Jets in Noncommutative Geometry

M. Mantegazza
joint collaboration with K. J. Flood and H. Winther
Charles University, Prague

Introduction

Jet bundles in differential geometry provide an intrinsic formulation of the notion of Taylor approximation. Moreover, jet bundles are representing object for functors of differential operators and can be used to define differential equations.
The aim of this project is to extend to the noncommutative setting the notion of jet bundles and objects related to them. Ultimately we aim at generalising the theory of differential equations.

Jet bundles in differential geometry

Definition 1 *Given a manifold M and a vector bundle $E \to M$, the n-jet bundle is the vector bundle $J^nE \to M$ whose fibre at $p \in M$ consists of the n-jets at p, i.e. equivalence classes of sections of E that agree at p up to n-th order Taylor approximation.*
For $n \geq m$ we have vector bundle maps $\pi^{n,m}\colon J^nE \to J^mE$, called *jet projections*, that map an n-jet at p to the corresponding m-jet at p for all $p \in M$.
We also have a map $j^n\colon \Gamma(E) \to \Gamma(J^nE)$ assigning to a section σ of E the section $j^n(\sigma)$ mapping p to the n-jet of σ at p. This map is called *n-jet prolongation*.
For all $n \geq 1$, we have the following short exact sequence of bundles

$$0 \longrightarrow S^nT^*M \otimes_{\mathbb{R}} E \longrightarrow J^nE \xrightarrow{\pi^{n,n-1}} J^{n-1}E \longrightarrow 0.$$

This induces a short exact sequence on sections

$$0 \longrightarrow S^n(M) \otimes_{\mathcal{C}^\infty(M)} \Gamma(E) \longrightarrow \Gamma(J^nE) \xrightarrow{\pi^{n,n-1}} \Gamma(J^{n-1}E) \longrightarrow 0.$$

Noncommutative differential geometry

In our setting, the objects of differential geometry translate as follows

Geometry	Algebra	NCDG	Structure
Scalars, "point": $\mathbb{R}$ (or $\mathbb{C}$)	$\mathbb{R}$ (or $\mathbb{C}$)	$\mathbb{k}$	commutative unital ring
Manifold: M	$\mathcal{C}^\infty(M)$	A	unital assoc. $\mathbb{k}$-algebra
Vector bundle: $E \to M$	$\Gamma(E)$	E	f.g.p. left A-module
Vector bundle map: $E \to F$	$\Gamma(E) \to \Gamma(F)$	$E \to F$	left A-linear map

We denote the category of left (right) A-modules by ${}_A\mathrm{Mod}$ (Mod_A), and that of $\mathbb{k}$-modules by Mod. The module of differential forms, is generalised by

Definition 2 (Exterior algebra) *Let A be a k-algebra, an* exterior algebra *on A is the data of a graded algebra $\Omega^\bullet_d = \oplus_{n\geq 0}\Omega^n_d$, with product $\wedge$ and a $\mathbb{k}$-linear map $d\colon \Omega^\bullet_d \to \Omega^\bullet_d$, called* differential*, such that*

1. *$\Omega^0_d = A$, $d(\Omega^n_d) \subseteq \Omega^{n+1}_d$, $d^2 = 0$;*
2. *$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^n \alpha \wedge d\beta$ for all $\alpha \in \Omega^n_d$, $\beta \in \Omega^\bullet_d$ (Leibniz);*
3. *A and dA generate $\Omega^\bullet_d$ as an algebra (surjectivity).*

Example 3 (Universal exterior algebra) *Given a $\mathbb{k}$-algebra A, we define Ω^n_u to be the intersection in $A^{\otimes(n+1)}$ of the kernels corresponding to adjacent products.*

1. *$(a_0 \otimes \cdots \otimes a_n) \wedge (b_0 \otimes \cdots \otimes b_m) = a_0 \otimes \cdots \otimes a_n b_0 \otimes \cdots \otimes b_m)$;*
2. *$d_u(a_0 \otimes \cdots \otimes a_n) = \Sigma_{i=0}^{n+1}(-1)^i a_0 \otimes \cdots \otimes a_{i-1} \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_n)$.*

Example 4 (Classical exterior algebra) *Given a smooth manifold M, the exterior algebra $\Omega^\bullet(M)$ of differential forms on M, equipped with the standard $\wedge$ and differential, is an exterior algebra over the $\mathbb{R}$-algebra $A = \mathcal{C}^\infty(M)$.*

Symmetric forms

We can interpret the exterior algebra and each of its components as endofunctors

$$\Omega^n_d := \Omega^n_d \otimes_A -\colon {}_A\mathrm{Mod} \longrightarrow {}_A\mathrm{Mod}, \qquad \Omega^\bullet_d := \Omega^\bullet_d \otimes_A -\colon {}_A\mathrm{Mod} \longrightarrow {}_A\mathrm{Mod}\,.$$

Using the exterior algebra we can define a non commutative analogue of the bundle of symmetric forms.

