $\quad \pi = $ ANGLE

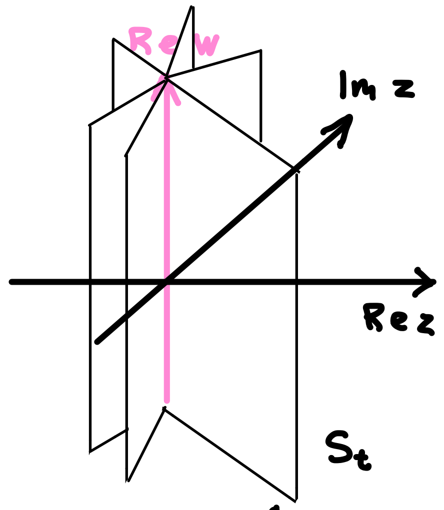
DEF: $S_t := \pi^{-1}(t)$ ARE THE **PAGES** OF (B,π)

E.G.: $M = S^3 = \{|z|^2 + |w|^2 = 1\} \subseteq \mathbb{C}^2$

$B = \{|z| = 0\} \cong S^1$, $\quad \pi : S^3 \setminus B \longrightarrow S^1$

$$(z,w) \longmapsto \frac{z}{|z|}$$

ON $\mathbb{R}^3$ WITH COORDINATES: $\dfrac{1}{1 - \operatorname{Im} w}(z, \operatorname{Re} w)$



$\{\arg(z) = \frac{z}{|z|} = t\}$

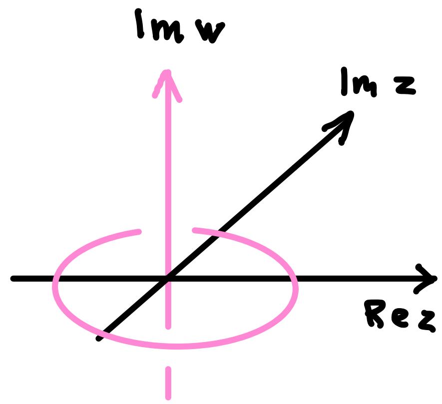
HERE $S_t \cong D^2$ $\quad \forall\, t$

OPEN BOOK DECOMPOSITIONS – ANOTHER EXAMPLE

E.G.: $M = S^3 = \{|z|^2 + |w|^2 = 1\} \subseteq \mathbb{C}^2$

$B = \{|z| = 0\} \sqcup \{|w| = 0\}$

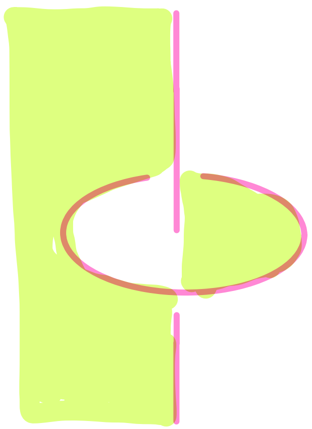
$$\pi : S^3 \setminus B \longrightarrow S^1$$
$$(z, w) \longmapsto \frac{zw}{|zw|}$$



ON $\mathbb{R}^3$ WITH COORDINATES: $\frac{1}{1 - \operatorname{Im} w}(z, \operatorname{Re} w)$

HERE $B \cong S^1 \sqcup S^1$

$S_t \cong S^1 \times I \quad \forall t$



Video

THM (ALEXANDER): ANY 3-MANIFOLD ADMITS AN OPEN BOOK DECOMPOSITION

PROOF LATER

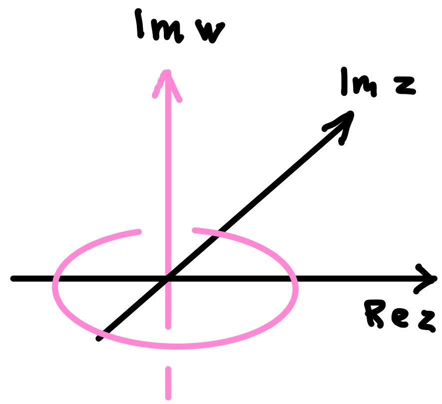
Open Book Decompositions - Another Example

E.G.: $M = S^3 = \{|z|^2 + |w|^2 = 1\} \subseteq \mathbb{C}^2$

$B = \{|z| = 0\} \amalg \{|w| = 0\}$

$\pi : S^3 \setminus B \longrightarrow S^1$

$(z, w) \longmapsto \dfrac{zw}{|zw|}$



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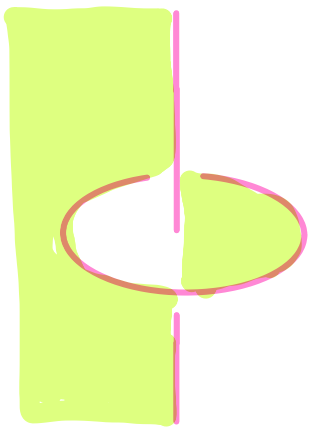
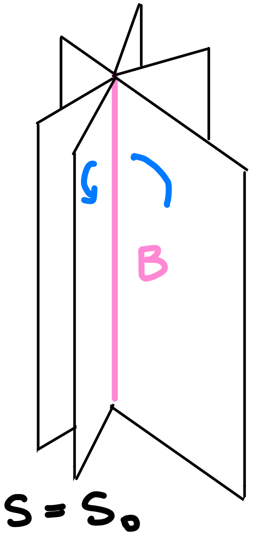


Video

THM (ALEXANDER): ANY 3-MANIFOLD ADMITS AN OPEN BOOK DECOMPOSITION

PROOF LATER

ABSTRACT OPEN BOOKS

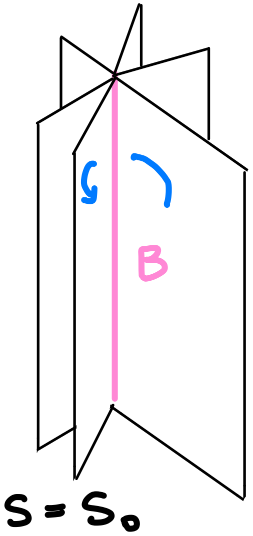


FIX $S := S_0$ & LOOK AT THE FIRST RETURN-MAP OF $\pi : M - B \longrightarrow S^1$ $\rightsquigarrow$ GET (S, φ) WHERE

- S IS AN ORIENTED SURFACE WITH BOUNDARY
- & $\varphi : S \circlearrowleft S$ HOMEOMORPHISM THAT FIXES $N(\partial S)$

DEF: THE PAIR (S, φ) IS AN ABSTRACT OPEN BOOK

ABSTRACT OPEN BOOKS



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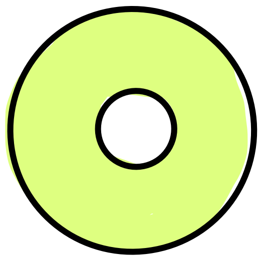
- & $\varphi : S \circlearrowleft S$ HOMEOMORPHISM THAT FIXES $N(\partial S)$

DEF: THE PAIR (S, φ) IS AN **ABSTRACT OPEN BOOK**

E.G.: THE PREVIOUS EXAMPLE GIVES

$S =$ 

Abstract Open Books

FIX $S := S_0$ & LOOK AT THE FIRST RETURN-MAP OF $\pi : M - B \longrightarrow S^1$ $\rightsquigarrow$ GET (S, φ) WHERE

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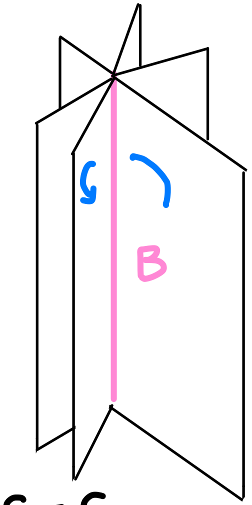
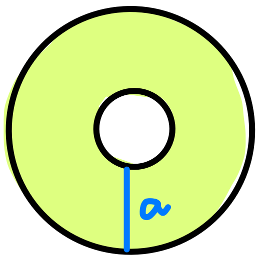


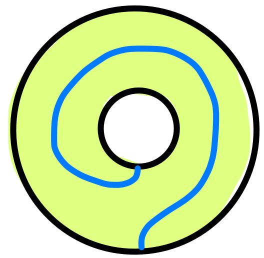
$S = S_0$

DEF: THE PAIR (S, φ) IS AN **ABSTRACT OPEN BOOK**

E.G.: THE PREVIOUS EXAMPLE GIVES

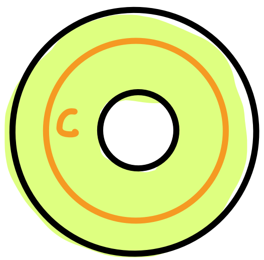
$S =$ 

$S^1 \times I$

& φ MAPPING a TO $\varphi(a)$ 

(THIS DETERMINES φ UP TO ISOTOPY)

DEF: THE ABOVE MAP IS A **RIGHT HANDED DEHN-TWIST** ALONG c

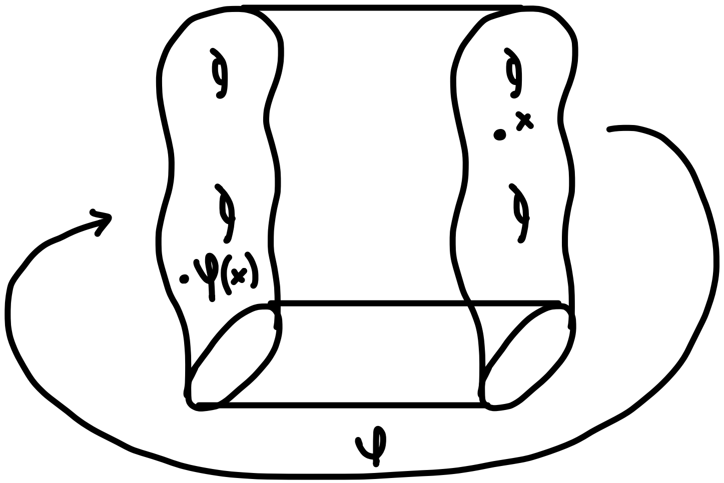


ABSTRACT OPEN BOOKS

CONVERSALY: AN ABSTRACT OB (S,φ) DETERMINES A 3-MANIFOLD M TOGETHER WITH AN OPEN BOOK DECOMPOSITION

PROOF: • TAKE THE MAPPING TORUS OF φ:

$$M_\varphi = S\times I \Big/ (x,1)\sim(\varphi(x),0)$$



• (AS φ FIXES ∂S) $\partial M_\varphi = \partial S \times S^1$

Abstract Open Books

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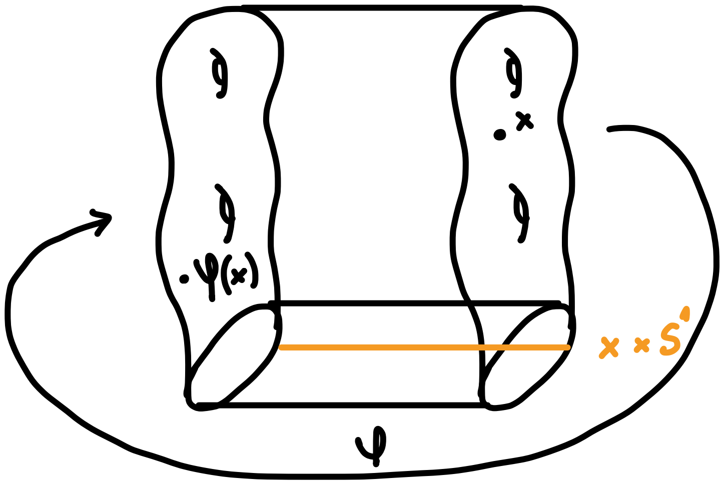
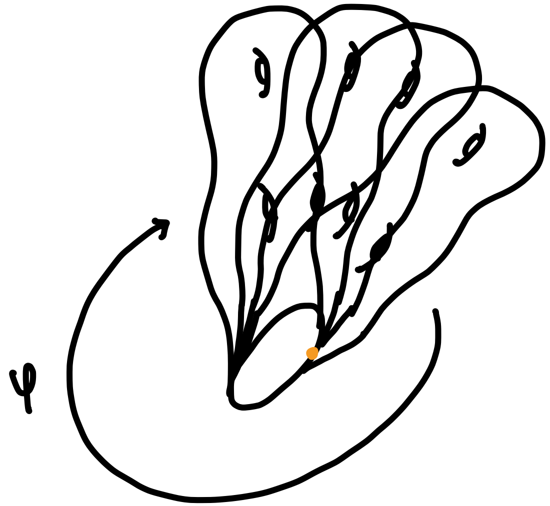


• (AS φ FIXES ∂S) $\partial M_\varphi = \partial S\times S^1$

• & WE COLLAPSE EACH CIRCLE $x\times S^1$ $(x\in\partial S)$

$$M := M_\varphi \Big/ (x,t)\sim(x,t') \quad x\in\partial S$$



THEN: $B = \partial S/\sim$

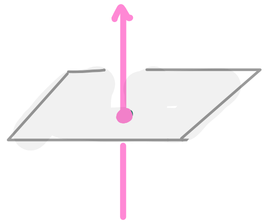
& $\pi : M\setminus B \longrightarrow S^1$

$(x,t) \longmapsto t$

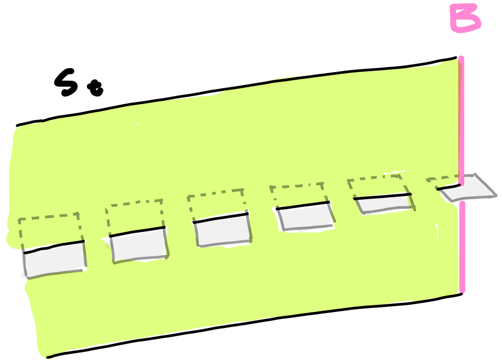
GIVES AN OBD

Open Books & Contact Structures

DEF: AN OBD (B,π) SUPPORTS A CONTACT STRUCTURE IF

- B IS TRANSVERSE ($TB \pitchfork \xi$ ) : $\boxed{\alpha|_B > 0}$
- $\boxed{d\alpha|_{T\mathring{S}_t} > 0}$

$\pi^{-1}(t)$ POSITIVE AREA FORM ON S_t:



ξ IS "NEAR" TS_t

Open Books & Contact Structures

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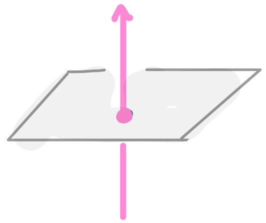
- B IS TRANSVERSE ($TB \pitchfork \xi$ ) : $\boxed{\alpha|_B > 0}$
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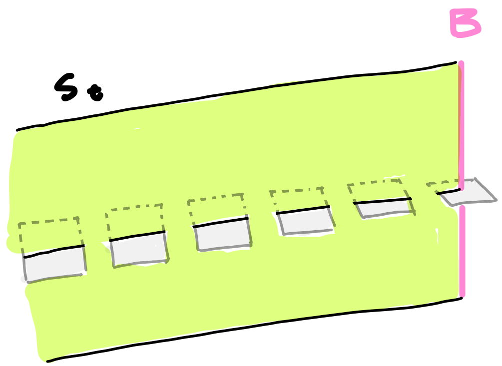


ξ IS "NEAR" TS_t

WE WILL HAVE A MORE TOPOLOGICAL DEF LATER

CONSTRUCTION (THURSTON - VINKENKEMPLER): ANY OBD SUPPORTS A CONTACT STRUCTURE THAT IS UNIQUE UP TO ISOTOPY

IDEA: • 1 - PARAMETER FAMILY OF AREA FORM $\{\beta_t\}$

$\rightsquigarrow \alpha = \beta_t + K\, dt$ WHERE $K \gg 0$

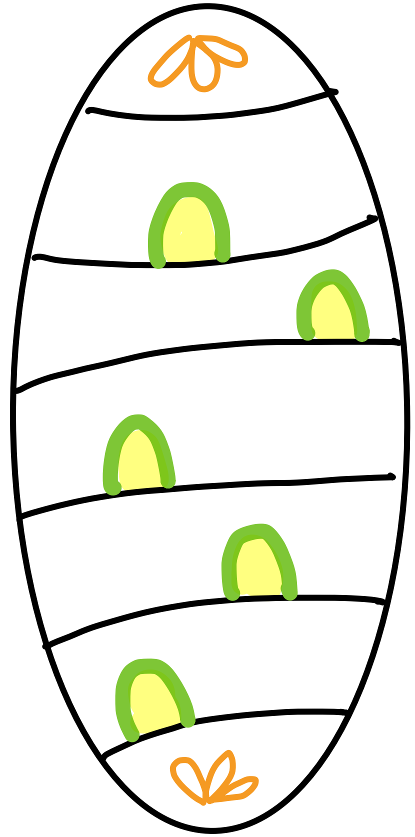
• + LOCAL CONSTRUCTION NEAR BINDING ∎

SRNI - LECTURE 3

PROOF OF GIROUX CORRESPONDENCE

- OPEN BOOK DECOMPOSITIONS & CONTACT HEEGAARD SPLITTINGS
- STABILISATION
- STATEMENT OF GIROUX CORRESPONDENCE
- IDEA OF PROOF
- FURTHER DIRECTIONS

THE GIROUX CORRESPONDENCE VIA CONVEX SURFACES



VERA VÉRTESI

JOINT WORK WITH JOAN LICATA

UNIVERSITY OF VIENNA

LECTURE 3

PROOF OF GIROUX'S CORRESPONDENCE

- OPEN BOOK DECOMPOSITIONS & CONTACT HEEGAARD SPLITTINGS
- STABILISATION
- STATEMENT OF GIROUX CORRESPONDENCE
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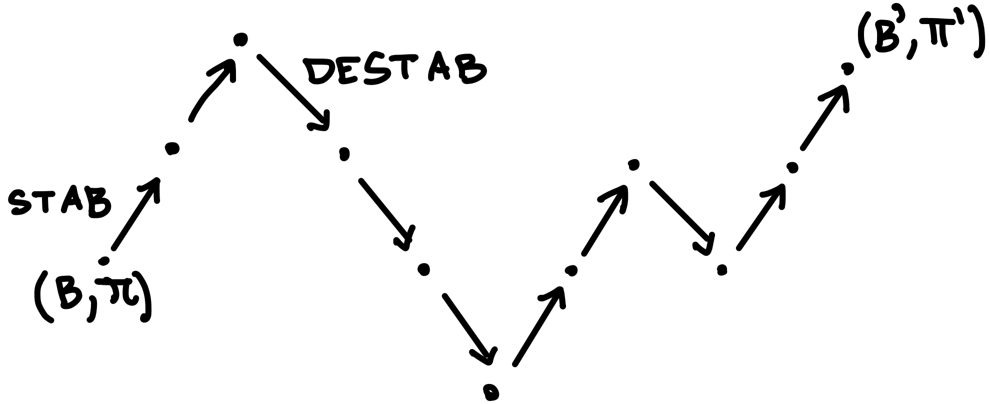
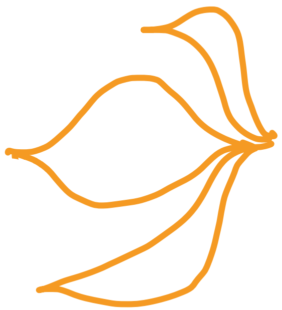
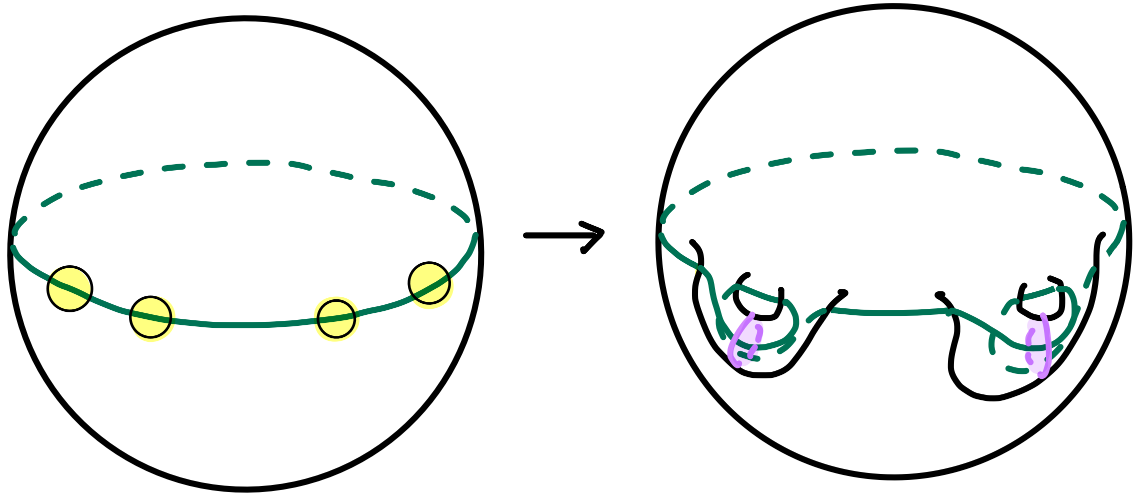


