



The spinor bundle on loop space

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Overview

1. Tangential structures
2. The spinor bundle construction
3. ... on loop spaces
4. Ongoing work

Tangential structures

A tangential structure on a manifold M is an alteration of the structure group of $GL(M)$ by a homomorphism

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In terms of classifying spaces, this is a lift

$$\begin{array}{ccc} & & BG \\ & \nearrow & \downarrow B\rho \\ M & \longrightarrow & BGL(d). \end{array}$$

Tangential structures

Important examples arise from the Whitehead-tower

$$\dots \longrightarrow \text{String}(d) \longrightarrow \text{Spin}(d) \longrightarrow \text{SO}(d) \longrightarrow \text{O}(d).$$

Tangential structures

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$$\begin{array}{ccccc} & & \text{BSO}(d) & & \\ & \nearrow \text{dashed} & \downarrow & & \\ M & \longrightarrow & \text{BO}(d) & \xrightarrow{w_1} & \text{B}\mathbb{Z}. \end{array}$$

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This yields a diagram

$$\begin{array}{ccccc} & & \mathrm{BSpin}(d) & & \\ & \nearrow & \downarrow & & \\ & & \mathrm{BSO}(d) & \xrightarrow{w_2} & \mathrm{B}^2\mathbb{Z} \\ & \nearrow & \downarrow & & \\ M & \longrightarrow & \mathrm{BO}(d) & \xrightarrow{w_1} & \mathrm{B}\mathbb{Z}. \end{array}$$

Tangential structures

This yields a diagram

$$\begin{array}{ccccc} & & \text{BString}(d) & & \\ & \nearrow & \downarrow & & \\ & & \text{BSpin}(d) & \xrightarrow{\frac{1}{2}p_1} & \text{B}^4\mathbb{Z} \\ & \nearrow & \downarrow & & \\ & & \text{BSO}(d) & \xrightarrow{w_2} & \text{B}^2\mathbb{Z} \\ & \nearrow & \downarrow & & \\ M & \xrightarrow{\quad} & \text{BO}(d) & \xrightarrow{w_1} & \text{B}\mathbb{Z}. \end{array}$$

The spinor bundle construction

Let M be an orientable Riemannian manifold with a spin structure.

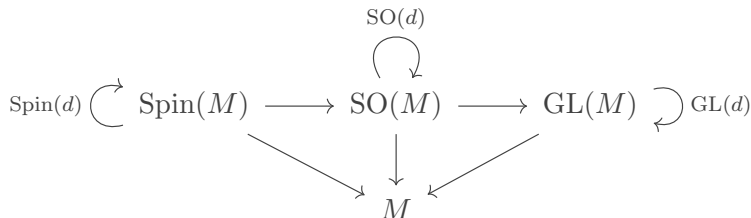
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Let M be an orientable Riemannian manifold with a spin structure. As M is orientable, the structure group of the frame bundle can be reduced to $SO(d)$.

$$\begin{array}{ccc} & SO(d) & \\ \curvearrowright & & \\ SO(M) & \longrightarrow & GL(M) \curvearrowright GL(d) \\ \downarrow & \swarrow & \\ & M & \end{array}$$

The spinor bundle construction

Let M be an orientable Riemannian manifold with a spin structure. As M is orientable, the structure group of the frame bundle can be reduced to $SO(d)$. As we also have a spin structure we can further pass to a $\text{Spin}(d)$ -bundle.



The spinor bundle construction

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The group $\mathrm{Spin}(d)$ has an interesting unitary representation Δ called the spinor representation. The spinor bundle on M is defined to be the associated vector bundle

$$\mathcal{S}_M := \mathrm{Spin}(M) \times_{\mathrm{Spin}(d)} \Delta.$$

... on loop spaces

We try to do an analogous construction on

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LM has a Riemannian metric given by

$$\tilde{g}(\gamma_1, \gamma_2) = \int_{S^1} g(\gamma_1(\theta), \gamma_2(\theta)) d\theta.$$

... on loop spaces

The bundle TLM has fibers $L\mathbb{R}^n$.

