Bimodule connections for relative Hopf modules over irreducible quantum flag manifolds $\mathcal{O}_q(G/L_S)$

-Joint work with Réamonn Ó Buachalla

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Overview

1. First order differential calculus, connections and bimodule connections

2. Differential calculus over $\mathcal{O}_q(G/L_S)$

3. Main result

First order differential calculus, connections and bimodule connections

First order differential calculus, connection over an algebra B

Definition

A first order differential calculus over an algebra B is a pair $(\Omega^1(\mathrm{B}), \mathrm{d})$, where $\Omega^1(\mathrm{B})$ is a B-bimodule and $\mathrm{d} : \mathrm{B} \to \Omega^1(\mathrm{B})$ is a derivation such that $\Omega^1(\mathrm{B})$ is generated as a left B-module by elements of the form db, for $\mathrm{b} \in \mathrm{B}$.

Definition

For a left B-module $\mathcal{F}$, a left connection on $\mathcal{F}$ is a linear map $\nabla : \mathcal{F} \to \Omega^1(\mathrm{B}) \otimes_{\mathrm{B}} \mathcal{F}$, satisfying the left Leibniz rule $\nabla(\mathrm{bf}) = \mathrm{db} \otimes \mathrm{f} + \mathrm{b}\nabla\mathrm{f}$, for $\mathrm{b} \in \mathrm{B}$, $\mathrm{f} \in \mathcal{F}$.

Bimodule connections

Definition

A left connection ∇ is called bimodule connection on a bimodule $\mathcal{F}$ if there exists a bimodule map $\sigma : \mathcal{F} \otimes_{\mathrm{B}} \Omega^1(\mathrm{B}) \to \Omega^1(\mathrm{B}) \otimes_{\mathrm{B}} \mathcal{F}$ satisfying $\sigma(\mathrm{f} \otimes \mathrm{db}) = \nabla(\mathrm{fb}) - \nabla(\mathrm{f})\mathrm{b}$.

- This bimodule map σ reduces to the flip map if B is commutative.

$$\sigma(\mathrm{f} \otimes \mathrm{db}) = \mathrm{db} \otimes \mathrm{f} \qquad \text{for } \mathrm{f} \in \mathcal{F}, \mathrm{b} \in \mathrm{B}$$

Differential calculus over $\mathcal{O}_q(G/L_S)$

Description of $U_q(\mathfrak{g})$

Let $\mathfrak{g}$ be a complex semi-simple lie algebra with rank r. For $q \in \mathbb{R}$ such that $q \notin \{-1, 0, 1\}$, $q_i := q^{d_i}$, and (a_{ij}) denotes the Cartan matrix of $\mathfrak{g}$.

Then $U_q(\mathfrak{g})$ is defined as an associative algebra with unit 1 with generators E_i, F_i, K_i and K_i^{-1}, for $i = 1, 2, .., r$ subject to the relations:

$$K_i K_i^{-1} = K_i^{-1} K_i = 1$$

$$K_i E_j K_i^{-1} = q_i^{a_{ij}} E_j, \qquad K_i F_j K_i^{-1} = q_i^{-a_{ij}} F_j$$

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$$

and the quantum Serre relations, which we omit.

Quantum Flag Manifold $\mathcal{O}_q(G/L_S)$

Definition

For $\{\alpha_i\}_{i\in S}$, a subset of simple roots, we consider a Hopf *-subalgebra,

$$U_q(\mathfrak{l}_S) := \langle E_i, F_i, K_j^{\pm 1} : i \in S, j = 1, 2, .., r \rangle$$

.

Definition (Quantum flag manifold associated to S)

$$\mathcal{O}_q(G/L_S) := {}^{U_q(\mathfrak{l}_S)}\mathcal{O}_q(\mathrm{G}) = \{a \in \mathcal{O}_q(G) : X \triangleright a = \epsilon(X)a, \ \ X \in U_q(\mathfrak{l}_S)\}$$

Definition

A quantum flag manifold $\mathcal{O}_q(G/L_S)$ is irreducible if the defining subset of simple roots is of the form $S = \{1, 2, .., r\} \setminus \{s\}$, where α_s has coefficient 1 in the expansion of the highest root of $\mathfrak{g}$.

