Elements of Graded Lie Theory

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What awaits

- A quick refresher on $\mathbb{Z}$-graded manifolds
- Group objects in a category
- Group objects in $\mathbb{Z}$-graded manifolds — graded Lie groups
- Left-invariant vector fields and the graded Lie algebra

There will also be some **action** (and fundamental VFs)

$\mathbb{Z}$-graded Refresher

- A way to work globally and consistently with graded-commutative variables
- Basically a smooth manifold M with an added sheaf of graded smooth functions, $\mathcal{M} := (M, C^\infty_{\mathcal{M}})$
- On coordinate patches, graded smooth functions behave like formal power series

$$f = \sum_{\boldsymbol{p}} f_{\boldsymbol{p}}\, \xi_1^{p_1} \cdots \xi_{\tilde{n}}^{p_{\tilde{n}}},$$

where

- $f_{\boldsymbol{p}} = f_{\boldsymbol{p}}(x^1, \dots, x^{n_0})$ are smooth functions on M
- $n_j := \#\{i \in \mathbb{Z} \mid |\xi_i| = j\}$ for $j \neq 0$. The sequence $(n_j)_{j \in \mathbb{Z}}$ is the graded dimension of $\mathcal{M}$
- $fg = (-1)^{|f||g|} gf$
- Both x^i and ξ_μ are called coordinates on $\mathcal{M}$, and are sometimes denoted together as $\{x^i, \xi_\mu\} \sim x^i$

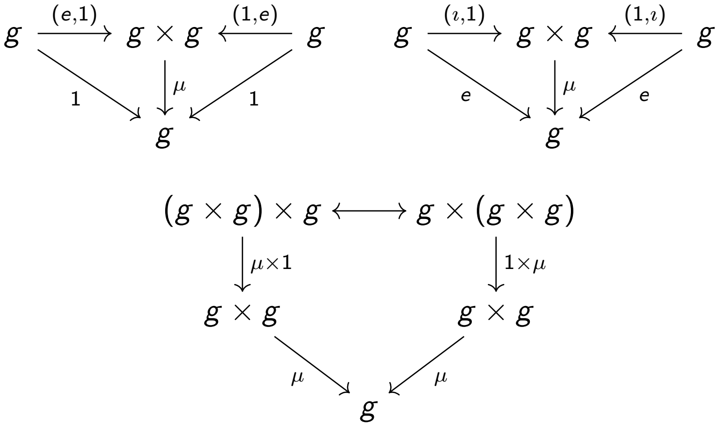
Example:

- Let $\mathbb{R}^{(n_j)}$ be the graded vector space $(\mathbb{R}^{(n_j)})_i = \mathbb{R}^{n_i}$ with $\sum_j n_j < +\infty$.
- We can make it into a graded manifold $\mathrm{g}\mathbb{R}^{(n_j)} := (\mathbb{R}^{n_0}, C^\infty_{(n_{-j})})$.
- With global coordinates on $\mathrm{g}\mathbb{R}^{(n_j)}$ given by some dual basis of $\mathbb{R}^{(n_j)}$.

Group Objects in Cats

- The notion of a group object makes sense in every category C with products and a terminal object $t \in$ C, [1].
- A group object in a category C is any $(g, \mu, e, \imath)$, where
  - $g \in$ C
  - $\mu : g \times g \to g$
  - $e : t \to g$
  - $\imath : g \to g$

Such that the diagrams:



all commute.

Can we actually say something about these things?

- We can apply the fully faithful Yoneda functor $Y : \mathsf{C} \to \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$
  - $Ya := \mathsf{C}(\cdot, a), \quad a \in \mathsf{C}$
  - $Yf := f_*, \quad f : a \to b$

Lemma

Let C be a locally small category. Then $(g, \mu, e, \imath)$ is a group object in C if and only if $(Yg, \mu_*, e_*, \imath_*)$ is a group object in $\mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$.

So What?

Group objects in categories of functors valued in Set are just collections of groups and their morphisms:

Lemma

Let B be any category. Then $(G, \mu, e, \imath)$ is a group object in $\mathrm{Set}^{\mathrm{B}}$ if and only if $(Gb, \mu_b, e_b, \imath_b)$ is a group for every $b \in \mathrm{B}$ and Gh is a group morphism for every arrow h in B.