LAST TIME

- CONTACT HANDLEBODY:
  - → CTCT 0-h ∪ 1-h's
  - → N(LEGENDRIAN GRAPH)
  - → TIGHT + PRODUCT DISC DECOMPOSABLE

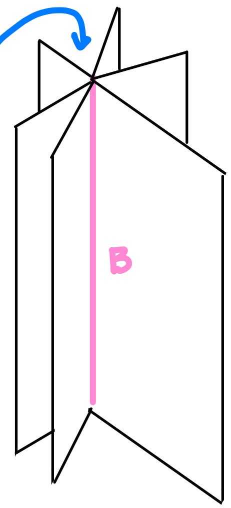




- CONTACT HEEGAARD DECOMPOSITION $M = U \cup_{\Sigma} V$
  - → Σ CONVEX
  - → U & V CONTACT HANDLEBODY

- OPEN BOOK DECOMPOSITION (B, π)

$B^1 \hookrightarrow M$ BINDING

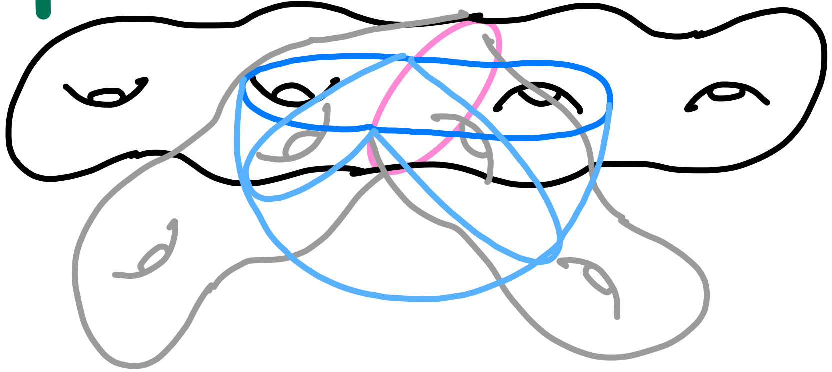


$S_t = \overline{\pi^{-1}(t)}$ PAGE

- ABSTRACT OPEN BOOK

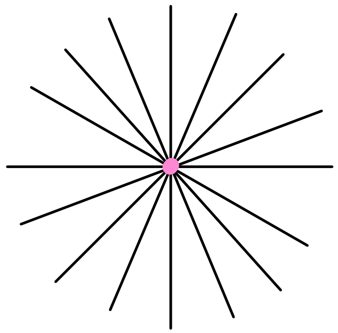
S

OPEN BOOKS & CONTACT HEEGAARD DECOMPOSITIONS



Open Book - & Contact Heegaard Decompositions

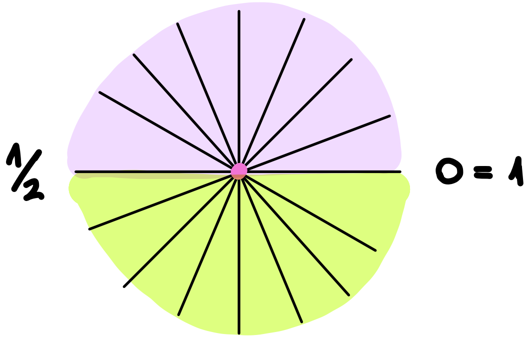
Given an OBD (B,π)



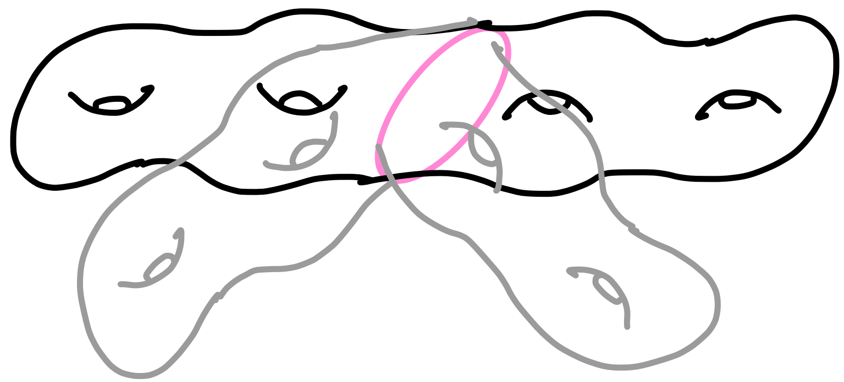
Open Book Decompositions $\rightsquigarrow$ Heegaard Decompositions

GIVEN AN OBD (B,π) $\rightsquigarrow$ CONSIDER $M = U \cup_\Sigma V$ WHERE

$U = \overline{\pi^{-1}([0,1/2])}$ $\quad$ $V = \overline{\pi^{-1}([1/2,1])}$



HERE $\Sigma = S_0 \cup S_1$



PROP: U & V ARE HANDLEBODIES

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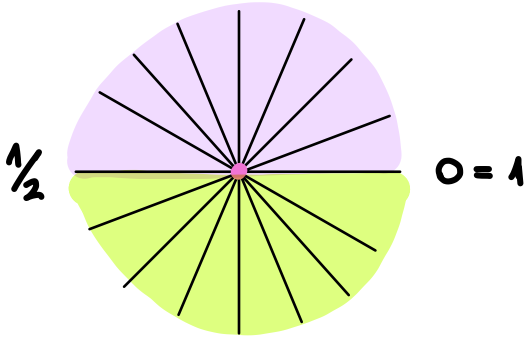

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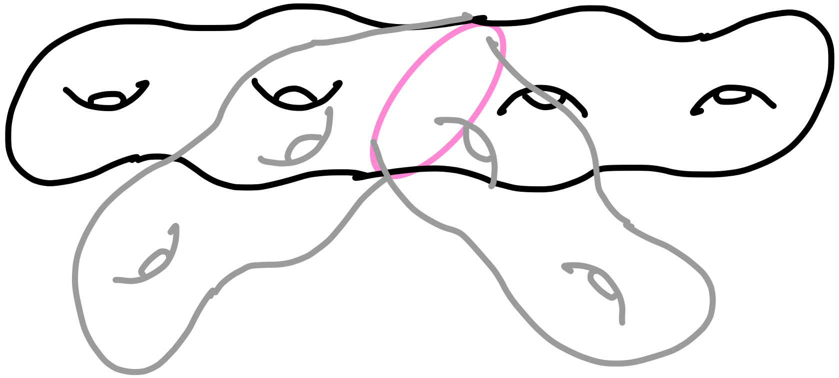
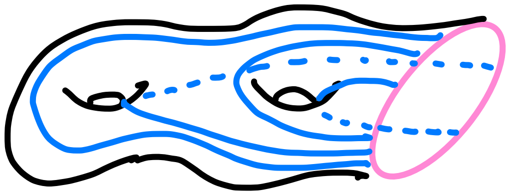


PROP: U & V ARE HANDLEBODIES

PROOF: FOR U:

- LET $a_1, \dots, a_{2g+b}$ BE ARCS ON S SUCH THAT $S \setminus \cup a_i \cong D^2$



OPEN BOOK DECOMPOSITIONS $\rightsquigarrow$ HEEGAARD DECOMPOSITIONS

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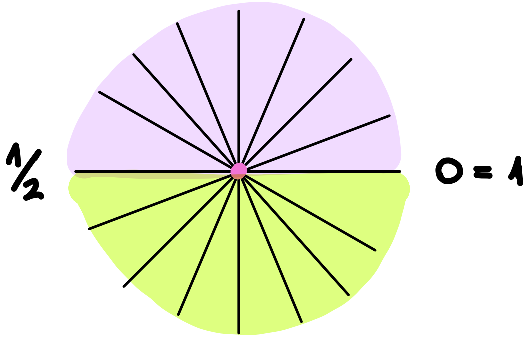

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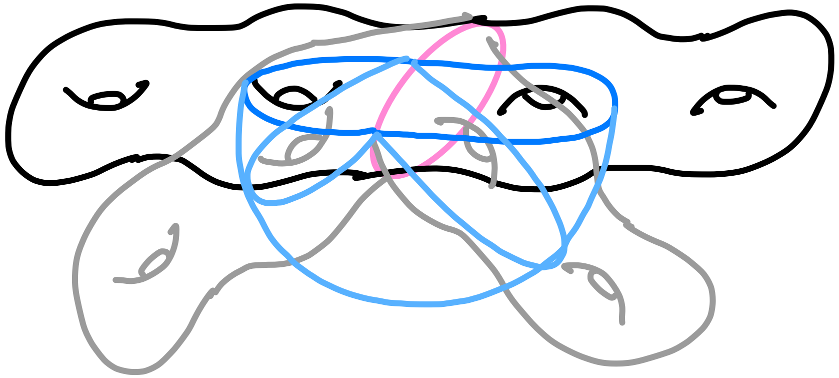

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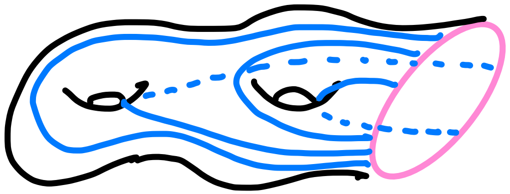
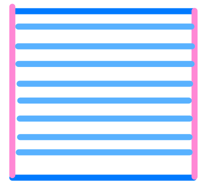
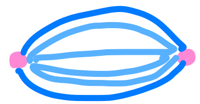


THEN $D_i := a_i \times [0,1/2] /_\sim$ =  $\cong$  $\cong D^2$

Open Book Decompositions $\rightsquigarrow$ Heegaard Decompositions

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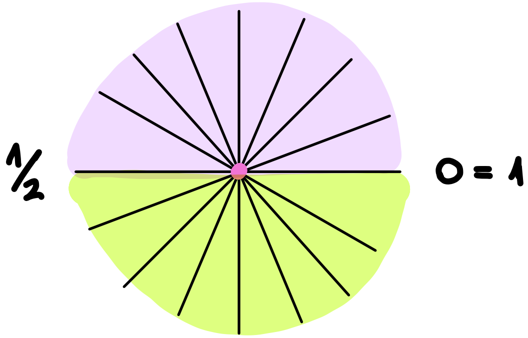

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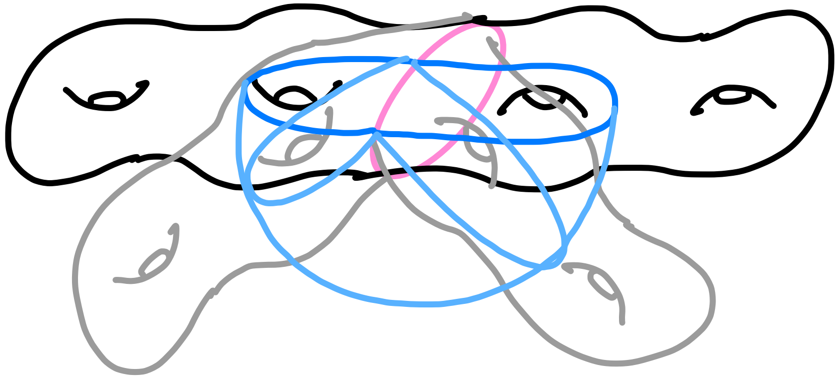

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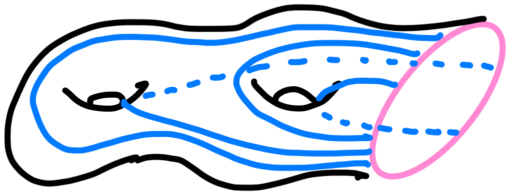
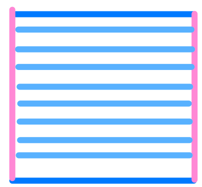
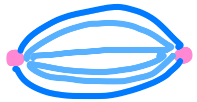


THEN $D_i := a_i \times [0,1/2]/\sim$ =  =  $\cong D^2$

- & $U \setminus \cup D_i = (S \setminus \cup a_i) \times [0,1/2]/\sim \;\cong D^2 \times [0,1/2]/\sim \;\cong D^3$

Open Book Decompositions $\rightsquigarrow$ Heegaard Decompositions

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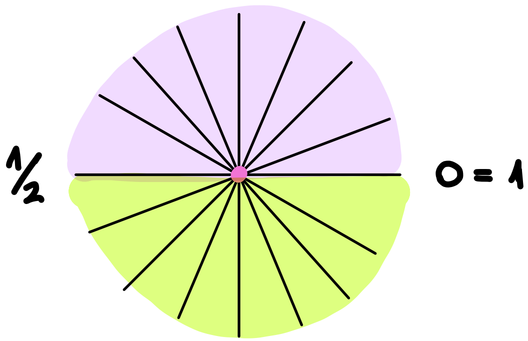

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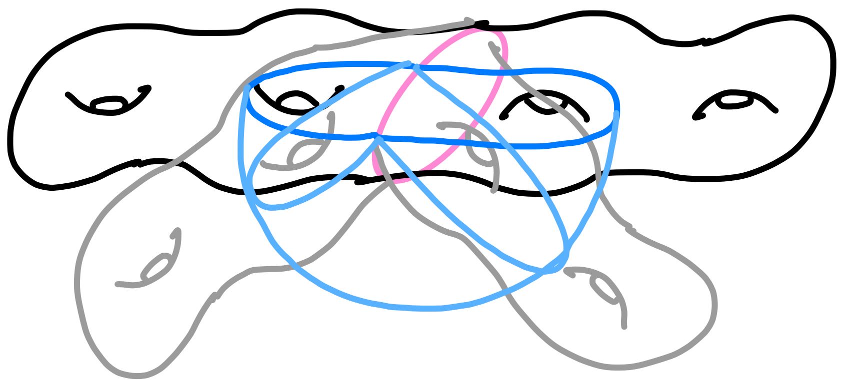

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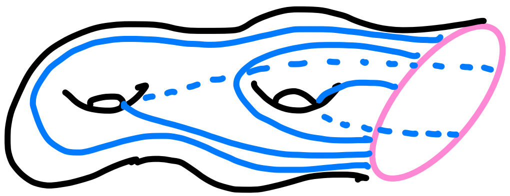
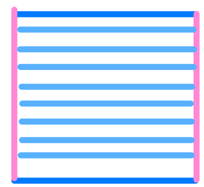
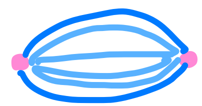


THEN $D_i := a_i \times [0,1/2]/\sim$ =  =  $\cong D^2$

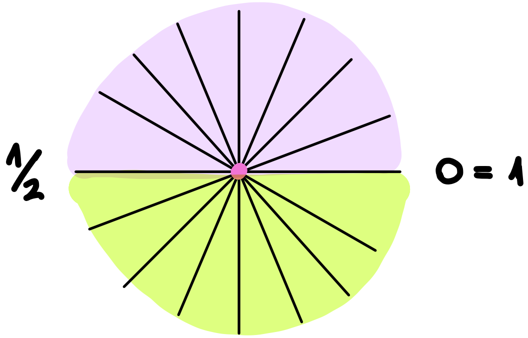
- & $U \setminus \cup D_i = (S \setminus \cup a_i) \times [0,1/2]/\sim \;\cong\; D^2 \times [0,1/2]/\sim \;\cong\; D^3$

$\Longrightarrow$ U IS A HANDLEBODY ∎

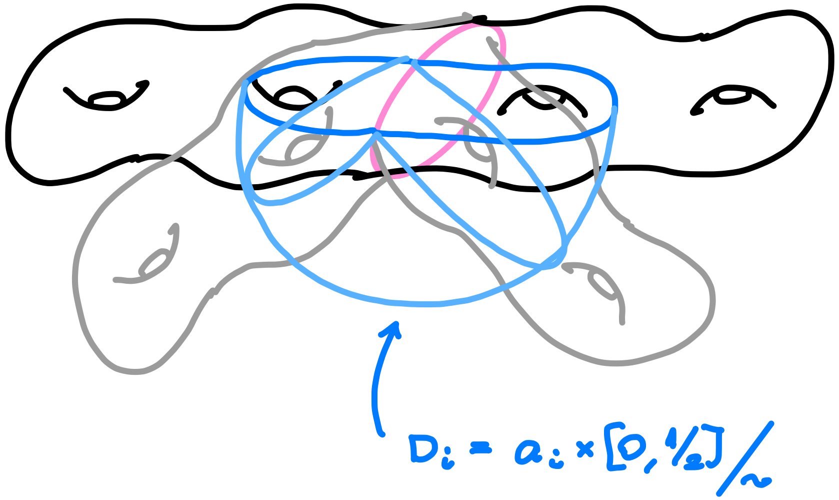
Open Book Decompositions $\rightsquigarrow$ Heegaard Decompositions

Given an OBD $(B,\pi) \rightsquigarrow$ consider $M = U \cup_\Sigma V$ where

$U = \overline{\pi^{-1}([0,1/2])}$ $\qquad$ $V = \overline{\pi^{-1}([1/2,1])}$



Here $\Sigma = S_0 \cup S_1$



Prop: U & V are handlebodies

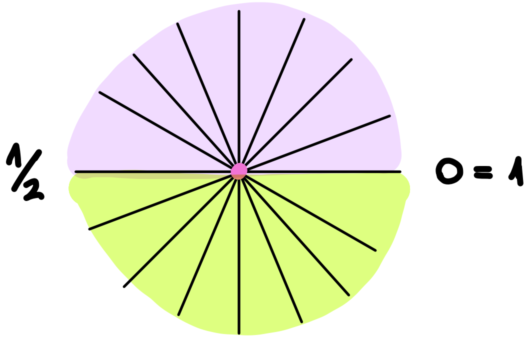
Moreover: $(U, \Sigma, \Gamma = B)$ is

Product disc decomposable

$\boxed{\begin{array}{l} \exists \text{ discs } D_i \text{ such that} \\ \bullet\ U - \cup D_i \cong D^3 \\ \bullet\ \&\ |\partial D_i \cap \Gamma| = 2 \ \forall i \end{array}}$

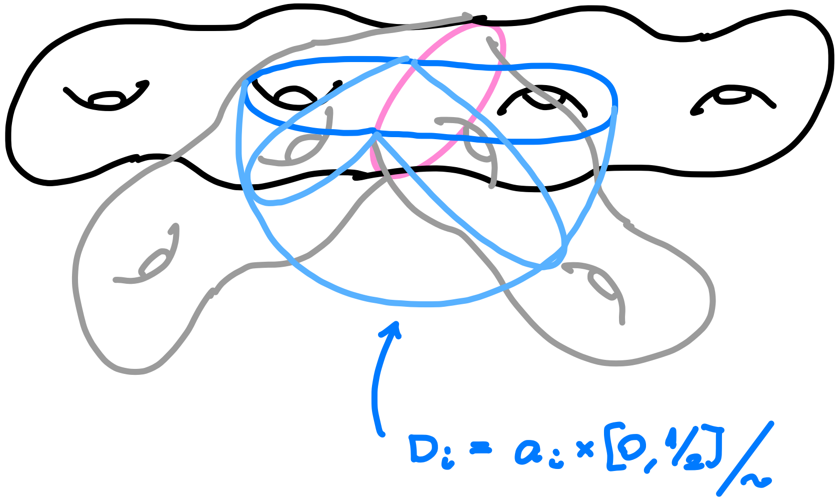
Open Book Decompositions $\rightsquigarrow$ Heegaard Decompositions

GIVEN AN OBD $(B,\pi) \rightsquigarrow$ CONSIDER $M = U \cup_\Sigma V$ WHERE



$U = \overline{\pi^{-1}([0,1/2])}$ $\quad$ $V = \overline{\pi^{-1}([1/2,1])}$

HERE $\Sigma = S_0 \cup S_1$



PROP: U & V ARE HANDLEBODIES

MOREOVER: $(U, \Sigma, \Gamma = B)$ IS PRODUCT DISC DECOMPOSABLE

$\exists$ DISCS D_i SUCH THAT
- $U - \cup D_i \cong D^3$
- & $|\partial D_i \cap \Gamma| = 2 \quad \forall i$

WE HAVE SEEN: AN OPEN BOOK DECOMPOSITION DEFINES A HEEGAARD DECOMPOSITION WITH PRODUCT DECOMPOSABLE HANDLEBODIES

WARNING: NEED Γ ON Σ

Open Book - & Contact Heegaard Decompositions

THM: AN OPEN BOOK DECOMPOSITION DEFINES A HEEGAARD DECOMPOSITION WITH PRODUCT DECOMPOSABLE HANDLEBODIES

$$(B,\pi) \rightsquigarrow M = U \cup_{(\Sigma,\Gamma)} V$$

LET'S LOOK AT THE CONTACT STRUCTURE SUPPORTED BY (B,π)

THM (TORISU): THE SURFACE Σ IS CONVEX WITH DIVIDING CURVE Γ. THE CONTACT STRUCTURES $\xi|_U$ & $\xi|_V$ ARE TIGHT.