Construction of $\mathcal{O}_q(L_S)$

- The hopf $*$ algebra embedding $U_q(\mathfrak{l}_S) \hookrightarrow U_q(\mathfrak{g})$ induces dual inclusion map $U_q(\mathfrak{g})^\circ \hookrightarrow U_q(\mathfrak{l}_S)^\circ$
- The restriction of dual inclusion map on $\mathcal{O}_q(G)$ gives $\pi_S : \mathcal{O}_q(G) \to U_q(\mathfrak{l}_S)^\circ$, and

$$\pi_S(\mathcal{O}_q(G)) =: \mathcal{O}_q(L_S)$$

.

Relative Hopf module

- We consider ${}^{\mathcal{O}_q(G)}_{\mathcal{O}_q(G/L_S)}\mathrm{Mod}_0$ be the category of left $\mathcal{O}_q(G/L_S)$-module $\mathcal{F}$ endowed with the left $\mathcal{O}_q(G)$- coaction Δ_L such that $\mathcal{O}_q(G/L_S)^+\mathcal{F} = \mathcal{F}\mathcal{O}_q(G/L_S)^+$

$$\Delta_L(bf) = b_{(1)}f_{(-1)} \otimes b_{(2)}f_{(0)}$$

for $b \in \mathcal{O}_q(G/L_S)$, $f \in \mathcal{F}$.
- ${}^{\mathcal{O}_q(L_S)}\mathrm{Mod}_0$ be the category of left $\mathcal{O}_q(L_S)$- comodules.
- If $\mathcal{F} \in {}^{\mathcal{O}_q(G)}_{\mathcal{O}_q(G/L_S)}\mathrm{Mod}_0$, then $\mathcal{F}/\mathcal{O}_q(G/L_S)^+\mathcal{F} \in {}^{\mathcal{O}_q(L_S)}\mathrm{Mod}_0$, where $\mathcal{O}_q(G/L_S)^+ = \ker(\epsilon)$

$$\Delta_{\mathrm{L}}^{\mathcal{O}_q(L_S)}([f]) = \pi(f_{(-1)}) \otimes [f_{(0)}]$$

for $f \in \mathcal{F}$.

Where $\pi : \mathcal{O}_q(G) \to \mathcal{O}_q(L_S)$ is a surjection and $[f]$ be the coset of f in $\mathcal{F}/\mathcal{O}_q(G/L_S)^+\mathcal{F}$.

Co-tensor product

Definition

We consider $\mathrm{V} \in {}^{\mathcal{O}_q(L_S)}\mathrm{Mod}_0$, then co-tensor product between $\mathcal{O}_q(G)$ and V is defined as

$$\mathcal{O}_q(G)\Box_{\mathcal{O}_q(L_S)}\mathrm{V} := \ker(\Delta_{\mathrm{R}} \otimes \mathrm{id} - \mathrm{id} \otimes \Delta_{\mathrm{L}}^{\mathcal{O}_q(L_S)} : \mathcal{O}_q(G) \otimes \mathrm{V} \to \mathcal{O}_q(G) \otimes \mathcal{O}_q(L_S) \otimes \mathrm{V})$$

- $\mathcal{O}_q(G)\Box_{\mathcal{O}_q(L_S)}\mathrm{V} \in {}^{\mathcal{O}_q(G)}_{\mathcal{O}_q(G/L_S)}\mathrm{Mod}_0$, as left $\mathcal{O}_q(G)$- coaction is given by

$$\Delta_L(\sum_i a_i \otimes v_i) = \sum_i (a_i)_{(1)} \otimes (a_i)_{(2)} \otimes v_{(i)}$$

- **Takeuchi equivarience**
  $\Phi : {}^{\mathcal{O}_q(G)}_{\mathcal{O}_q(G/L_S)}\mathrm{Mod}_0 \to {}^{\mathcal{O}_q(L_S)}\mathrm{Mod}_0$ given by

$$\Phi(\mathcal{F}) = \mathcal{F}/\mathcal{O}_q(G/L_S)^+\mathcal{F}$$

  $\Psi : {}^{\mathcal{O}_q(L_S)}\mathrm{Mod}_0 \to {}^{\mathcal{O}_q(G)}_{\mathcal{O}_q(G/L_S)}\mathrm{Mod}_0$ given by

$$\Psi(\mathrm{V}) = \mathcal{O}_q(G)\square_{\mathcal{O}_q(L_S)}\mathrm{V}$$

Heckenberger- Kolb Calculus on $\mathcal{O}_q(G/L_S)$

Theorem (Heckenberger, Kolb '06)