These two lemmas have a bunch of nice corollaries for any group object $(g, \mu, e, \imath)$ in a locally small category, e.g:

- For a given μ, e and $\imath$ are unique.
- $\imath \circ \imath = 1$

Group Object Action

We can also define action in this general setting:

- Let $(g, \mu, e, \imath)$ be a group object in C, $m \in$ C. A left action of g on m is any arrow $\theta : g \times m \to m$ such that

$$\begin{array}{ccc} (g \times g) \times m & \longleftrightarrow & g \times (g \times m) \\ \downarrow{\scriptstyle \mu \times 1} & & \downarrow{\scriptstyle 1 \times \theta} \\ g \times m & & g \times m \\ & {\scriptstyle \theta} \searrow \quad \swarrow {\scriptstyle \theta} & \\ & m & \end{array} \qquad \begin{array}{ccc} m & \xrightarrow{(e,1)} & g \times m \\ & {\scriptstyle 1} \searrow & \downarrow{\scriptstyle \theta} \\ & & m \end{array}$$

both commmute. Note that the multiplication arrow is automatically both left and right action.

Graded Lie Groups

- A graded Lie group can be defined as a group object $(\mathcal{G}, \mu, e, \imath)$ in the category gMan$^\infty$.
- The definition via diagrams ensures that $(G, \underline{\mu}, \underline{e}, \underline{\imath})$ is a Lie group.
- The terminal object in gMan$^\infty$ is the trivially graded point manifold $\{*\}$.
- An arrow $\{*\} \to \mathcal{M}$ just corresponds to a choice of a point in M.

Graded Lie Groups

Example

The **graded general linear group** $\mathrm{GL}(\mathbb{R}^{(n_j)})$ for a graded vector space $\mathbb{R}^{(n_j)}$, $\sum_j n_j = n < +\infty$.

- Covered by global graded coordinates $\{x^i{}_j\}_{i,j=1}^n$ of degree $|x^i{}_j| = |e_j| - |e_i|$, where e_i are the standard basis vectors on $\mathbb{R}^{(n_j)}$.
- The underlying smooth manifold is $\times_j \mathrm{GL}(n_j)$.
- Multiplication arrow $\ldots \mu^*(x^i{}_j) := p_2^*(x^k{}_j)\, p_1^*(x^i{}_k)$.
- Inversion arrow ... is a little ugly.
- Can act on $\mathrm{g}\mathbb{R}^{(n_j)}$ via $\theta^*(y^k) = y^j x^k{}_j$, where y^j are standard coordinates on $\mathrm{g}\mathbb{R}^{(n_j)}$.

Left-Invariant Vector Fields

- For $\mathcal{M}, \mathcal{N} \in \mathsf{gMan}^\infty$, $\phi : \mathcal{M} \to \mathcal{N}$, $X \in \Gamma(T\mathcal{M})$, $Y \in \Gamma(T\mathcal{N})$ se say that X and Y are ϕ-related, $X \sim_\phi Y$, iff $X \circ \phi^* = \phi^* \circ Y$.
- We can say that a vector field $X \in \Gamma(T\mathcal{G})$ is left-invariant, iff $1 \otimes X \sim_\mu X$.
- Note that for any $g \in G \iff g : \{*\} \to \mathcal{G}$ we have the isomorphism $L_g : \mathcal{G} \to \mathcal{G}$, $L_g := \mu \circ (g, 1)$. The requirement $(L_g)_* X = X$ for all $g \in G$ is now only neccesary, but not sufficient, for X to be left invariant.
    - An example of this is the Euler vector field $Ef := |f| f$ on $\mathrm{GL}(\mathbb{R}^{(n_j)})$.

Left-Invariant Vector Fields

- Left-invariant vector fields are closed under the graded commutator $\implies$ they form a degree zero graded Lie algebra.
- For any $v \in T_e\mathcal{G}$ one obtains a global left-invariant vector field as $\Gamma^L(T\mathcal{G}) \ni v^L := (1, e)^* \circ 1 \otimes X \circ \mu^*$, for any X such that $X|_e = v$.
- This yields the expected isomorphism $T_e\mathcal{G} \simeq \Gamma^L(T\mathcal{G})$ in gVec. Furthermore, if $(v_i)_{i=1}^n$ is a basis for $T_e\mathcal{G}$, then $(v_i^L)_{i=1}^n$ is a frame for $\Gamma(T\mathcal{G})$.

Infinitesimal Generator

In a *very* similar way we can define fundamental vector fields. Let $\theta : \mathcal{M} \times \mathcal{G} \to \mathcal{M}$ be a right action of a graded Lie group.

- For any left-invariant $X \in \Gamma^L(T\mathcal{G})$ we can find a fundamental vector field

  $$\#X := (1, e)^* \circ (1 \otimes X) \circ \theta^* \in \Gamma(T\mathcal{M}),$$

  which satisfies

  $$1 \otimes X \sim_\theta \#X.$$

  As expected, the infinitesimal generator $\#$ is an injective graded Lie algebra morphism.

The end... for now.

References and Acknowledgements

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