Open Book & Contact Heegaard Decompositions

> **THM:** AN OPEN BOOK DECOMPOSITION DEFINES A HEEGAARD DECOMPOSITION WITH PRODUCT DECOMPOSABLE HANDLEBODIES

$$(B,\pi) \rightsquigarrow M = U \cup_{(\Sigma,\Gamma)} V$$

LET'S LOOK AT THE CONTACT STRUCTURE SUPPORTED BY (B,π)

> **THM (TORISU):** THE SURFACE Σ IS CONVEX WITH DIVIDING CURVE Γ. THE CONTACT STRUCTURES $\xi|_U$ & $\xi|_V$ ARE TIGHT.

SO: (U,ξ_U) & (V,ξ_V) ARE CONTACT HANDLEBODIES

THUS $\boxed{M = U \cup_{(\Sigma,\Gamma)} V \text{ IS A CONTACT HEEGAARD DECOMPOSITION}}$

OPEN BOOK - & CONTACT HEEGAARD DECOMPOSITIONS

THM: AN OPEN BOOK DECOMPOSITION DEFINES A HEEGAARD DECOMPOSITION WITH PRODUCT DECOMPOSABLE HANDLEBODIES

$$(B,\pi) \rightsquigarrow M = U \cup_{(\Sigma,\Gamma)} V$$

LET'S LOOK AT THE CONTACT STRUCTURE SUPPORTED BY (B,π)

THM (TORISU): THE SURFACE Σ IS CONVEX WITH DIVIDING CURVE Γ. THE CONTACT STRUCTURES $\xi|_U$ & $\xi|_V$ ARE TIGHT.

SO: (U,ξ_U) & (V,ξ_V) ARE CONTACT HANDLEBODIES

THUS $M = U \cup_{(\Sigma,\Gamma)} V$ IS A CONTACT HEEGAARD DECOMPOSITION

THIS GIVES RISE TO AN EQUIVALENT DEFINITION:

DEF: ξ IS SUPPORTED BY THE OPEN BOOK (B,π) IF THE HEEGARD DECOMPOSITION DEFINED BY (B,π) IS A CONTACT HEEGAARD DECOMPOSITION

HEEGAARD DECOMPOSITIONS $\rightsquigarrow$ OPEN BOOK DECOMPOSITIONS

PROP: (U,Γ) PRODUCT DISC DECOMPOSABLE HANDLEBODY

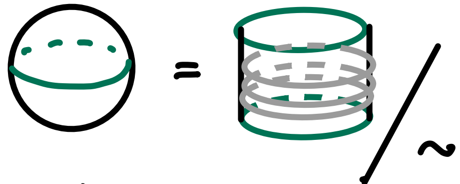
$$\Longrightarrow \quad U = S\times I \big/ (x,t)\sim(x,t') \qquad x\in\partial S,\ t,t'\in I$$

SUCH THAT $\quad \partial U = S\times 0 \cup_{\Gamma} -S\times 1$

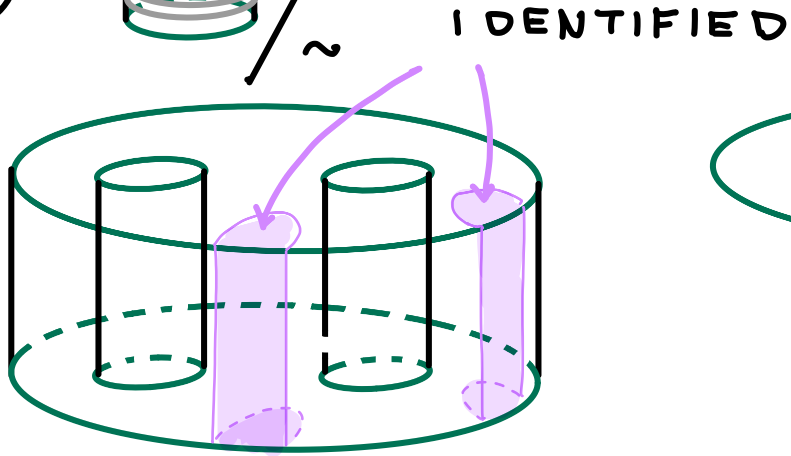
& $\Gamma = \partial S/\sim$

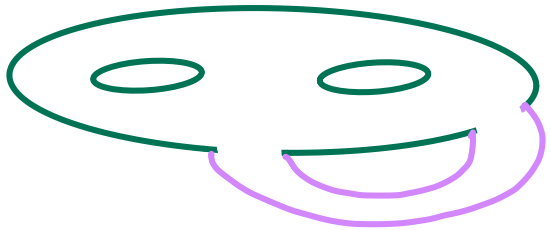
IDEA: INDUCTION ON THE # OF PRODUCT DISCS

- $n=0$



- $n \rightsquigarrow n+1$





$\times I/\sim$

∎

HEEGAARD DECOMPOSITIONS $\rightsquigarrow$ OPEN BOOK DECOMPOSITIONS

PROP: (U,Γ) PRODUCT DISC DECOMPOSABLE HANDLEBODY

$$\Longrightarrow U = S\times I\Big/_{(x,t)\sim(x,t')} \quad x\in\partial S,\ t,t'\in I$$

SUCH THAT $\partial U = S\times 0 \cup_{\Gamma} -S\times 1$

& $\Gamma = \partial S/_{\sim}$

GIVEN A CONTACT HEEGAARD DECOMPOSITION $M = U \cup_{(\Sigma,\Gamma)} V$

$$\begin{array}{lllll} U & \qquad \partial U = \Sigma = & R_+ & \cup_{\Gamma} & -R_- \\ & & \updownarrow & & \updownarrow \\ V & \qquad \partial V = -\Sigma = & -R_+ & \cup_{-\Gamma} & R_- \end{array}$$

Heegaard Decompositions $\rightsquigarrow$ Open Book Decompositions

Prop: (U,Γ) product disc decomposable handlebody

$$\Longrightarrow \quad U = S\times I /_{(x,t)\sim(x,t')} \quad x\in\partial S,\ t,t'\in I$$

such that $\partial U = S\times 0 \cup_\Gamma -S\times 1$

& $\Gamma = \partial S/_\sim$

Given a contact Heegaard decomposition $M = U \cup_{(\Sigma,\Gamma)} V$

By **Prop**: $U = S\times[0,\frac{1}{2}]$ $\quad \partial U = \Sigma = \underset{S\times 0}{R_+} \cup_\Gamma \underset{-S\times \frac{1}{2}}{-R_-}$

$V = S\times[\frac{1}{2},1]$ $\quad \partial V = -\Sigma = \underset{-S\times 1}{-R_+} \cup_{-\Gamma} \underset{S\times\frac{1}{2}}{R_-}$

$\Longrightarrow$ glues to a full fibration (B,π), where $B = \Gamma$

given by projection onto $[0,\frac{1}{2}]\cup[\frac{1}{2},1]$

Open Book & Contact Heegaard Decompositions

SO WE GET A ONE-TO-ONE CORRESPONDANCE

$$\left\{\text{OPEN BOOK DECOMPOSITIONS OF } (M,\xi)\right\}/\text{ISOT} \longleftrightarrow \left\{\text{CONTACT HEEGAARD DECOMPOSITIONS OF } (M,\xi)\right\}/\text{ISOT}$$

SO WE CAN WORK WITH WHICHEVER IS MORE CONVENIENT

RECALL:

THM: EVERY CONTACT MANIFOLD (M,ξ) ADMITS A CONTACT HEEGAARD DECOMPOSITION

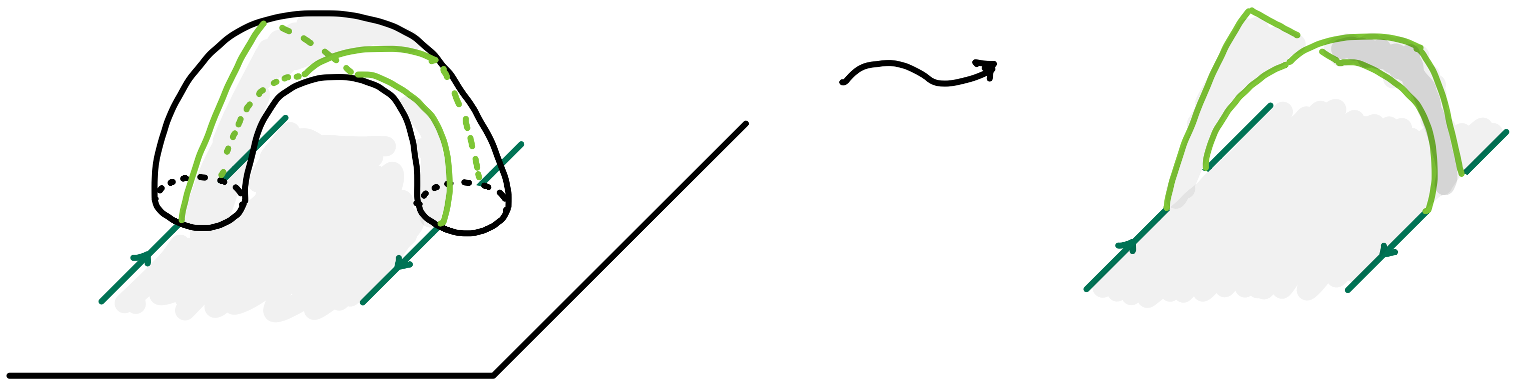
COR: EVERY CONTACT MANIFOLD (M,ξ) ADMITS AN OPEN BOOK DECOMPOSITION

STABILISATION

OF HEEGAARD DECOMPOSITIONS

- SMOOTH
- CONTACT

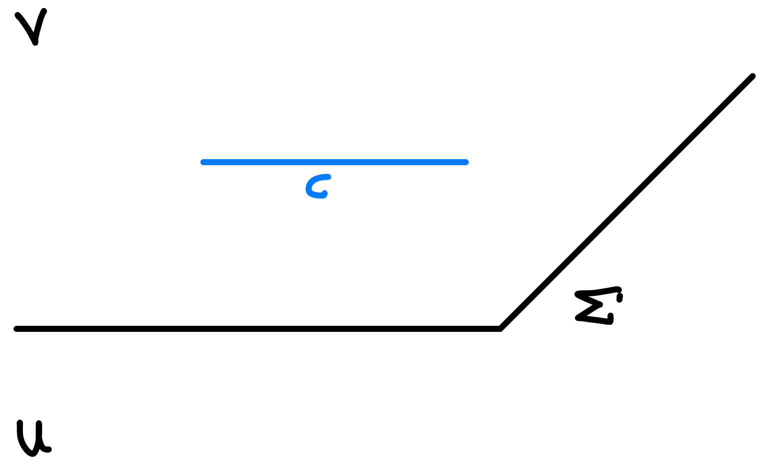
OPEN BOOK DECOMPOSITIONS



STABILISATION OF HEEGAARD DECOMPOSITIONS

$M = U \cup_\Sigma V$ HEEGAARD DECOMPOSITION

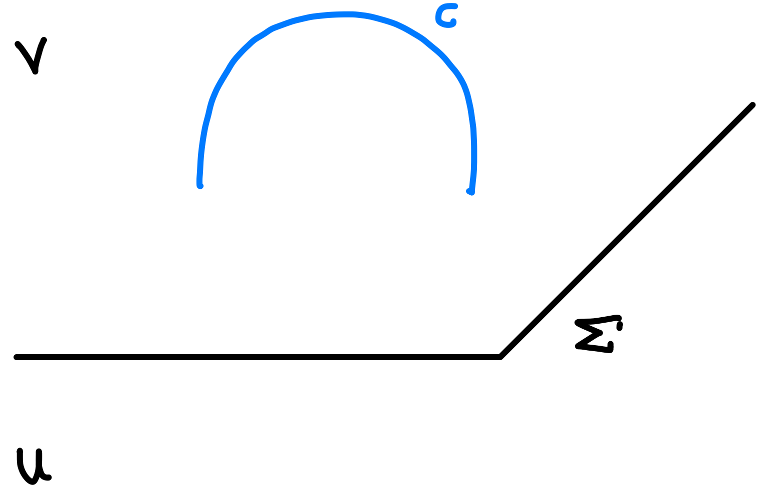
- c ARC ON Σ



Stabilisation of Heegaard Decompositions

$M = U \cup_\Sigma V$ Heegaard decomposition

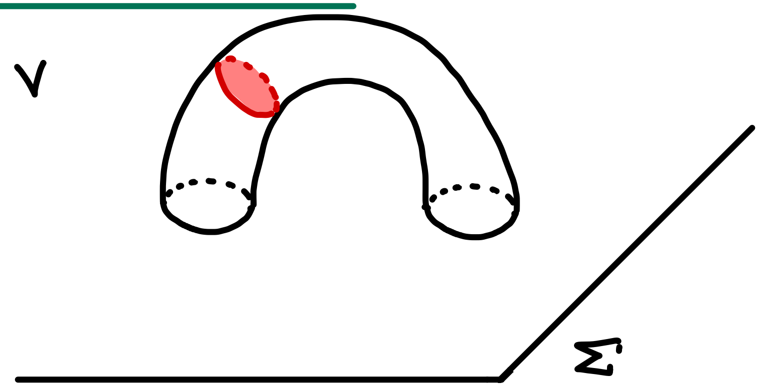
- c arc on Σ
- isotope $\mathring{c}$ into V



STABILISATION OF HEEGAARD DECOMPOSITIONS

$M = U \cup_\Sigma V$ HEEGAARD DECOMPOSITION

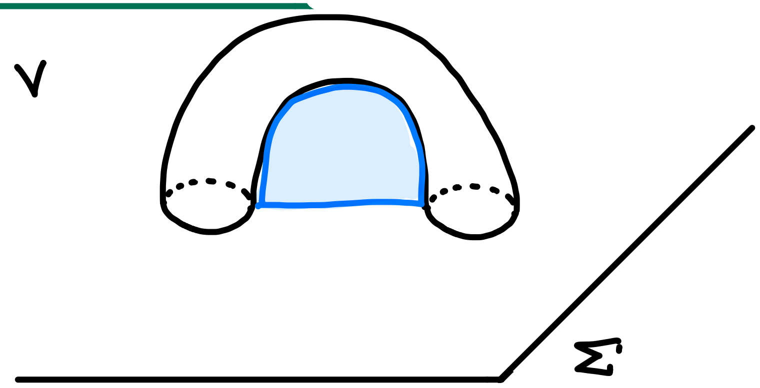
- c ARC ON Σ
- ISOTOPE $\mathring{c}$ INTO V



$\rightsquigarrow$ - $U' := U \cup N(c)$ IS A HANDLEBODY U

Stabilisation of Heegaard Decompositions

$M = U \cup_\Sigma V$ Heegaard decomposition



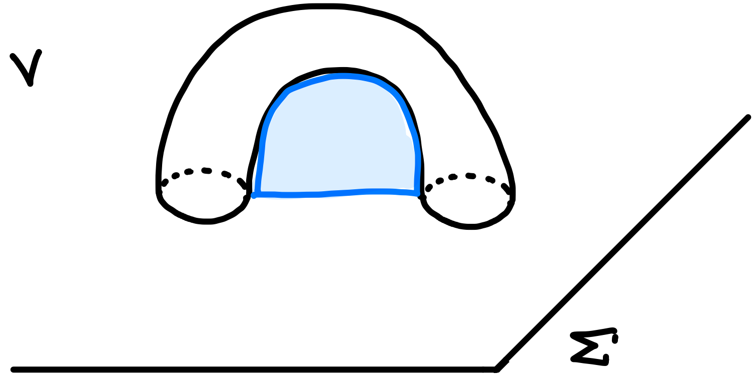
- c arc on Σ
- isotope $\mathring{c}$ into V

$\rightsquigarrow$ - $U' := U \cup N(c)$ is a handlebody

- $V' := V \setminus N(c)$ is also a handlebody
- $\Sigma' := \partial U'$

STABILISATION OF HEEGAARD DECOMPOSITIONS

$M = U \cup_\Sigma V$ HEEGAARD DECOMPOSITION



- c ARC ON Σ
- ISOTOPE $\mathring{c}$ INTO V

$\rightsquigarrow$
- $U' := U \cup N(c)$ IS A HANDLEBODY
- $V' := V \setminus N(c)$ IS ALSO A HANDLEBODY
- $\Sigma' := \partial U'$

DEF: $M = U \cup_{(\Sigma,\Gamma)} V \;\underset{\text{DESTABILISATION}}{\overset{\text{STABILISATION}}{\rightleftarrows}}\; M = U' \cup_{(\Sigma',\Gamma')} V'$

REIDEMEISTER-SINGER THEOREM: ANY TWO HD-S OF M^3 ARE RELATED BY STABILISATIONS & DESTABILISATIONS

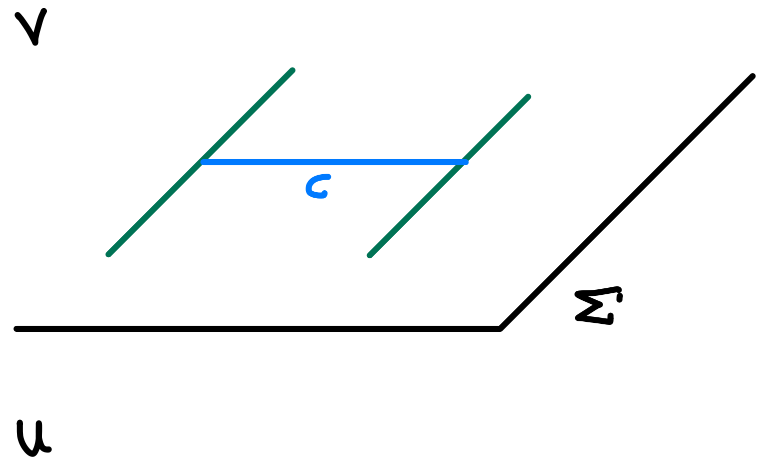
Stabilisation of Contact Heegaard Decompositions

(M,ξ) contact 3-manifold

$M = U \cup_{(\Sigma,\Gamma)} V$ contact HD

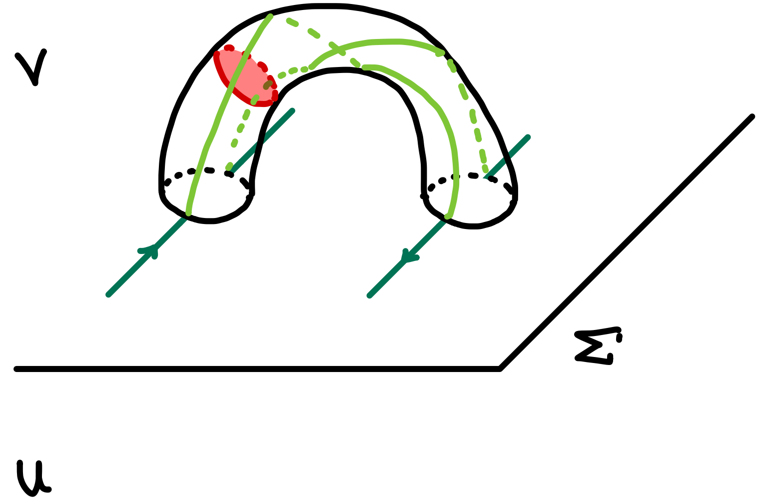
- c Legendrian arc on Σ

  with $tw_c(\xi, T\Sigma) = -1/2$



Stabilisation of Contact Heegaard Decompositions

(M, ξ) contact 3-manifold



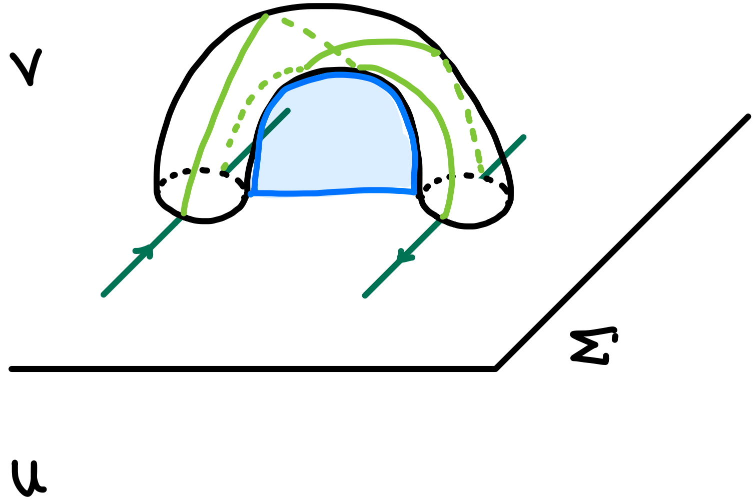
$M = U \cup_{(\Sigma, \Gamma)} V$ contact HD

- c Legendrian arc on Σ
  with $tw_c(\xi, T\Sigma) = -\frac{1}{2}$
- Legendrian isotope $\mathring{c}$ into V