For every irreducible quantum flag manifold $\mathcal{O}_q(G/L_S)$, there exist precisely two irreducible $\mathcal{O}_q(G)$ left covarient first order differential calculi for $\Omega^1(\mathcal{O}_q(G/L_S))$:

$$\Omega^1(\mathcal{O}_q(G/L_S)) := \Omega^{(1,0)}(\mathcal{O}_q(G/L_S)) \oplus \Omega^{(0,1)}(\mathcal{O}_q(G/L_S))$$

- Classically, for $q = 1$,

$$\Omega^1((G/L_S)) := \Omega^{(1,0)}((G/L_S)) \oplus \Omega^{(0,1)}((G/L_S))$$

- $\Omega^{(1,0)}(\mathcal{O}_q(G/L_S)) = \mathcal{O}_q(G)\square_{\mathcal{O}_q(L_S)}\mathrm{V}^{(1,0)} \in {}^{\mathcal{O}_q(G)}_{\mathcal{O}_q(G/L_S)}\mathrm{Mod}_0$
- $\Omega^{(0,1)}(\mathcal{O}_q(G/L_S)) = \mathcal{O}_q(G)\square_{\mathcal{O}_q(L_S)}\mathrm{V}^{(0,1)} \in {}^{\mathcal{O}_q(G)}_{\mathcal{O}_q(G/L_S)}\mathrm{Mod}_0$

Differential Calculus over $\mathcal{O}_q(G/L_s)$

We have first order differential calculi $\Omega^{(0,1)}(\mathcal{O}_q(G/L_S))$ and $\Omega^{(1,0)}(\mathcal{O}_q(G/L_S))$. Given by

- $\mathrm{d}^{(0,1)} : \mathcal{O}_q(G/L_S) \to \Omega^{(0,1)}(\mathcal{O}_q(G/L_S))$, defined as

$$\mathrm{d}^{(0,1)}(a) = \sum_{X_i \in \mathcal{T}^{(0,1)}_{\mathcal{O}_q(G/L_S)}} (X_i \triangleright a) \otimes e_i$$

- $\mathrm{d}^{(1,0)} : \mathcal{O}_q(G/L_S) \to \Omega^{(1,0)}(\mathcal{O}_q(G/L_S))$, defined as

$$\mathrm{d}^{(1,0)}(a) = \sum_{Y_i \in \mathcal{T}^{(1,0)}_{\mathcal{O}_q(G/L_S)}} (Y_i \triangleright a) \otimes f_i$$

Main result

Some Motivating results

Theorem (Diaz Garcia, Krutov, Ó Buachalla, Somberg, Strung'20)

Let $\mathcal{O}_q(G/L_S)$ be irreducible and endowed with its Heckenberger-Kolb calculus, then for every $\mathcal{F} \in {}^{\mathcal{O}_q(G)}_{\mathcal{O}_q(G/L_S)}\mathrm{Mod}_0$, admits a left $\mathcal{O}_q(G)$)-covariant connection.

Theorem (Carotenuto, Ó Buachalla '22)

Let $(\mathrm{A}, \Omega^1(\mathrm{A}))$ be a quantum principal bundle, endowed with a strong principal connection, then for relative hopf module $\mathcal{F} \in {}^{\mathrm{A}}_{\mathrm{B}}\mathrm{Mod}_0$, it admits a bimodule connection.

Main result

Theorem (B., Ó Buachalla '24)

Let $\mathcal{O}_q(G/L_S)$ be irreducible and endowed with its Heckenberger-Kolb calculus, then every left-covariant connection on $\mathcal{F} \in {}^{\mathcal{O}_q(G)}_{\mathcal{O}_q(G/L_S)}\mathrm{Mod}_0$ is a bimodule connection.

Thank You!