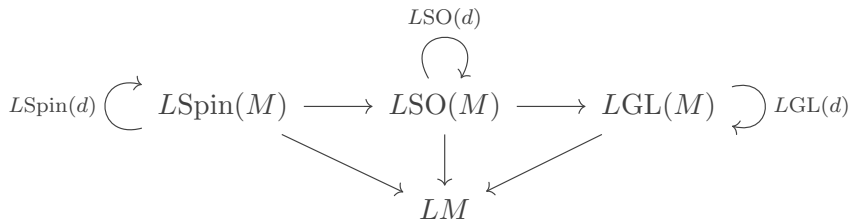
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The bundle TLM has fibers $L\mathbb{R}^n$. We now get a $LSO(d)$ -bundle.

$$\begin{array}{ccccc} & LSO(d) & & & \\ & \curvearrowright & & & \\ LSO(M) & \longrightarrow & LGL(M) & \curvearrowright & LGL(d) \\ \downarrow & & \swarrow & & \\ & & LM & & \end{array}$$

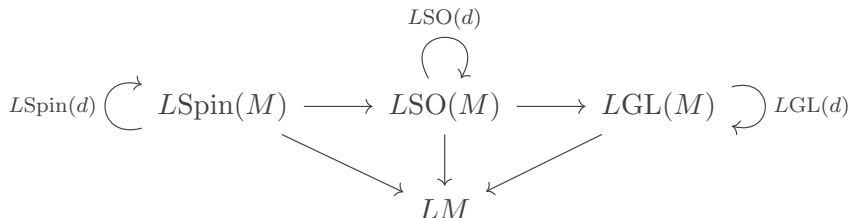
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The bundle TLM has fibers $L\mathbb{R}^n$. We now get a $LSO(d)$ -bundle. However, $LSO(d)$ has two connected components and the reduction to its identity component is an $LSpin(d)$ -bundle.



We interpret this as the oriented frame bundle of LM .

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$\mathrm{LSpin}(d)$ has no appropriate unitary representations. However, the so called basic extension

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has one. Thus we need a loop-spin structure, i.e. a lift to a $\widetilde{\mathrm{LSpin}}(d)$ -bundle

$$\begin{array}{ccccc} \widetilde{\mathrm{LSpin}}(d) & \begin{array}{c} \curvearrowright \\ \rightarrow \end{array} & \widetilde{\mathrm{LSpin}}(M) & \xrightarrow{\quad\quad\quad} & \mathrm{LSpin}(M) & \begin{array}{c} \curvearrowleft \\ \leftarrow \end{array} & \mathrm{LSpin}(d) \\ & & \searrow & & \swarrow & & \\ & & LM. & & & & \end{array}$$

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The obstruction to the existence of a loop-spin structure is a class

$$\tau_M(\frac{1}{2}p_1(M)) \in H^3(LM, \mathbb{Z}).$$

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In particular string manifolds are loop-spin.

$\widetilde{\text{LSpin}}(d)$ has a unitary representation Δ . The spinor bundle on the loop space is the associated vector bundle

$$\mathcal{S}_{LM} = \widetilde{\text{LSpin}}(M) \times_{\widetilde{\text{LSpin}}(d)} \Delta.$$

Ongoing work

For a spin manifold M one can define a Dirac-operator

$$D : \Gamma(M, \mathcal{S}_M) \rightarrow \Gamma(M, \mathcal{S}_M).$$

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$$\sum_{\mathbb{Z}} [V_n] q^n \in K(M)[[q]].$$

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We want to compare these two.

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$\mathrm{LSpin}(d)$ also has a natural $\mathrm{Diff}^+(S^1)$ -action and thus $\widetilde{\mathrm{LSpin}(d)}$ comes with an action of an extension $\widetilde{\mathrm{Diff}^+(S^1)}$.

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We want to look at how this relates to the construction from the previous slide.

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