$\rightsquigarrow$ - $U' := U \cup N(c)$ is a contact handlebody

STABILISATION OF CONTACT HEEGAARD DECOMPOSITIONS

(M, ξ) CONTACT 3-MANIFOLD



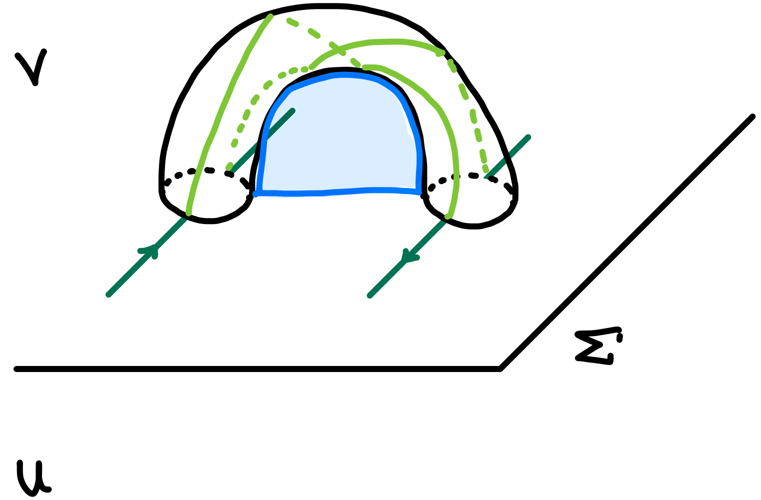
$M = U \cup_{(\Sigma,\Gamma)} V$ CONTACT HD

- c LEGENDRIAN ARC ON Σ
  WITH $tw_c(\xi, T\Sigma) = -1/2$

- LEGENDRIAN ISOTOPE $\mathring{c}$ INTO V

$\rightsquigarrow$ - $U' := U \cup N(c)$ IS A CONTACT HANDLEBODY
- $V' := V \setminus N(c)$ IS ALSO A CONTACT HANDLEBODY
- $\Sigma' := \partial U'$

Stabilisation of Contact Heegaard Decompositions



(M,ξ) contact 3-manifold

$M = U \cup_{(\Sigma,\Gamma)} V$ contact HD

- c Legendrian arc on Σ

  with $tw_c(\xi, T\Sigma) = -1/2$

- Legendrian isotope $\mathring{c}$ into V

$\rightsquigarrow$ - $U' := U \cup N(c)$ is a contact handlebody

- $V' := V \setminus N(c)$ is also a contact handlebody

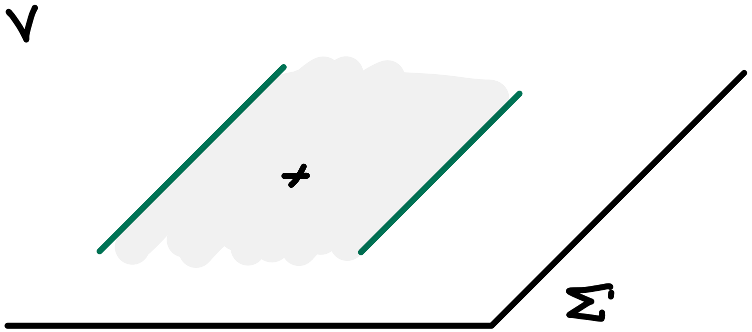
- $\Sigma' := \partial U'$

DEF: $M = U \cup_{(\Sigma,\Gamma)} V \; \underset{\text{contact destabilisation}}{\overset{\text{contact stabilisation}}{\rightleftarrows}} \; M = U' \cup_{(\Sigma',\Gamma')} V'$

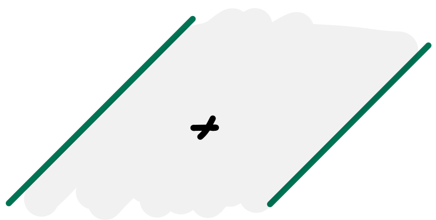
Stabilisation of Open Books

Recall: Contact Heegaard decomposition $\longleftrightarrow$ Open book decomposition

$$M = U \cup_{(\Sigma,\Gamma)} V \quad \rightsquigarrow \quad (S = R_+, h)$$



$\rightsquigarrow$

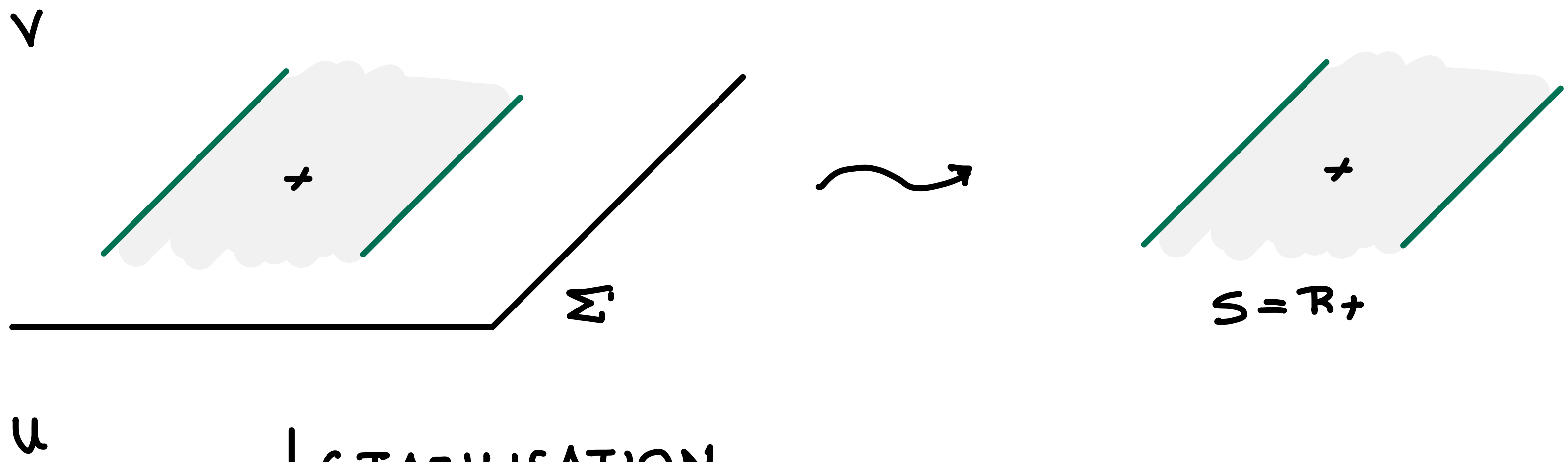


$S = R_+$

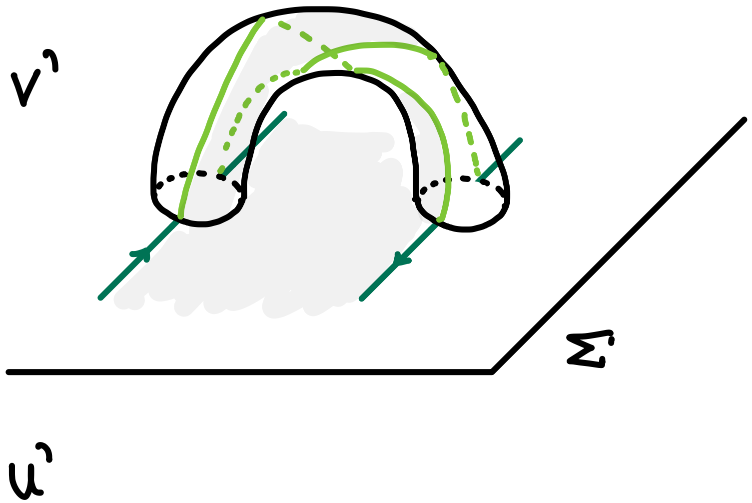
Stabilisation of Open Books

RECALL: CONTACT HEEGAARD DECOMPOSITION $\longleftrightarrow$ OPEN BOOK DECOMPOSITION

$$M = U \cup_{(\Sigma,\Gamma)} V \rightsquigarrow (S = R_+, h)$$



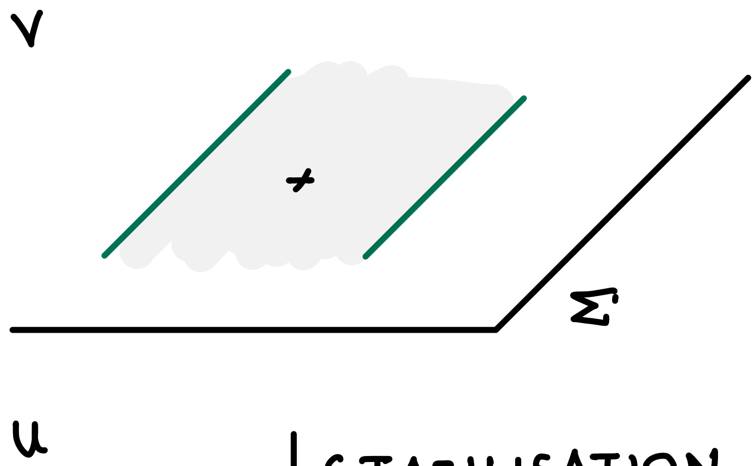
$\downarrow$ STABILISATION

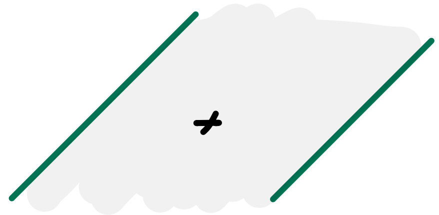


STABILISATION OF OPEN BOOKS

RECALL: CONTACT HEEGAARD DECOMPOSITION $\longleftrightarrow$ OPEN BOOK DECOMPOSITION

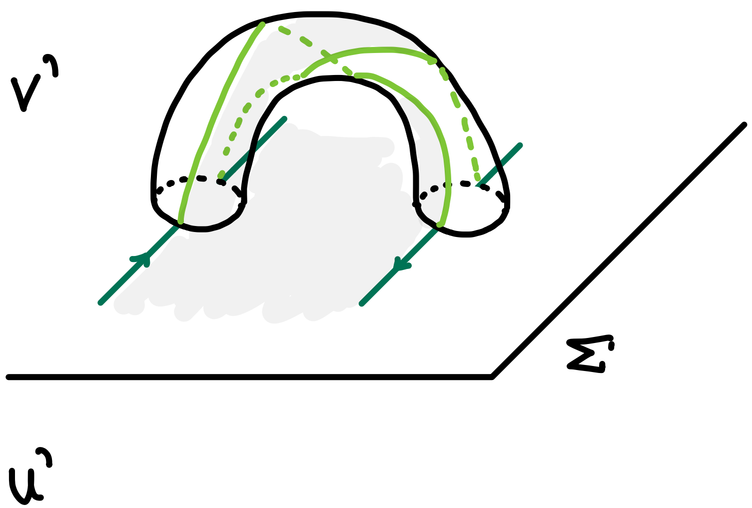
$$M = U \cup_{(\Sigma,\Gamma)} V \rightsquigarrow (S = R_+, \varphi)$$

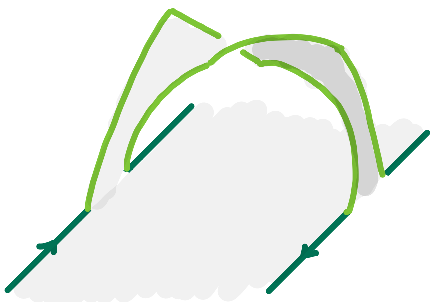




$S = R_+$

$\downarrow$ STABILISATION

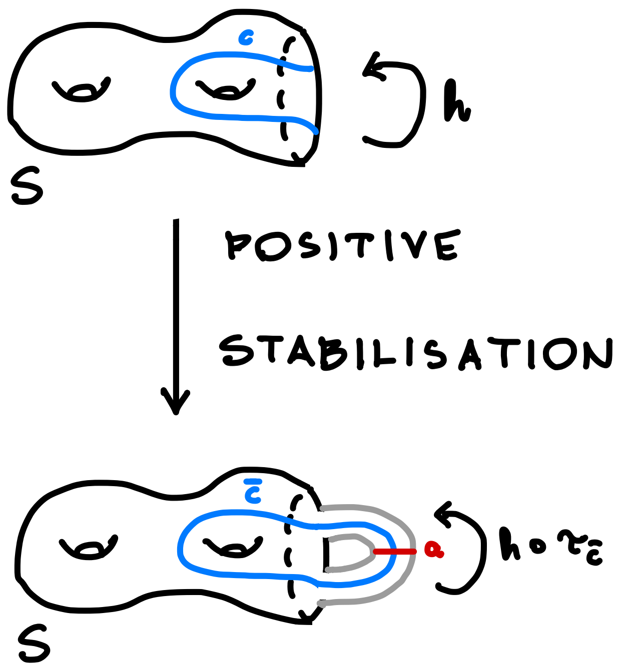


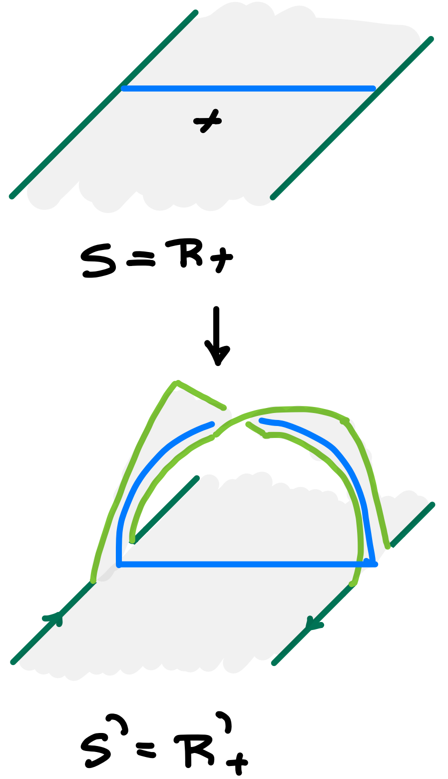


$S' = R'_+$

STABILISATION OF OPEN BOOKS

THE MONODROMIES CAN ALSO BE READ OFF & WE GET



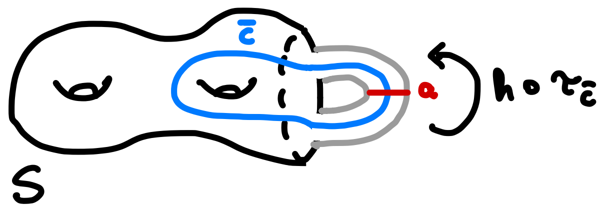


SO UNDER THE CORRESPONDENCE

CONTACT HEEGAARD DECOMPOSITION $\longleftrightarrow$ OPEN BOOK DECOMPOSITION

STABILISATION $\longleftrightarrow$ POSITIVE STABILISATION

(RE)STATEMENT OF GIROUX CORRESPONDENCE



Giroux Correspondence – Rephrased

Giroux Correspondance

Open books / positive stabilisation $\longleftrightarrow$ Contact 3-manifolds / contactomorphism

Giroux Correspondance (Rephrased)

Contact Heegaard decompositions / contact stabilisation $\longleftrightarrow$ Contact 3-manifolds / contactomorphism

GIROUX CORRESPONDENCE – REPHRASED

GIROUX CORRESPONDANCE

OPEN BOOKS / POSITIVE STABILISATION $\longleftrightarrow$ CONTACT 3-MANIFOLDS / CONTACTOMORPHISM

GIROUX CORRESPONDANCE (REPHRASED)

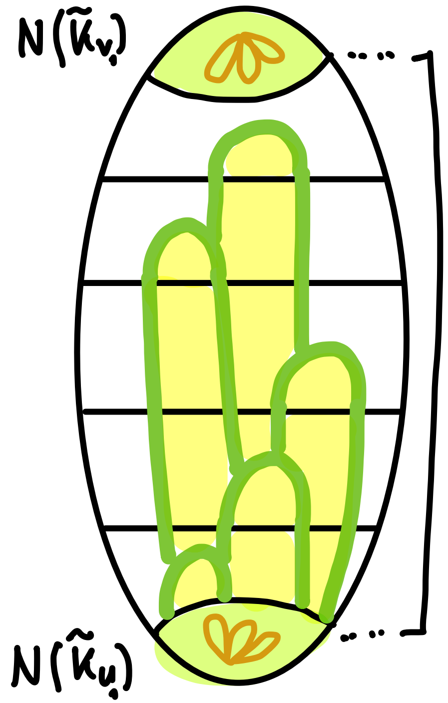
CONTACT HEEGAARD DECOMPOSITIONS / CONTACT STABILISATION $\longleftrightarrow$ CONTACT 3-MANIFOLDS / CONTACTOMORPHISM

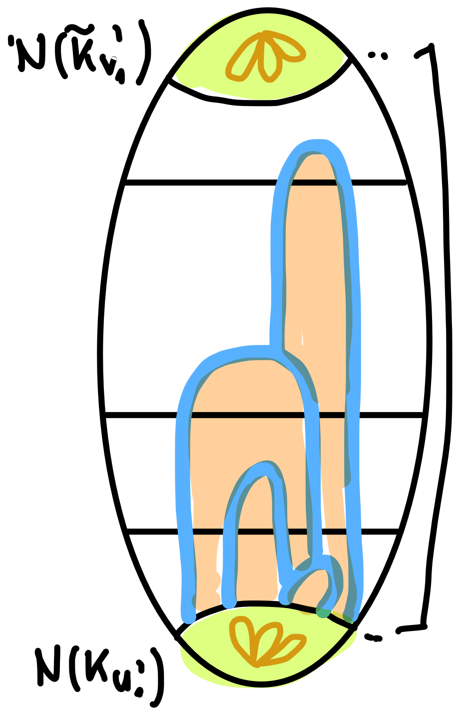
FROM SMOOTH TOPOLOGY:

REIDEMEISTER-SINGER THEOREM

HEEGAARD DECOMPOSITIONS / STABILISATION $\longleftrightarrow$ SMOOTH 3-MANIFOLDS / DIFFEOMORPHISM

IDEA OF PROOF





What do we Want to Prove?

<u>Giroux Correspondance</u> (Rephrased)

Contact Heegaard Decompositions / Contact Stabilisation $\longleftrightarrow$ Contact 3-Manifolds / Contactomorphism

What do we want to prove?

Giroux correspondance (rephrased)

$$\text{Contact Heegaard decompositions} \,/\, \text{contact stabilisation} \longleftrightarrow \text{contact 3-manifolds} \,/\, \text{contactomorphism}$$

More precisely: given two contact Heegaard decompositions

$\mathcal{H}'$

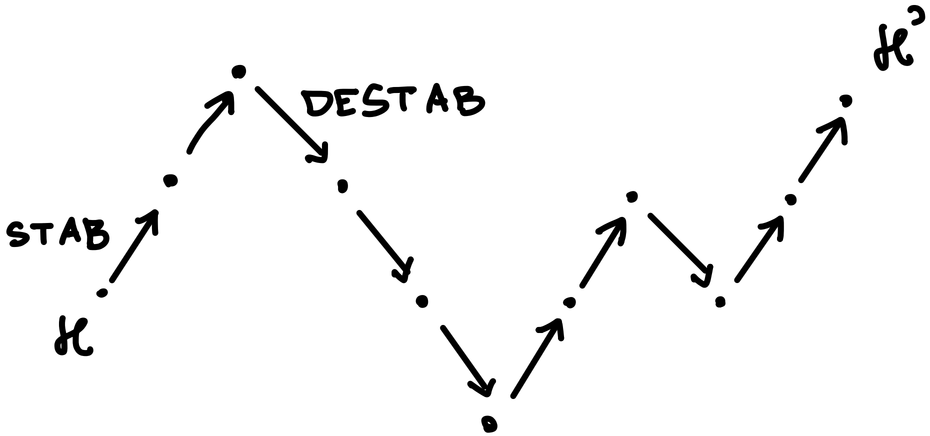
$\mathcal{H}$

WHAT DO WE WANT TO PROVE?