Definition 5 (Functor of symmetric forms) *Given a $\mathbb{k}$-algebra A endowed with an exterior algebra $\Omega^\bullet_d$, and E in ${}_A\mathrm{Mod}$, we define $S^0_d(E) = \Omega^0_d(E) = E$, $S^1_d = \Omega^1_d(E)$, with $\iota^0_{\wedge,E} := \mathrm{id}_E$ and $\iota^1_{\wedge,E} = \mathrm{id}_{\Omega^1_d E}$. By induction, $S^n_d(E)$ is the kernel of the following composition*

$$\Omega^1_d S^{n-1}_d E \xrightarrow{\Omega^1_d(\iota^{n-1}_{\wedge,E})} \Omega^1_d\Omega^1_d S^{n-2}_d E \xrightarrow{\wedge\otimes_A \mathrm{id}_{S^{n-2}_d E}} \Omega^2_d S^{n-2}_d E$$

with kernel morphism $\iota^n_{\wedge,E}\colon S^n_d(E) \longrightarrow \Omega^1 \otimes_A S^n_d E$.

Example 6 (Universal symmetric forms) *For the universal exterior algebra, the functor of symmetric form is $S^n_u E = 0$ for $n > 1$.*

Example 7 (Classical symmetric forms) *In the classical case, $S^n_d E$ it is the module of global symmetric n-tensors.*

We can now define the so-called *Spencer δ-complex* as follows. For all $n > 0$ and $m \geq 0$, the component n, m is $\Omega^m_d S^n_d E$, and we define $\delta^{n,m}_d$ as the following composition

$$\Omega^m_d S^n_d E \xrightarrow{\Omega^m_d(\iota^n_{\wedge,E})} \Omega^m_d \Omega^1_d S^{n-1}_d E \xrightarrow{(-1)^m \wedge^{m,1}_{S^{n-1}_d E}} \Omega^{m+1}_d S^{n-1}_{d,E}, \qquad \delta^{n,m}_{d,E}\colon \Omega^m_d S^n_d E \to \Omega^{m+1}_d S^{n-1}_{d,E}$$

We call its cohomology $H^{n,m}_{\delta_d}(E)$ the Spencer δ-cohomology.

Jet functors

We look for a generalisation of the n-jet functor so that they fit in a jet exact sequence:

$$0 \longrightarrow S^n_d E \xrightarrow{\iota^n_{d,E}} J^n_d E \xrightarrow{\pi^{n,n-1}_{d,E}} J^{n-1}_d E \longrightarrow 0. \qquad \text{(JES)}$$

The universal exterior algebra has the property that for all exterior algebras $\Omega^\bullet_d$ there exists a unique epimorphism $\Omega^\bullet_u \twoheadrightarrow \Omega^\bullet_d$. Let $p_d\colon \Omega^1_u \twoheadrightarrow \Omega^1_d$ and let N_d be its kernel. We find that $J^1_d A$ can be identified with the A-bimodule $A \otimes A/N_d$. In general:

Definition 8 (Holonomic n-jet functor) *We define $J^0_d E := E$, $J^1_d E := J^1_d A \otimes_A E$, $l^1_{d,E} := \mathrm{id}_{J^1_d E}$, and by induction, $J^n_d E$ as the kernel of the composition*

$$J^1_d J^{n-1}_d E \xrightarrow{J^1_d(l^{n-1}_{d,E})} J^1_d J^1_d J^{n-2}_d E \xrightarrow{\widetilde{Ð}_d \otimes_A \mathrm{id}_{J^{n-2}_d E}} (\Omega^1_d \ltimes \Omega^2_d) J^{n-2}_d E$$

with natural inclusion $l^n_d\colon J^n_d \longrightarrow J^1_d \circ J^{n-1}_d$. Here, $\Omega^1_d \ltimes \Omega^2_d$ is a direct sum in Mod_A with left A-action $a(\alpha + \beta) = a\alpha + da \wedge \alpha + a\beta$ and $\widetilde{Ð}_d([a \otimes b] \otimes_A [c \otimes e]) = ad(bc)e + da \wedge d(bc)e$. We call J^n_d the (holonomic) n-jet functor*.*

We can also define compatible natural transformations: $\pi^{n,m}_{d,E}\colon J^n_d E \to J^m_d E$ (projections) and $\iota^n_{d,E}\colon S^n_d E \to J^n_d E$ in ${}_A\mathrm{Mod}$, and $j^n_{d,E}\colon E \to J^n_d E$ in Mod (prolongations).

