The spinor bundle on loop space

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Overview

1. Tangential structures
2. The spinor bundle construction
3. ... on loop spaces
4. Ongoing work

Tangential structures

A tangential structure on a manifold M is an alteration of the structure group of $GL(M)$ by a homomorphism

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In terms of principal fiber bundles, this is a G-bundle P together with a G-equivariant map $P \to \mathrm{GL}(M)$ which commutes with the projection to M.
In terms of classifying spaces, this is a lift

$$\begin{tikzcd} & \mathrm{B}G \arrow[d, "\mathrm{B}\rho"] \\ M \arrow[ur, dashed] \arrow[r] & \mathrm{BGL}(d). \end{tikzcd}$$

Tangential structures

Important examples arise from the Whitehead-tower

$$\ldots \longrightarrow \mathrm{String}(d) \longrightarrow \mathrm{Spin}(d) \longrightarrow \mathrm{SO}(d) \longrightarrow \mathrm{O}(d).$$

Tangential structures

This yields a diagram

$$M \longrightarrow \mathrm{BO}(d).$$

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$$\begin{tikzcd}
 & \mathrm{BSO}(d) \arrow[d] & \\
M \arrow[ur, dashed, bend left] \arrow[r] & \mathrm{BO}(d) \arrow[r, "w_1"] & \mathrm{B}\mathbb{Z}.
\end{tikzcd}$$

Tangential structures

This yields a diagram

$$
\begin{tikzcd}
 & \mathrm{BSpin}(d) \arrow[d] & \\
 & \mathrm{BSO}(d) \arrow[r, "w_2"] \arrow[d] & \mathrm{B}^2\mathbb{Z} \\
M \arrow[r] \arrow[ur, dashed, bend left] \arrow[uur, dashed, bend left] & \mathrm{BO}(d) \arrow[r, "w_1"] & \mathrm{B}\mathbb{Z}.
\end{tikzcd}
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Tangential structures

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$$\begin{tikzcd}
& \mathrm{BString}(d) \arrow[d] & \\
& \mathrm{BSpin}(d) \arrow[d] \arrow[r, "\frac{1}{2}p_1"] & \mathrm{B}^4\mathbb{Z} \\
& \mathrm{BSO}(d) \arrow[d] \arrow[r, "w_2"] & \mathrm{B}^2\mathbb{Z} \\
M \arrow[r] \arrow[ur, dashed, bend left] \arrow[uur, dashed, bend left] \arrow[uuur, dashed, bend left] & \mathrm{BO}(d) \arrow[r, "w_1"] & \mathrm{B}\mathbb{Z}.
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$$\begin{tikzcd}
\mathrm{SO}(M) \arrow[loop above, "\mathrm{SO}(d)"] \arrow[r] \arrow[d] & \mathrm{GL}(M) \arrow[loop right, "\mathrm{GL}(d)"] \arrow[dl] \\
M &
\end{tikzcd}$$

The spinor bundle construction

Let M be an orientable Riemannian manifold with a spin structure. As M is orientable, the structure group of the frame bundle can be reduced to $\mathrm{SO}(d)$. As we also have a spin structure we can further pass to a $\mathrm{Spin}(d)$-bundle.

$$\begin{tikzcd}
\mathrm{Spin}(M) \arrow[loop left, "\mathrm{Spin}(d)"] \arrow[r] \arrow[dr] & \mathrm{SO}(M) \arrow[loop above, "\mathrm{SO}(d)"] \arrow[r] \arrow[d] & \mathrm{GL}(M) \arrow[loop right, "\mathrm{GL}(d)"] \arrow[dl] \\
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The group $\mathrm{Spin}(d)$ has an interesting unitary representation Δ called the spinor representation. The spinor bundle on M is defined to be the associated vector bundle

$$\mathcal{S}_M := \mathrm{Spin}(M) \times_{\mathrm{Spin}(d)} \Delta.$$

... on loop spaces

We try to do an analogous construction on

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LM has a Riemannian metric given by

$$\tilde{g}(\gamma_1, \gamma_2) = \int_{S^1} g(\gamma_1(\theta), \gamma_2(\theta)) d\theta.$$

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$$\begin{tikzcd}
LSO(M) \arrow[loop above, "LSO(d)"] \arrow[r] \arrow[d] & LGL(M) \arrow[loop right, "LGL(d)"] \arrow[dl] \\
LM &
\end{tikzcd}$$

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$$
\begin{tikzcd}
 & & LSO(d) \arrow[d, phantom, "\circlearrowright"] & & \\
LSpin(d) \arrow[r, phantom, "\circlearrowleft"] & L\mathrm{Spin}(M) \arrow[r] \arrow[dr] & L\mathrm{SO}(M) \arrow[r] \arrow[d] & L\mathrm{GL}(M) \arrow[dl] \arrow[r, phantom, "\circlearrowright"] & L\mathrm{GL}(d) \\
 & & LM & &
\end{tikzcd}
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$$\begin{tikzcd}
L\mathrm{Spin}(M) \arrow[loop left, "L\mathrm{Spin}(d)"] \arrow[r] \arrow[dr] & L\mathrm{SO}(M) \arrow[loop above, "L\mathrm{SO}(d)"] \arrow[r] \arrow[d] & L\mathrm{GL}(M) \arrow[loop right, "L\mathrm{GL}(d)"] \arrow[dl] \\
& LM &
\end{tikzcd}$$

We interpret this as the oriented frame bundle of LM.

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$\mathrm{LSpin}(d)$ has no appropriate unitary representations. However, the so called basic extension

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has one. Thus we need a loop-spin structure, i.e. a lift to a $\widetilde{\mathrm{LSpin}(d)}$-bundle

$$\begin{tikzcd}
\widetilde{\mathrm{LSpin}(d)} \arrow[r, phantom, "\circlearrowright"] & \widetilde{\mathrm{LSpin}(M)} \arrow[rr] \arrow[dr] & & \mathrm{LSpin}(M) \arrow[dl] \arrow[r, phantom, "\circlearrowleft"] & \mathrm{LSpin}(d) \\
& & LM. & &
\end{tikzcd}$$

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The obstruction to the existence of a loop-spin structure is a class

$$\tau_M(\frac{1}{2}p_1(M)) \in H^3(LM, \mathbb{Z}).$$

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In particular string manifolds are loop-spin.

$\widetilde{\mathrm{LSpin}(d)}$ has a unitary representation Δ. The spinor bundle on the loop space is the associated vector bundle

$$\mathcal{S}_{LM} = \widetilde{\mathrm{LSpin}(M)} \times_{\widetilde{\mathrm{LSpin}(d)}} \Delta.$$

Ongoing work

For a spin manifold M one can define a Dirac-operator

$$D : \Gamma(M, \mathcal{S}_M) \to \Gamma(M, \mathcal{S}_M).$$

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$\widetilde{\mathrm{LSpin}}(d)$ carries a S^1-action which decomposes $\Delta = \bigoplus_{\mathbb{Z}} E_n$. We can associate vector bundles V_n to these E_n and obtain another power series

$$\sum_{\mathbb{Z}} [V_n] q^n \in K(M)[[q]].$$

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We want to compare these two.

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We want to look at how this relates to the construction from the previous slide.

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