GIROUX CORRESPONDANCE (REPHRASED)

$$\text{CONTACT HEEGAARD DECOMPOSITIONS} \,\big/\, \text{CONTACT STABILISATION} \longleftrightarrow \text{CONTACT 3-MANIFOLDS} \,\big/\, \text{CONTACTOMORPHISM}$$

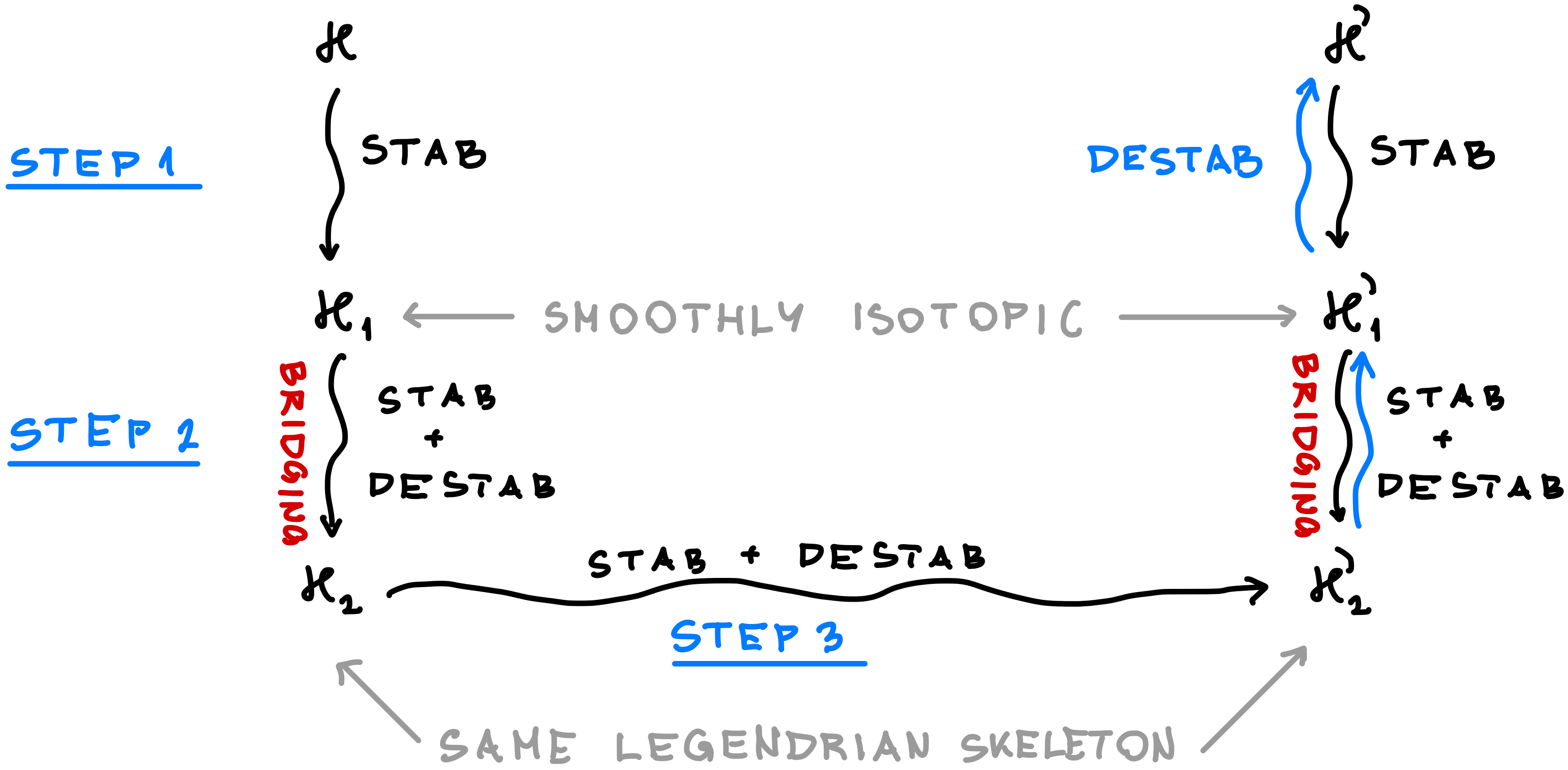
MORE PRECISELY: GIVEN TWO CONTACT HEEGAARD DECOMPOSITIONS



THEN THERE IS A SEQUENCE OF CONTACT STABILISATIONS & CONTACT DESTABILISATIONS CONNECTING THEM

IDEA OF PROOF

IN THREE STEPS WE MAKE THEM MORE & MORE SIMILAR

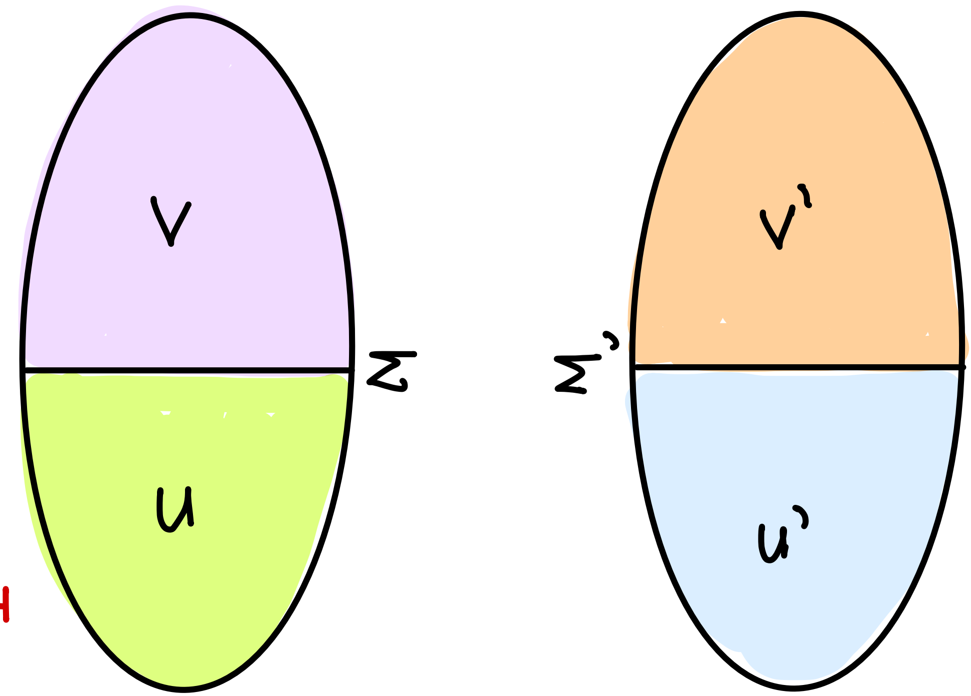


STEP 1

GIVEN TWO CTCT HD'S

$\underbrace{M = U \cup_\Sigma V}_{\mathcal{H}}$ & $\underbrace{M' = U' \cup_{\Sigma'} V'}_{\mathcal{H}'}$

$\xRightarrow[\text{SINGER}]{\text{REIDEMEISTER}}$ $\mathcal{H}$ & $\mathcal{H}'$ ADMITS A COMMON SMOOTH STABILISATION

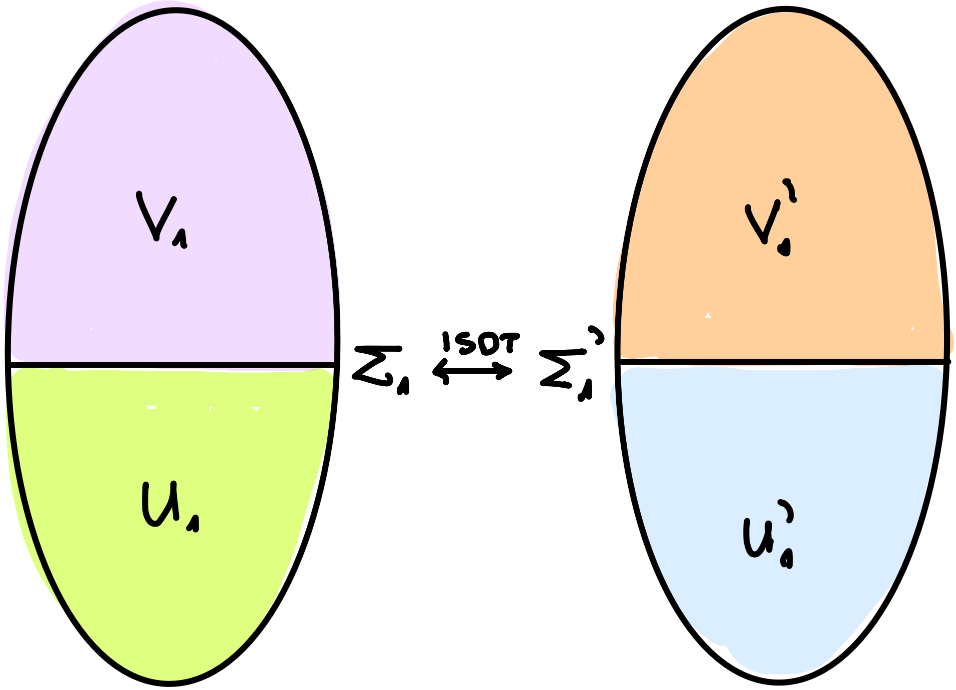


Step 1

GIVEN TWO CTCT HD'S

$\underbrace{M = U \cup_\Sigma V}_{\mathcal{H}}$ & $\underbrace{M' = U' \cup_{\Sigma'} V'}_{\mathcal{H}'}$

$\xrightarrow[\text{SINGER}]{\text{REIDEMEISTER}}$ $\mathcal{H}$ & $\mathcal{H}'$ ADMITS A COMMON SMOOTH STABILISATION



REALISE THESE STABILISATIONS WITH CTCT STABILISATIONS ON EACH

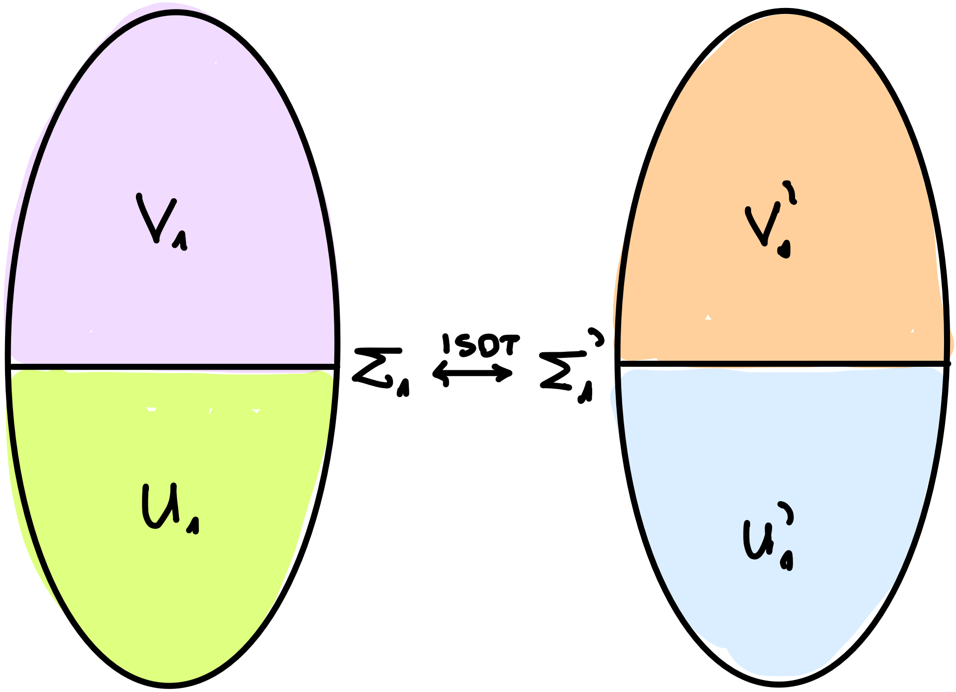
$\rightsquigarrow$ GET $\overbrace{M = U_1 \cup_{\Sigma_1} V_1}^{\mathcal{H}_1}$ & $\overbrace{M = U_1' \cup_{\Sigma_1'} V_1'}^{\mathcal{H}_2}$, WHERE Σ_1 & Σ_1'

ARE SMOOTHLY ISOTOPIC

STEP 2

$$\overbrace{M = U_1 \cup_{\Sigma_1} V_1}^{\mathcal{H}_1} \quad \& \quad \overbrace{M = U_1' \cup_{\Sigma_1'} V_1'}^{\mathcal{H}_2}$$

WHERE Σ_1 & Σ_1' ARE SMOOTHLY ISOTOPIC

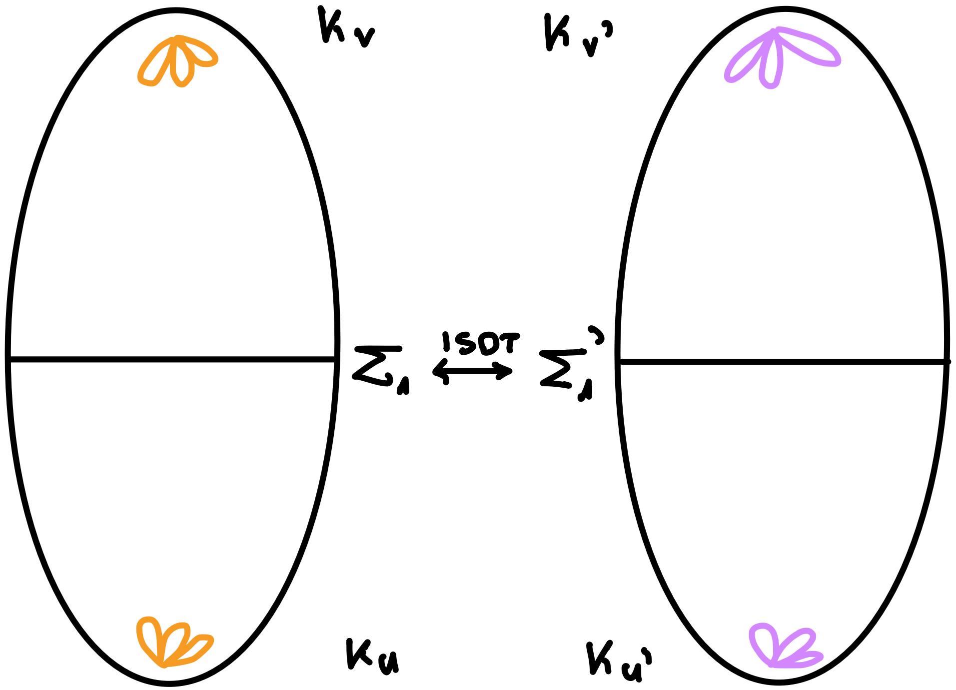


APPLY THE EXISTENCE PROOF

STEP 2

$$\overbrace{M = U_1 \cup_{\Sigma_1} V_1}^{\mathcal{H}_1} \quad \& \quad \overbrace{M = U_1' \cup_{\Sigma_1'} V_1'}^{\mathcal{H}_2}$$

WHERE Σ_1 & Σ_1' ARE SMOOTHLY ISOTOPIC



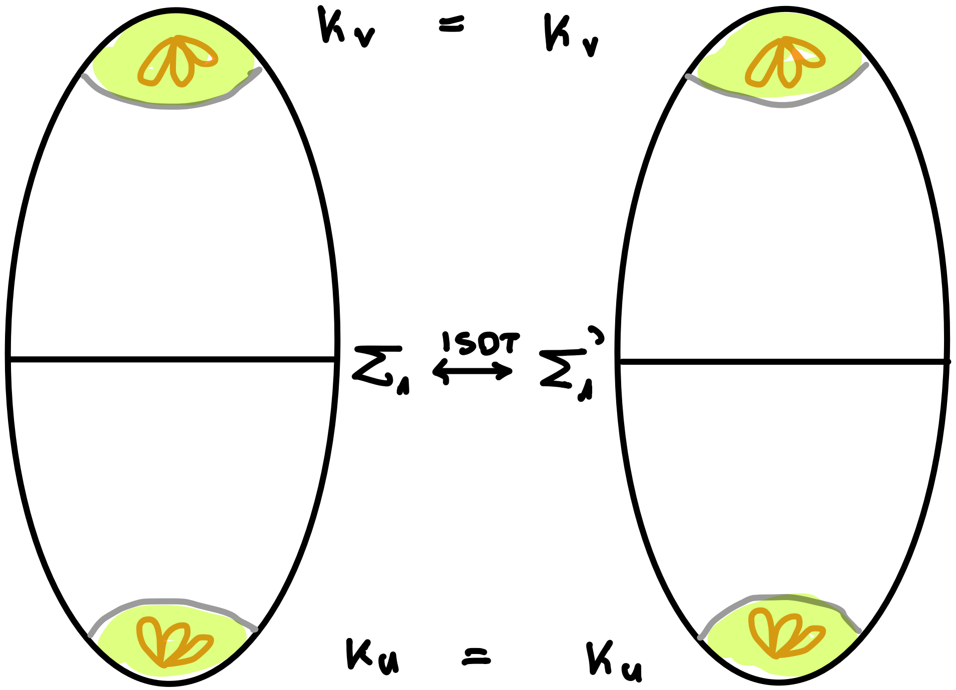
APPLY THE EXISTENCE PROOF

- TAKE LEGENDRIAN SKELETONS FOR THE HANDLEBODIES

STEP 2

$\overbrace{M = U_1 \cup_{\Sigma_1} V_1}^{\mathcal{H}_1}$ & $\overbrace{M = U_1' \cup_{\Sigma_1'} V_1'}^{\mathcal{H}_2}$

WHERE Σ_1 & Σ_1' ARE SMOOTHLY ISOTOPIC



APPLY THE EXISTENCE PROOF

- TAKE LEGENDRIAN SKELETONS FOR THE HANDLEBODIES

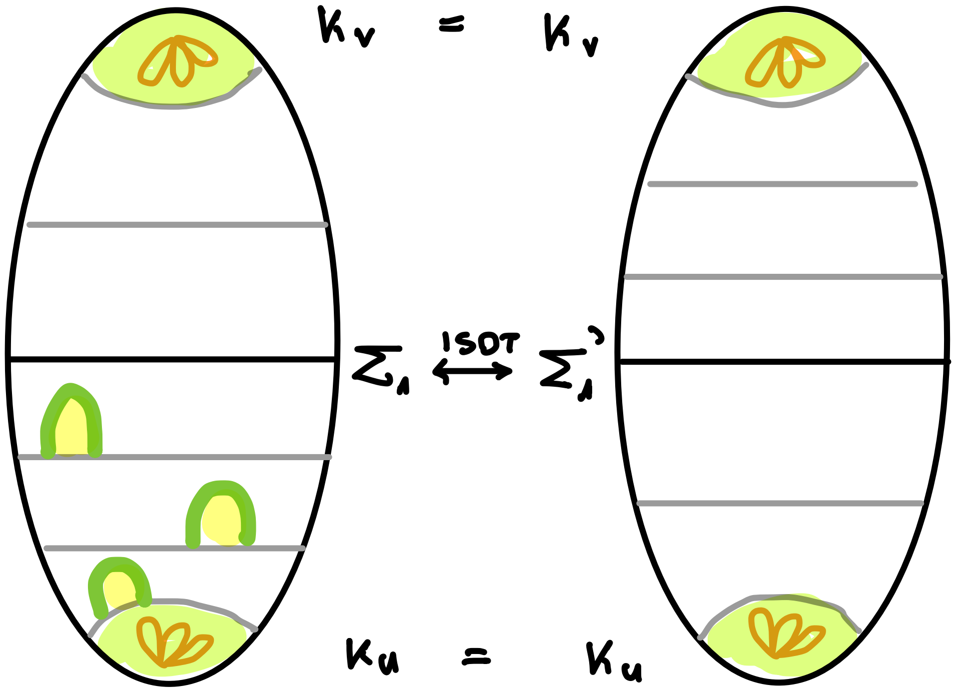
FUCHS-TABACHNIKOV: AFTER SUFFICIENTLY MANY LEGENDRIAN STAB. WE CAN ASSUME $K_u = K_u'$ & $K_v = K_v'$

- TAKE THE STANDARD NEIGHBOURHOOD $N(K_u)$, $N(K_v)$

Step 2

$\overbrace{M = U_1 \cup_{\Sigma_1} V_1}^{\mathcal{H}_1}$ & $\overbrace{M = U_1' \cup_{\Sigma_1'} V_1'}^{\mathcal{H}_2}$

WHERE Σ_1 & Σ_1' ARE SMOOTHLY ISOTOPIC



APPLY THE EXISTENCE PROOF

- TAKE LEGENDRIAN SKELETONS FOR THE HANDLEBODIES

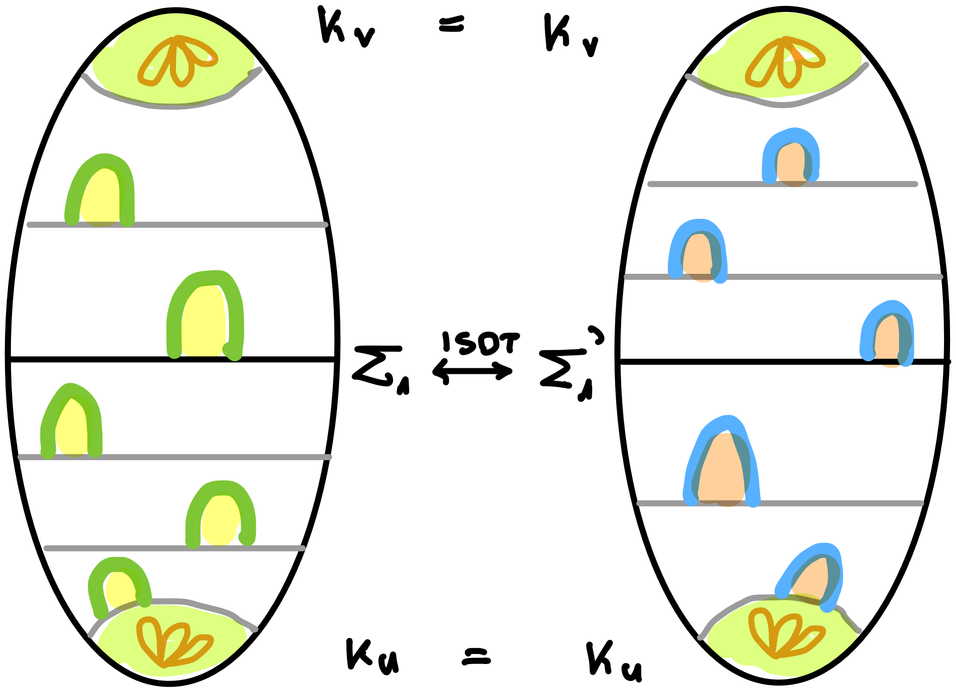
FUCHS-TABACHNIKOV: AFTER SUFFICIENTLY MANY LEGENDRIAN STAB. WE CAN ASSUME $K_U = K_{U'}$ & $K_V = K_{V'}$