Theorem 9 (Holonomic jet exact sequence) *Let $\Omega^\bullet_d$ be an exterior algebra over the $\mathbb{k}$-algebra A such that Ω^1_d, Ω^2_d, and Ω^3_d are flat in Mod_A. For $n \geq 1$, if the Spencer δ-cohomology $H^{m,2}_{\delta_d}$ vanishes, for all $1 \leq m \leq n-2$, then* (JES) *is exact.*

Differential operators

Definition 10 *Let E, F in ${}_A\mathrm{Mod}$. A $\mathbb{k}$-linear map $\Delta\colon E \to F$ is called a (holonomic) linear differential operator of order at most n with respect to the exterior algebra $\Omega^\bullet_d$, if there exists an A-module map $\widetilde{\Delta} \in {}_A\mathrm{Hom}(J^n_d E, F)$ such that the following diagram commutes:*

$$\begin{array}{ccc} J^n_d E & & \\ {\scriptstyle j^n_{d,E}}\uparrow & \searrow^{\widetilde{\Delta}} & \\ E & \xrightarrow{\Delta} & F \end{array}$$

If n is minimal, we say that Δ is a (holonomic) linear differential operator of order *n with respect to the exterior algebra $\Omega^\bullet_d$.*

Some properties of differential operators generalise to the noncommutative case

Proposition 11

1. *The n-jet prolongation $j^n_{d,E}$ is a differential operator of order at most n.*
2. *Partial derivatives for a parallelisable first order differential calculus are differential operators of order at most 1.*
3. *A connection $\nabla\colon E \to \Omega^1_d E$ is a differential operator of order at most 1. The differential is a particular case.*
4. *Let A be a $\mathbb{k}$-algebra and E be in ${}_A\mathrm{Mod}$. There is a bijective correspondence between connections on E and left A-linear splittings of the 1-jet short exact sequence corresponding to E.*
5. *Let $\Delta_1\colon E \to F$ and $\Delta_2\colon F \to G$ be differential operators of order at most n and m, respectively. Then the composition $\Delta_2 \circ \Delta_1\colon E \to G$ is a differential operator of order at most $n + m$.*

We can also generalise the notion of vector field as a differential operator.

Definition 12 *A* vector field *X is a differential operator of order at most 1 such that if $da = 0$, then $X(a) = 0$.*

A vector field X corresponds to an element $\langle -, X\rangle \in {}_A\mathrm{Hom}(\Omega^1_d, A)$ such that $\langle \alpha, X\rangle := \widetilde{X} \circ \iota^1_{d,A}(\alpha)$ and this correspondence is bijective.
One can also show that the module of vector fields together with the natural embedding in $\mathrm{End}(A)$, form a (canonical) *left Cartan pair*.

Results

Theorem 13 (Stability) *Let $\Omega^\bullet_d$ be an exterior algebra over a $\mathbb{k}$-algebra A such that the n-jet sequences are exact and $S^m_d A$ is flat, projective, or finitely generated projective in ${}_A\mathrm{Mod}$ for all $1 \leq m \leq n$, then J^n_d preserves the subcategory of flat, projective, or finitely generated projective A-modules, respectively.*

Theorem 14 (Classical case) *Let $A = \mathcal{C}^\infty(M)$ for a smooth manifold M, let $\Omega^\bullet_d = \Omega^\bullet(M)$ equipped with the de Rham differential d, and let E be the space of global smooth sections of a vector bundle. Then the module of global sections of classical holonomic n-jet bundle of E is isomorphic to $J^n_d E$ in ${}_A\mathrm{Mod}$, and the prolongation maps and jet projections are compatible with the isomorphisms.*

Theorem 15 (Representability) *The jet module $J^n_d E$ is a representing object for linear differential operators of order at most n on E, i.e. for the functor $\mathrm{Diff}^n_d(E, -)$, if and only if $J^n_d E = \{a j^n_{d,E}(e) | a \in A, e \in E\} = A j^n_{d,E}(E)$.*