- TAKE THE STANDARD NEIGHBOURHOOD $N(K_U)$, $N(K_V)$
- $U' \setminus N(K_U)$ & $V' \setminus N(K_V)$ ARE $\cong \Sigma_1 \times I \overset{\text{HONDA}}{\Longrightarrow}$ = STACKS OF BYPASSES

STEP 2

$$\overbrace{M = U_1 \cup_{\Sigma_1} V_1}^{\mathcal{H}_1} \quad \& \quad \overbrace{M = U_1' \cup_{\Sigma_1'} V_1'}^{\mathcal{H}_2}$$

WHERE Σ_1 & Σ_1' ARE SMOOTHLY ISOTOPIC



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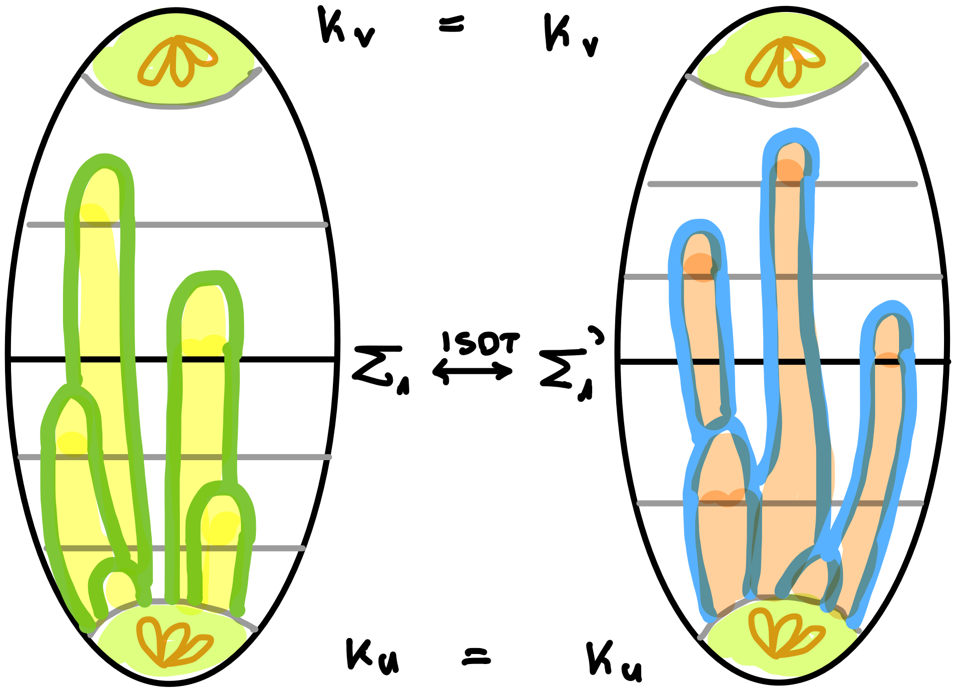

APPLY THE EXISTENCE PROOF

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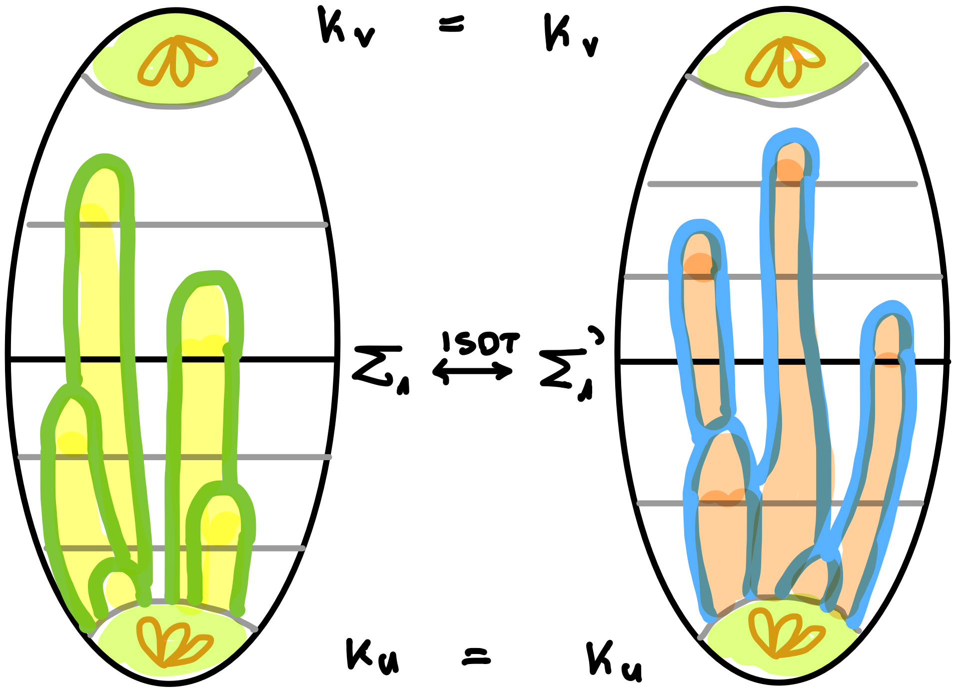
- EXTEND THE HANDLES

STEP 2

$$\underbrace{M = U_1 \cup_{\Sigma_1} V_1}_{\mathcal{H}_1} \quad \& \quad \underbrace{M = U_1' \cup_{\Sigma_1'} V_1'}_{\mathcal{H}_2}$$

WHERE Σ_1 & Σ_1' ARE SMOOTHLY ISOTOPIC



APPLY THE EXISTENCE PROOF

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- EXTEND THE HANDLES

$$U_2' := N(U_K) \cup (1\text{-}h\text{'s}) \qquad V_2' = M \setminus U_2'$$

$$\rightsquigarrow \quad \underbrace{M = U_2 \cup V_2}_{\mathcal{H}_2} \quad \& \quad \underbrace{M = U_2' \cup V_2'}_{\mathcal{H}_2'}$$

STEP 2

- $U' \setminus N(K_u)$ & $V' \setminus N(K_v)$ ARE $\cong \Sigma_1 \times I$

$\xRightarrow{\text{HONDA}}$ = STACKS OF BYPASSES

- EXTEND THE HANDLES

$U_2' := N(U_K) \cup (1\text{-}h's)$ $\qquad V_2' = M \setminus U_2'$

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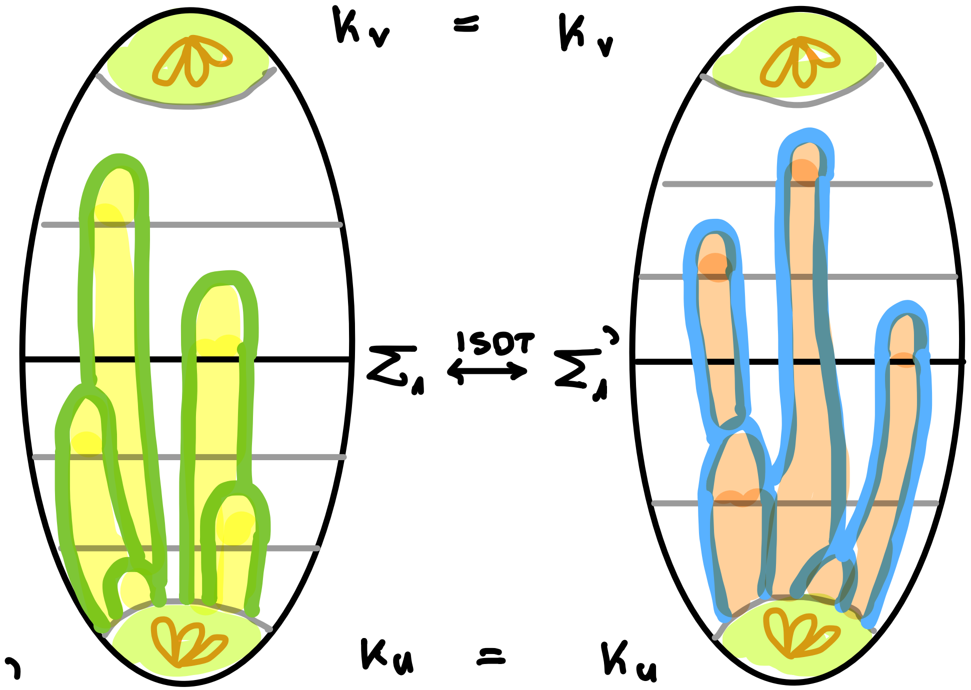


PROP (L-V) $\mathcal{H}_2'$ CAN BE OBTAINED FROM $\mathcal{H}_1'$ VIA A SEQUENCE OF CONTACT STAB & DESTAB

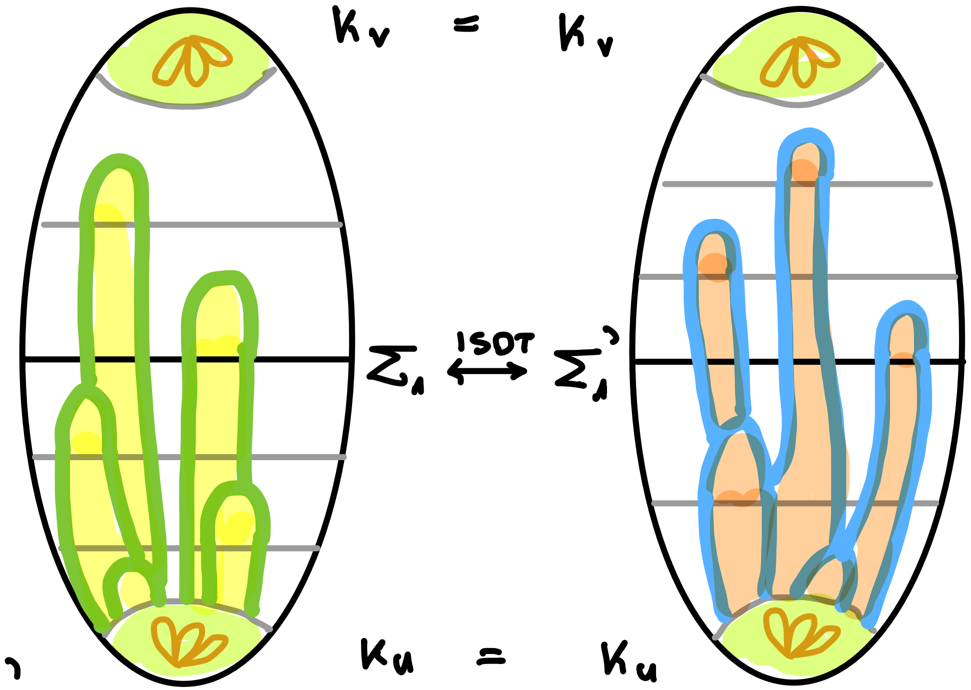
STEP 2

- $U' \setminus N(K_u)$ & $V' \setminus N(K_v)$ ARE $\cong \Sigma_1 \times I$

$\xRightarrow{\text{HONDA}}$ = STACKS OF BYPASSES

- EXTEND THE HANDLES

$U_2' := N(U_K) \cup (1\text{-}h'\text{s})$ $\qquad V_2' = M \setminus U_2'$



$\rightsquigarrow \underbrace{M = U_2 \cup V_2}_{\mathcal{H}_2}$ & $\underbrace{M = U_2' \cup V_2'}_{\mathcal{H}_2'}$

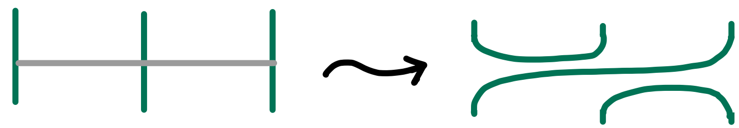
PROP (L~V) $\mathcal{H}_2'$ CAN BE OBTAINED FROM $\mathcal{H}_1'$ VIA A SEQUENCE OF CONTACT STAB & DESTAB

BEFORE NEXT STEP NOTICE

$$\mathcal{H}_2 = \hat{\mathcal{H}}(U_K, V_K, \mathcal{B}) \qquad \mathcal{H}_2' = \hat{\mathcal{H}}(U_K, V_K, \mathcal{B}')$$

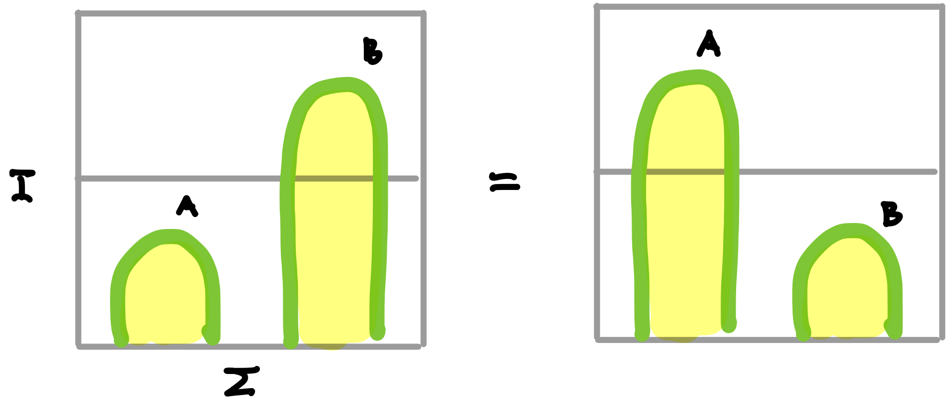
WHERE $\mathcal{B}$ & $\mathcal{B}'$ ARE DIFFERENT DECOMPOSITIONS OF $\xi|_{M - (N(U_K) \cup N(V_K))}$ AS BYPASS STACKS

INTERLUDE - STACKING BYPASSES



THE FOLLOWING MOVES DO NOT CHANGE ξ

COMMUTATION



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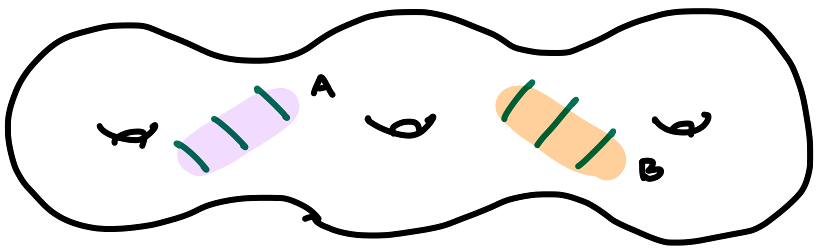
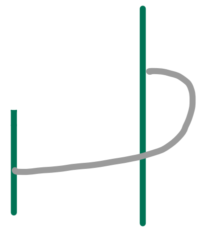
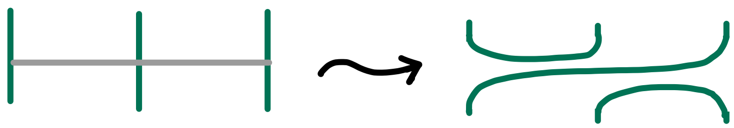


TRIVIAL BYPASS



Interlude - Stacking Bypasses



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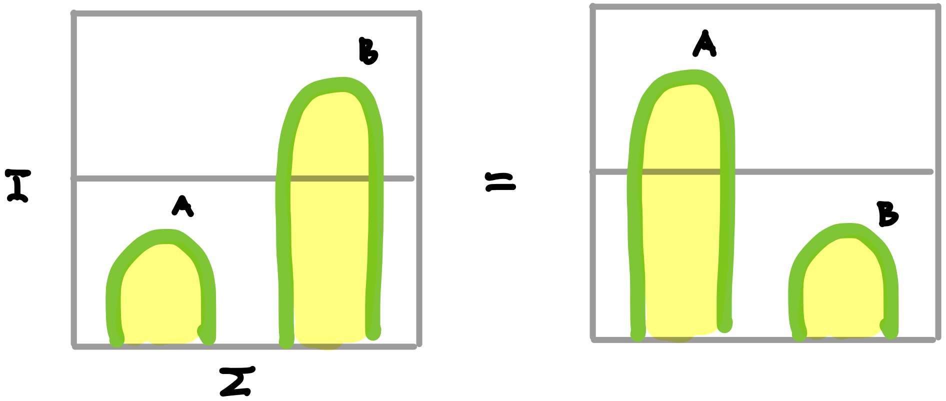

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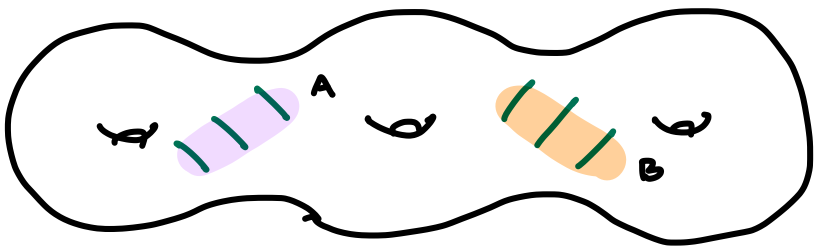

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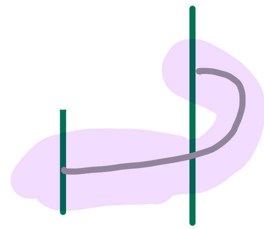

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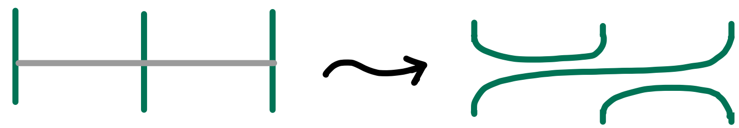

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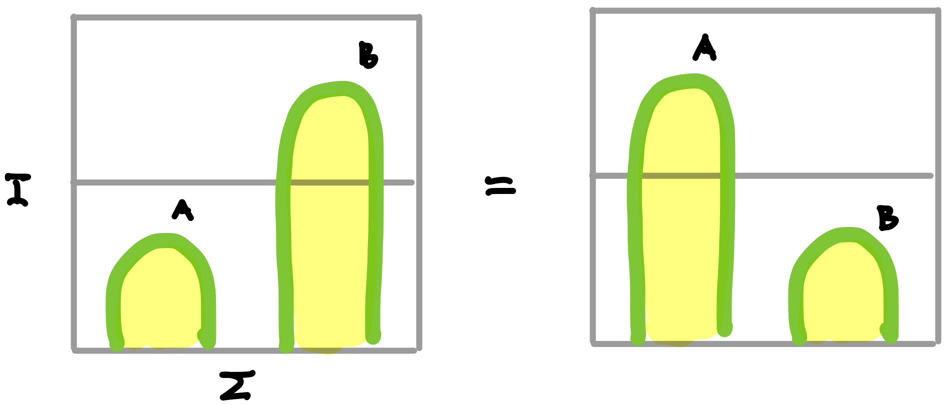

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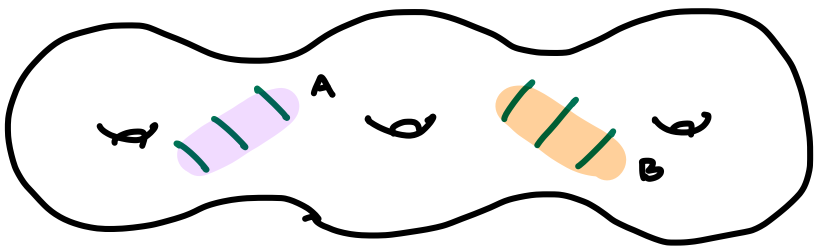

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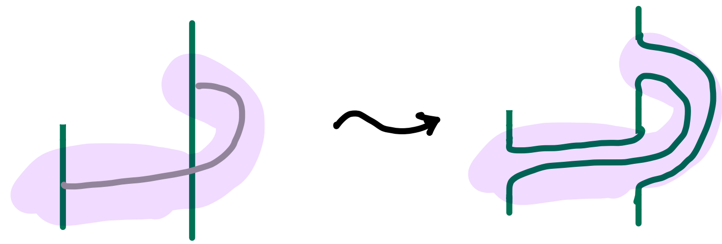

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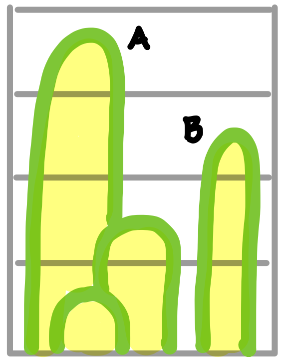

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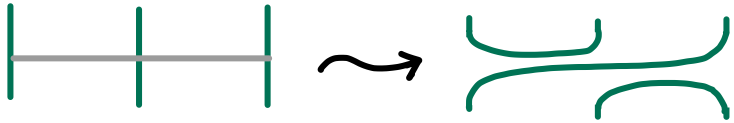

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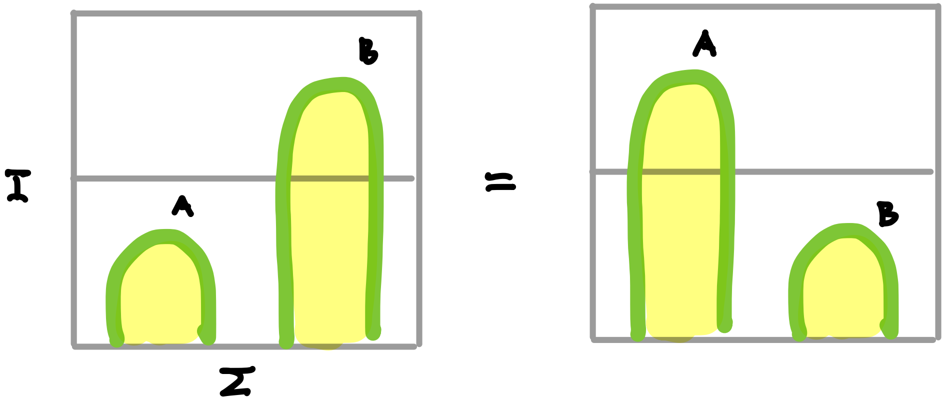

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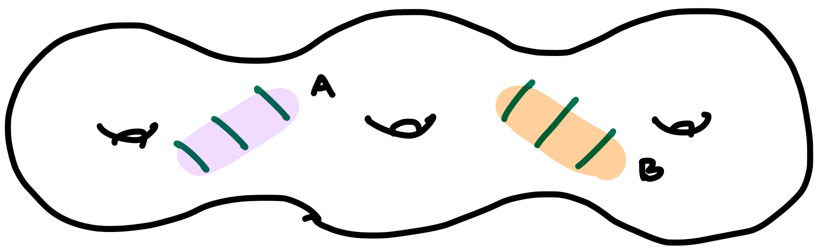

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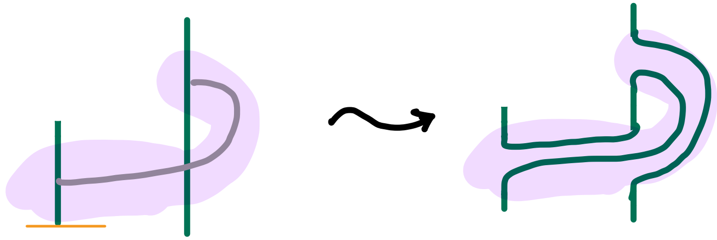

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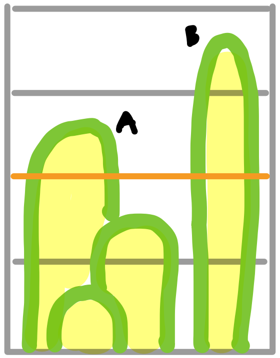

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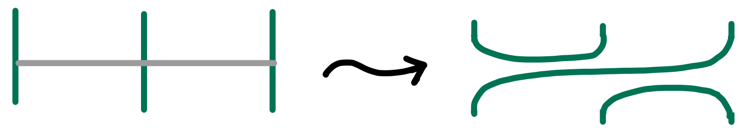

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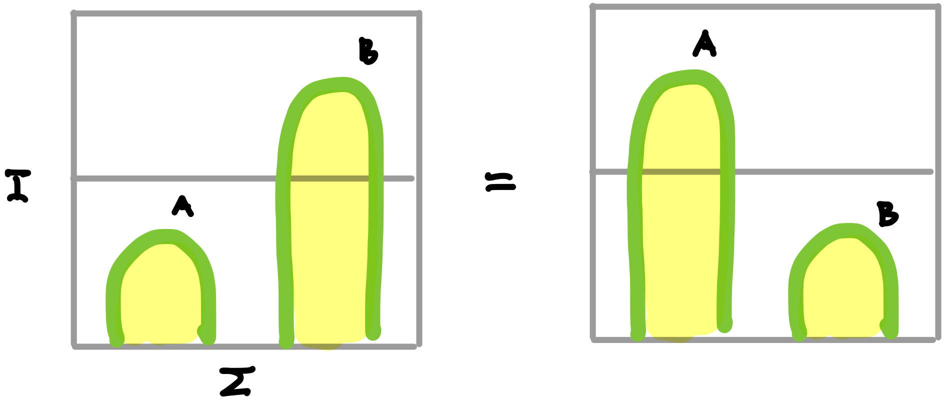

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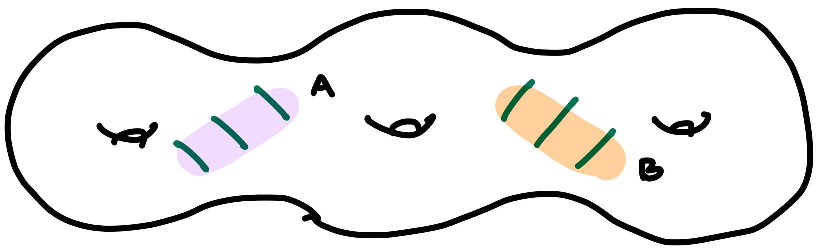

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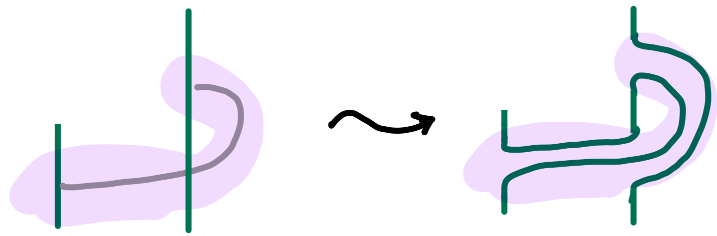

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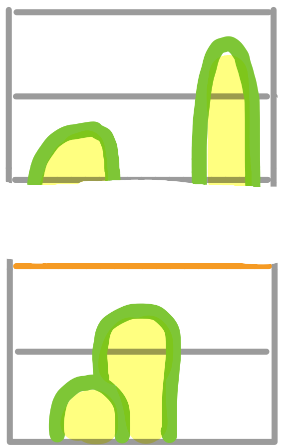
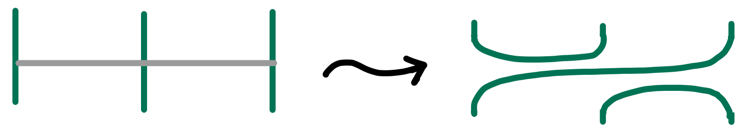


Interlude - Stacking Bypasses



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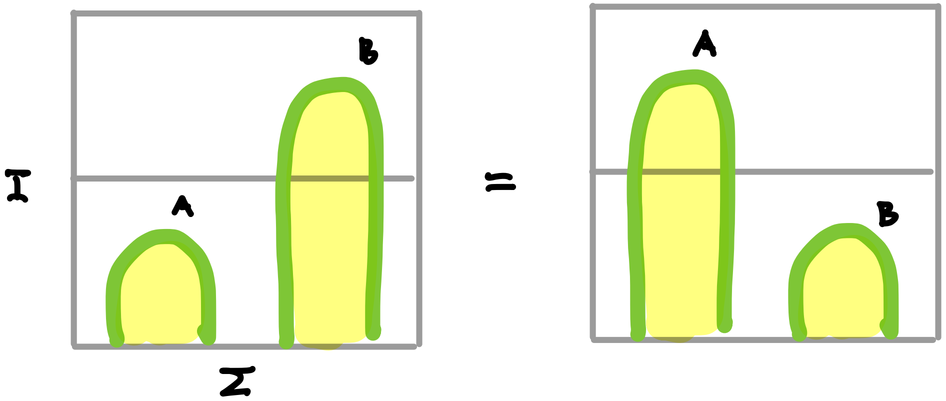

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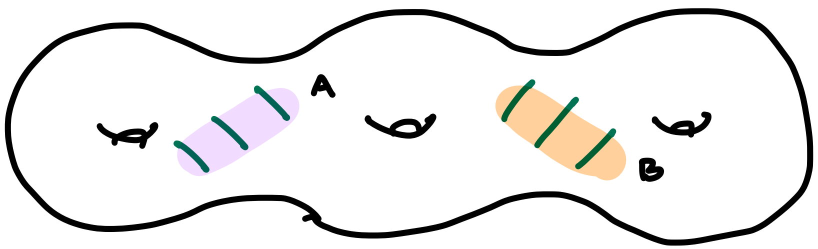

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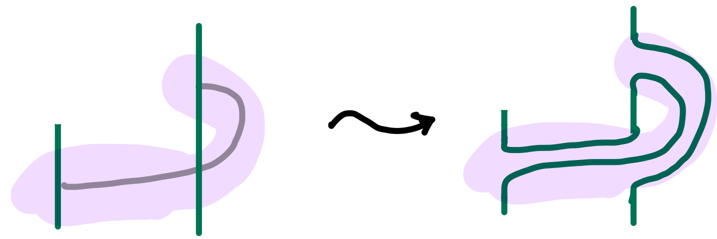

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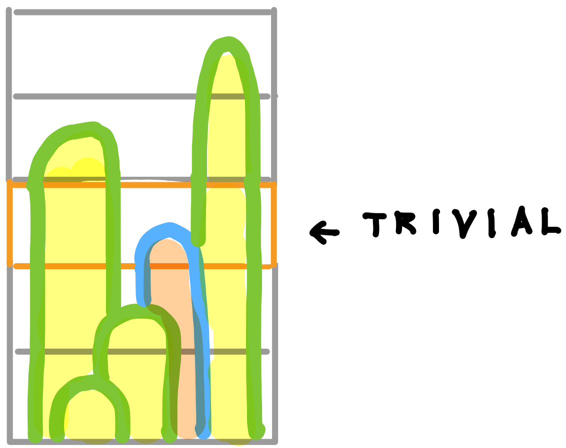
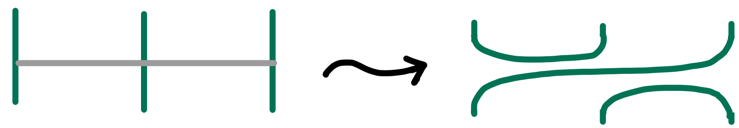


INTERLUDE - STACKING BYPASSES



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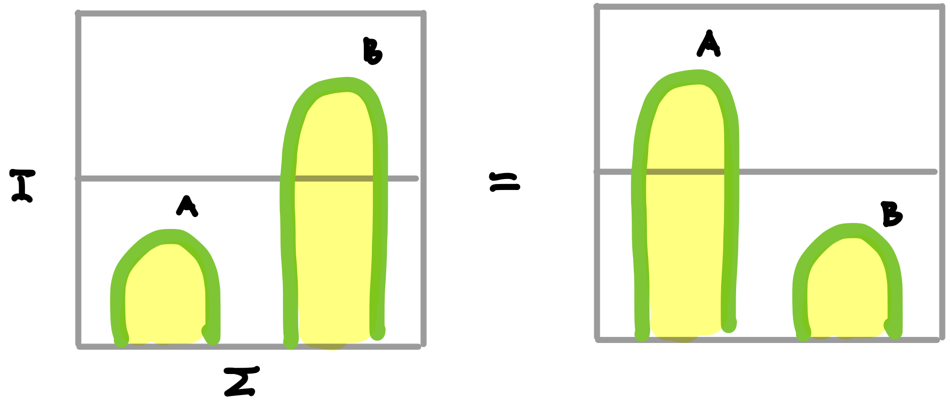

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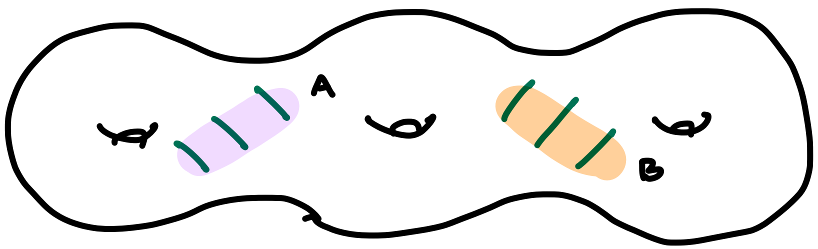

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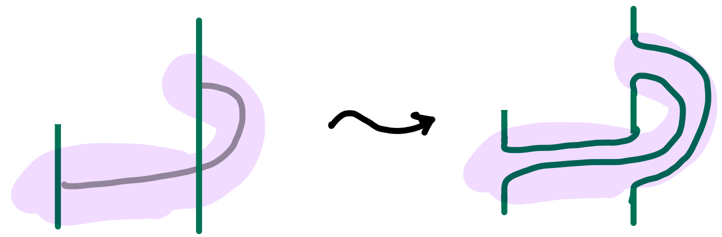

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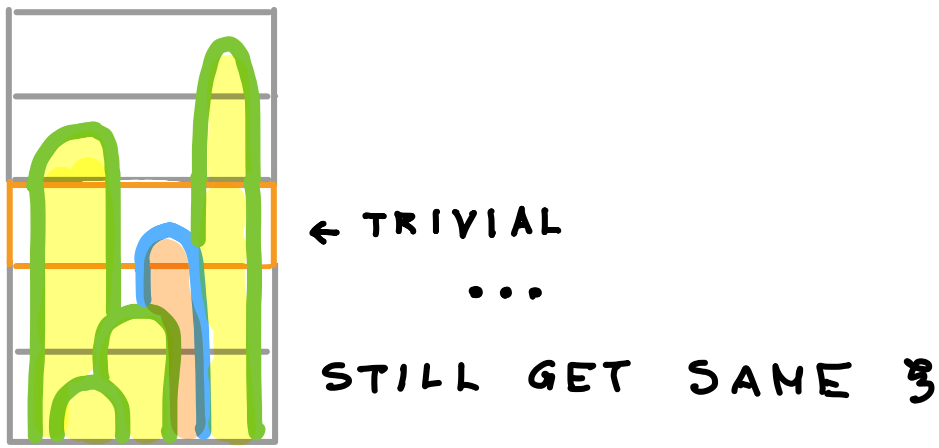
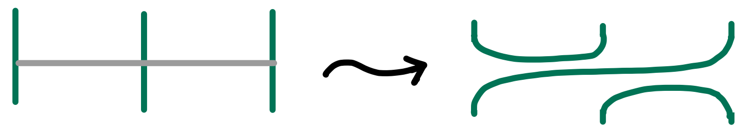


Interlude - Stacking Bypasses



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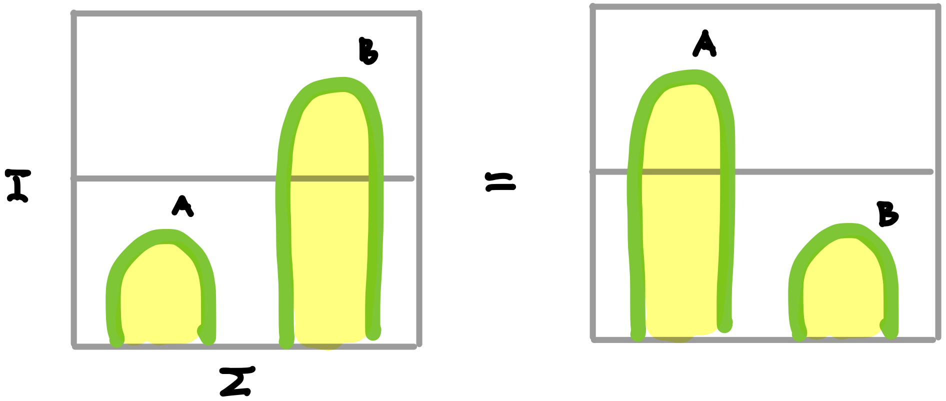

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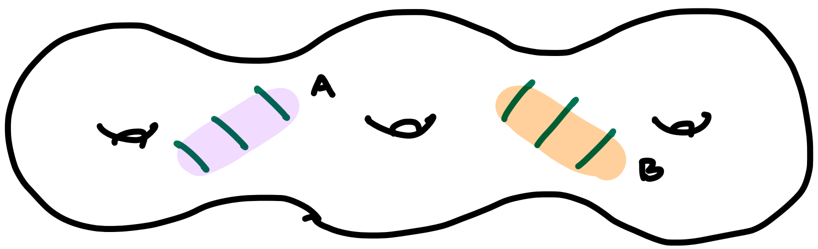

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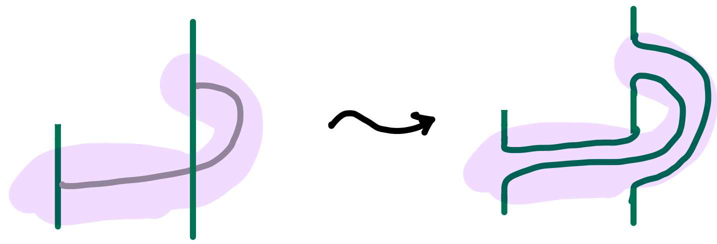

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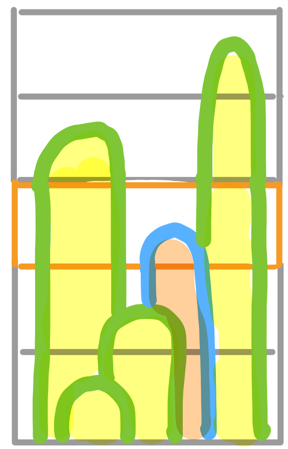


← TRIVIAL

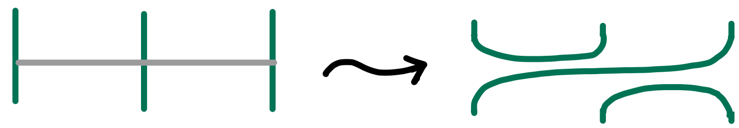
...

STILL GET SAME ξ

THM (TIAN, BREEN - HONDA - HUANG) ON $\Sigma \times I$ ANY TWO DECOMPOSITION OF ξ INTO STACKS OF BYPASS SLICES ARE RELATED VIA

- ADDING A TRIVIAL BYPASS
- COMMUTING DISJOINT BYPASSES

INTERLUDE - STACKING BYPASSES



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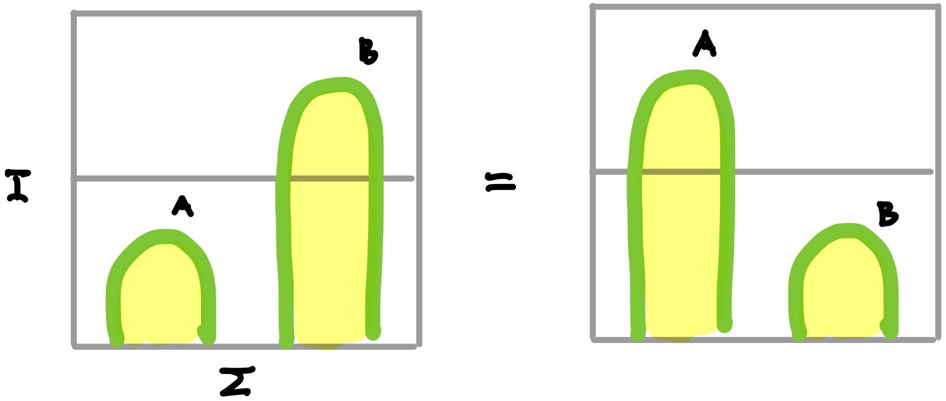
COMMUTATION



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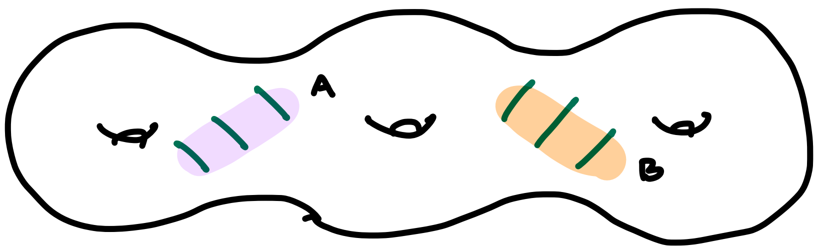
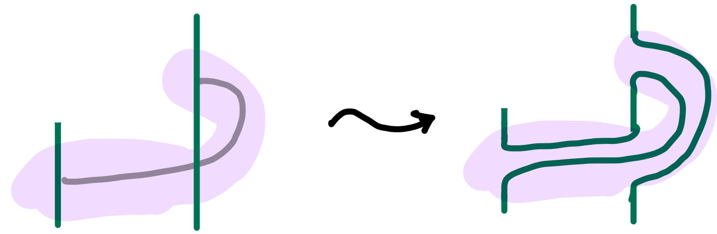
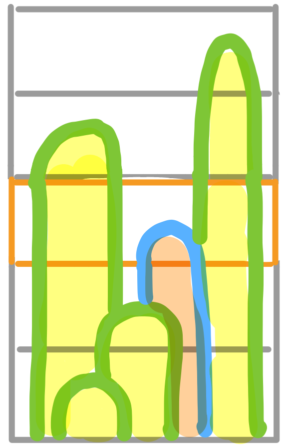


TRIVIAL BYPASS



E.G.:



← TRIVIAL

...

STILL GET SAME ξ

THM (TIAN, BREEN-HONDA-HUANG) ON $\Sigma \times I$ ANY TWO DECOMPOSITION OF ξ INTO STACKS OF BYPASS SLICES ARE RELATED VIA

- ADDING A TRIVIAL BYPASS
- COMMUTING DISJOINT BYPASSES

} **BYPASS MOVES**

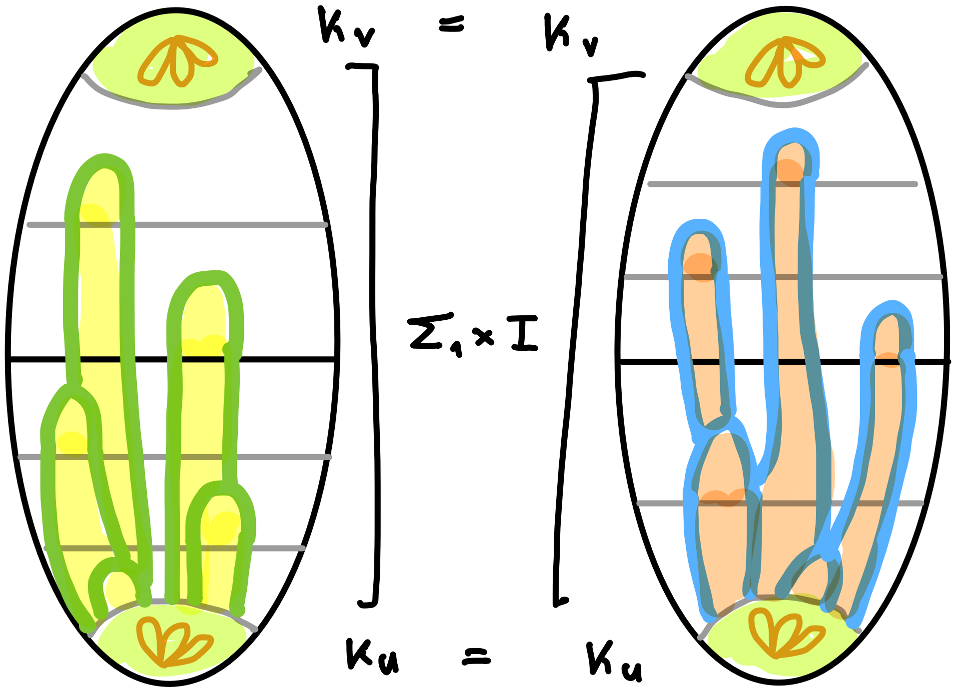
STEP 3

$\mathcal{H}_2 = \hat{\mathcal{H}}(U_K, V_K, \mathcal{B}) \quad \mathcal{H}_2' = \hat{\mathcal{H}}(U_K, V_K, \mathcal{B}')$

WHERE $\mathcal{B}$ & $\mathcal{B}'$ ARE TWO BYPASS - DECOMPOSITIONS OF $\xi|_{M-(N(U_K)\cup N(V_K))}$

$\xRightarrow[\text{B-H-H}]{\text{TIAN}}$ $\mathcal{B}$ & $\mathcal{B}'$ ARE RELATED VIA BYPASS MOVES

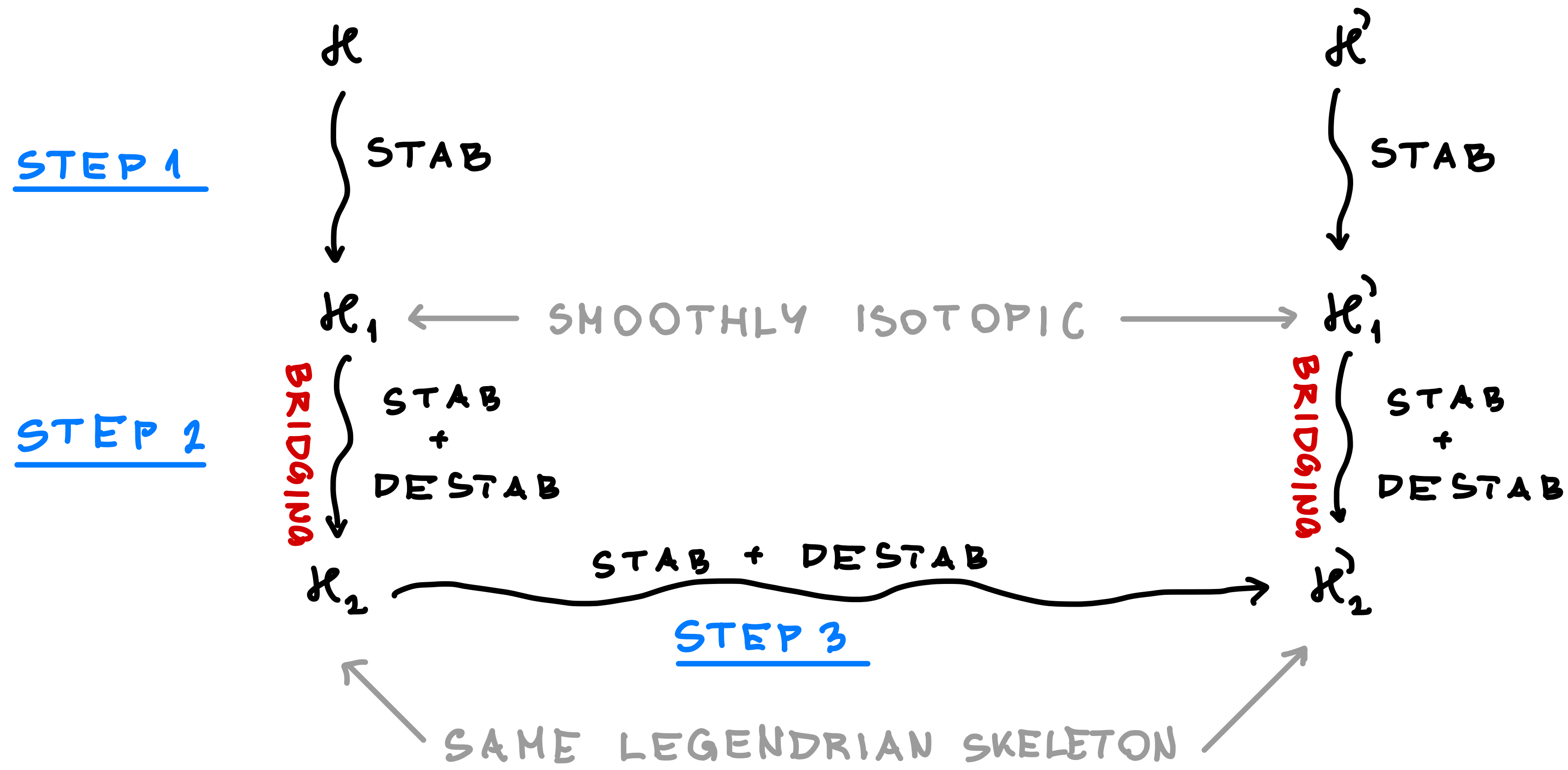
$\mathcal{B} = \mathcal{B}_0 \rightsquigarrow \mathcal{B}_1 \rightsquigarrow \cdots \rightsquigarrow \mathcal{B}_k = \mathcal{B}'$



THM (L-V): BYPASS MOVES ON $\Sigma_1 \times I$ CORRESPOND TO CONTACT STAB & DESTAB OF $\hat{\mathcal{H}}$

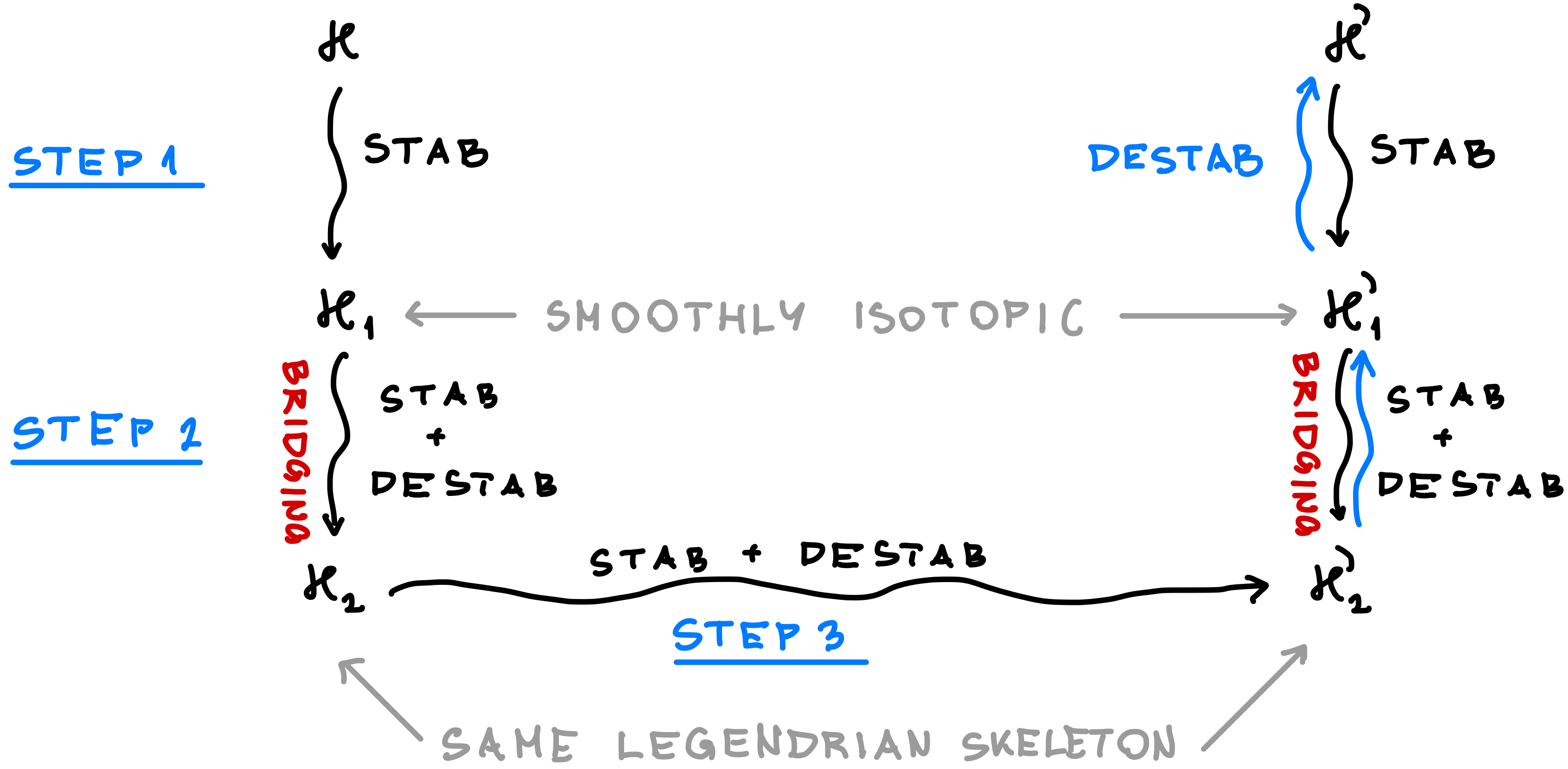
$$\underset{\mathcal{H}_2}{\overset{\parallel}{\hat{\mathcal{H}}(U_K,V_K,\mathcal{B})}} \xrightarrow{\text{STAB+DESTAB}} \hat{\mathcal{H}}(U_K,V_K,\mathcal{B}_1) \rightsquigarrow \cdots \xrightarrow{\text{STAB+DESTAB}} \underset{\mathcal{H}_2'}{\overset{\parallel}{\hat{\mathcal{H}}(U_K,V_K,\mathcal{B}')}}$$

Summary



$\mathcal{H}$
$\mathcal{H}'$
STEP 1
STAB
STAB
$\mathcal{H}_1$
SMOOTHLY ISOTOPIC
$\mathcal{H}'_1$
STEP 2
BRIDGING
STAB + DESTAB
BRIDGING
STAB + DESTAB
$\mathcal{H}_2$
STAB + DESTAB
$\mathcal{H}'_2$
STEP 3
SAME LEGENDRIAN SKELETON

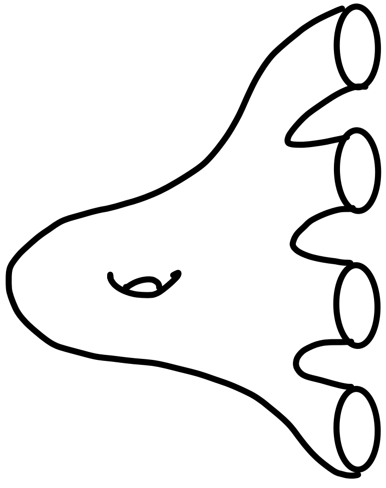
Summary

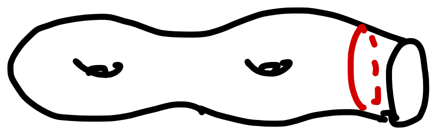


$\Longrightarrow$ $\mathcal{H}$ & $\mathcal{H}'$ ARE RELATED VIA A SEQUENCE OF CONTACT STAB. & DESTAB

$\blacksquare$

FURTHER DIRECTIONS





WHAT'S NEXT?

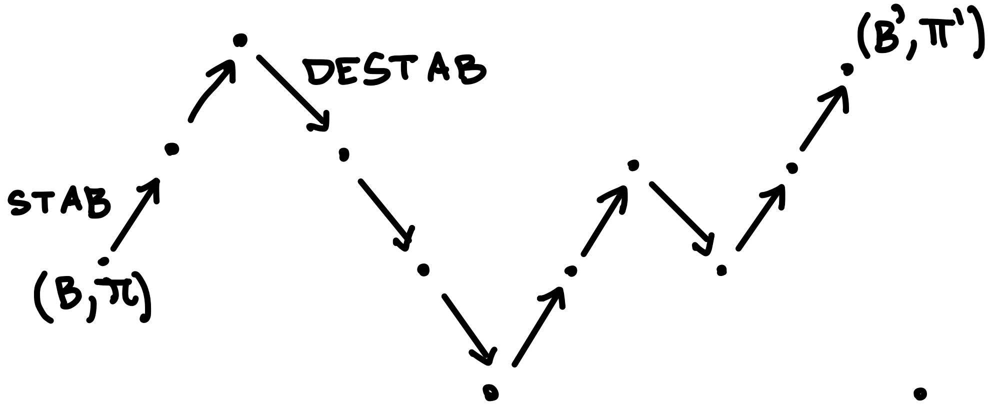
FEELS LIKE END OF A LONG STORY

BUT AS ALWAYS THERE IS A LOT TO DO:

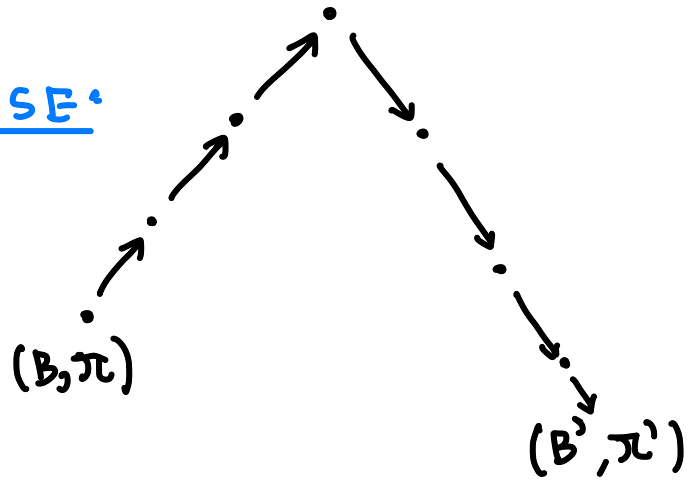
- (HOPEFULLY) MINOR ISSUE:

  COMMON STABILISATION VS. SEQUENCE OF (DE)STABILISATIONS

WHAT WE PROVED:



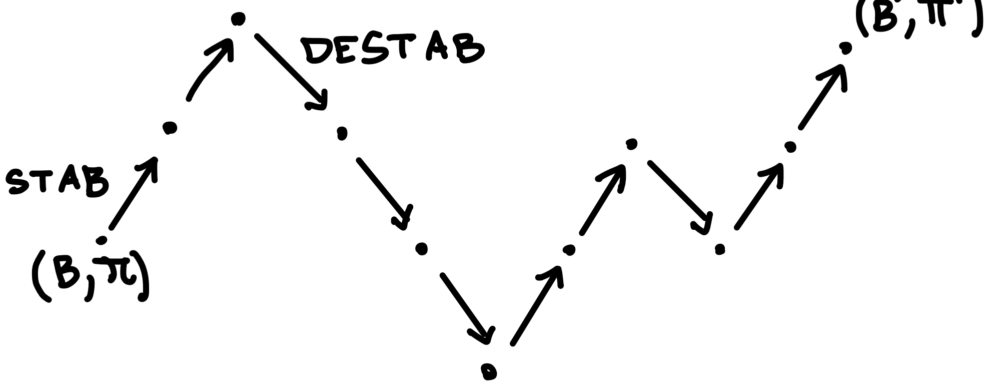
WHAT SOME OF THE APPLICATIONS USE:



ARE THEY EQUIVALENT?

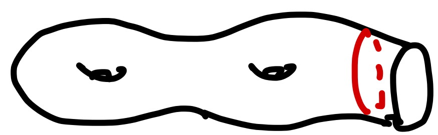
WHAT'S NEXT ?

- HOW MANY STABILISATIONS DO WE NEED?



- ETNYRE: DOES EVERY CONTACT STRUCTURE HAVE A GENUS 1 OPEN BOOK ?

  - OVERTWISTED CONTACT STRUCTURES HAVE PLANAR (= GENUS 0) OPEN BOOKS (ETNYRE)
  - THERE ARE CONTACT STRUCTURES THAT DO NOT HAVE PLANAR OPEN BOOKS (ETNYRE)
  - POSSIBLE COUNTEREXAMPLE (HASSOT)



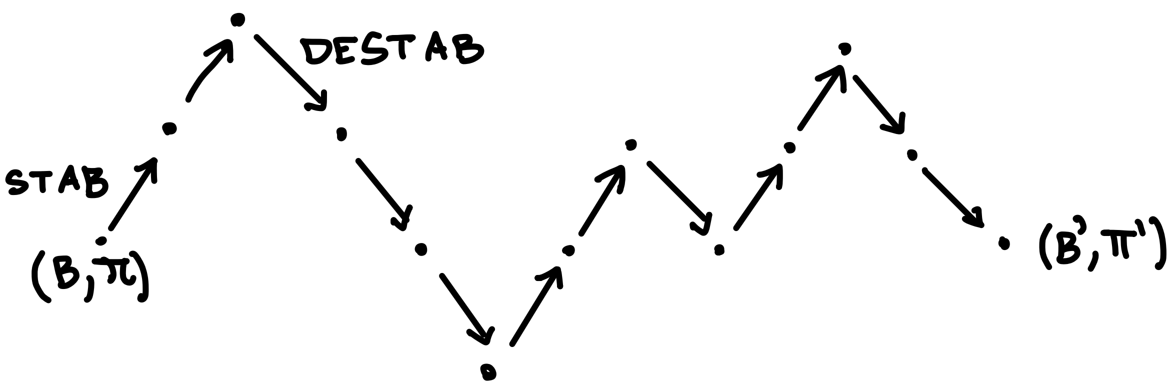
WHAT'S NEXT?

THM (WAND): LEGENDRIAN SURGERY PRESERVS TIGHTNESS

- THE PROOF RELIES ON AN EQUIVALENT CHARACTERISATION OF TIGHTNESS IN TERMS OF OPEN BOOKS
- THIS CHARACTERISATION IS GIVEN IN A SEQUENCE OF COMBINATORIAL DEFINITIONS THAT TAKE UP MULTIPLE PAGES

IS THERE A SIMPLER PROOF USING CONTACT HEEGAARD DECOMPOSITIONS?

- MOVES BETWEEN OPEN BOOKS OF THE SAME GENUS/ EULER CHARACTERISTIC



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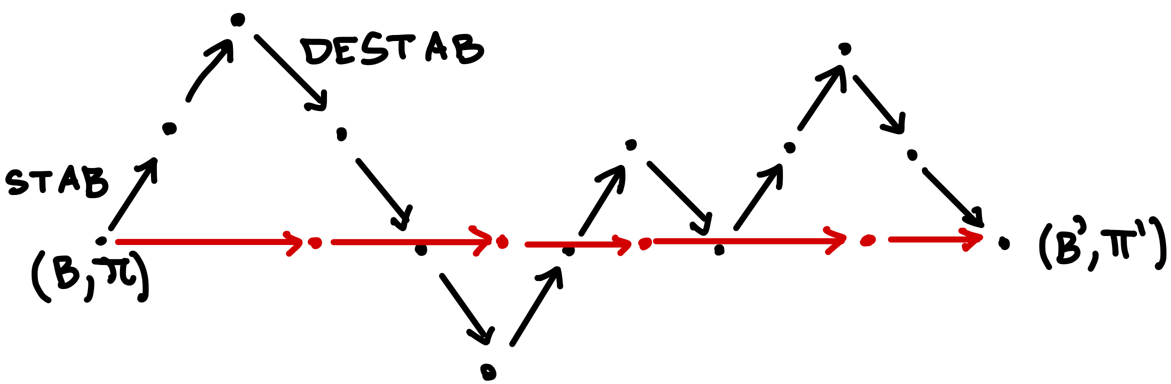

THANKS FOR
YOUR
ATTENTION!