Convex orderings on quantum root vectors and differential calculi

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January 20, 2025

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1. Drinfeld–Jimbo quantum groups and quantum flag manifolds

- Let $(a_{ij})_{ij}$ denote the Cartan matrix of $\mathfrak{g}$.

  Fix $q \in \mathbb{R} \setminus \{\pm 1, 0\}$.

  The **quantised enveloping algebra** $U_q(\mathfrak{g})$ is generated by

$$E_i, F_i, K_i, K_i^{-1}, \qquad i = 1, \ldots, r;$$

  subject to the relations

$$K_i E_j = q_i^{a_{ij}} E_j K_i, \qquad K_i F_j = q_i^{-a_{ij}} F_j K_i, \qquad K_i K_j = K_j K_i,$$

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- Hopf algebra, with coproduct defined on generators

$$\Delta(K_i) = K_i \otimes K_i, \quad \Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i,$$
$$\Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i$$

- We have a pairing of Hopf algebras

$$U_q(\mathfrak{g}) \otimes \mathcal{O}_q(G) \to \mathbb{C}$$

- Which induces left and right actions

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- For S a subset of simple roots, consider the Hopf subalgebra of $U_q(\mathfrak{g})$ given by

$$U_q(\mathfrak{l}_S) := \left\langle K_i, E_j, F_j \mid i = 1, \ldots, l; j \in S \right\rangle.$$

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$$\mathcal{O}_q\big(G/L_S\big) := {}^{U_q(\mathfrak{l}_S)}\mathcal{O}_q(G),$$

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with respect to the natural left $U_q(\mathfrak{g})$-module structure on $\mathcal{O}_q(G)$.
- We call $\mathcal{O}_q(G/L_S)$ the **quantum flag manifold associated to S**.
- This is an example of quantum homogeneous space and a q-deformation of classical flag manifolds.

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- (Ω^1, d) **a first-order differential calculus**(fodc) over an algebra B
  - Ω^1 is a B-bimodule
  - $\mathrm{d} : B \to \Omega^1$ is a derivation (**the exterior derivative**).
  - Ω^1 is generated as a left B-module by those elements of the form $\mathrm{d}b$, for $b \in B$.

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- Let B be a left $A-$comodule, we say that a (first order) differential calculus is A covariant if d is an $A-$comodule map.

- There exist no bicovariant calculus on $\mathcal{O}_q(G)$ of classical dimension.
- However most of our troubles arise from the action of K_i on $\mathcal{O}_q(G)$ and we can avoid them by looking at the quantum flag manifolds $\mathcal{O}_q(G/L_S)$ (We wipe out the worst noncommutativity that comes from the action of the K's!)

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- Differential calculi for **irreducible** quantum flag manifolds. (Heckenberger and Kolb 2006).
- Differential calculi for full flag manifolds in type A (Ó Buachalla and Somberg 2022, A.C.,Ó Buachalla and Razzaq 2022).

- The optimal strategy is to define a calculus at the level of $A = \mathcal{O}_q(G)$ and then restrict it to the quantum full flag manifold B. Here we have:

$$\Omega^1(A) = A \otimes \frac{A^+}{I} \quad A^+ = \ker(\varepsilon)$$

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$$I = \ker(T) := \bigcap_{X \in T} \ker(X).$$

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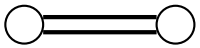
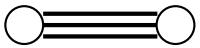
3. Lusztig root vectors and fodc

- $\mathcal{B}_{\mathfrak{g}}$ the **braid group of** $\mathfrak{g}$ is the group generated by $s_i \quad 1 \leq i \leq l$ with relations

$s_i s_j = s_j s_i$	
$s_i s_j s_i = s_j s_i s_j$	
$s_j s_i s_j s_i = s_i s_j s_i s_j$	
$s_j s_i s_j s_i s_j s_i = s_i s_j s_i s_j s_i s_j$	

- $\mathcal{W}_{\mathfrak{g}}$, the **Weyl group of** $\mathfrak{g}$, is generated by w_i with the same relations and additionally

$$w_i^2 = 1$$

Theorem

To every generator s_i $i = 1, \dots, l$ of $\mathcal{B}_{\mathfrak{g}}$ there corresponds an algebra automorphism T_i of $U_q(\mathfrak{g})$ which acts on the generators as

$$T_i(K_j) = K_j K_i^{-a_{ij}}, \quad T_i(E_i) = -F_i K_i, \quad T_i(F_i) = -K_i^{-1} E_i,$$

$$T_i(E_j) = \sum_{t=0}^{-a_{ij}} (-1)^{t-a_{ij}} q_i^{-t} (E_i)^{(-a_{ij}-t)} E_j (E_i)^{(t)}, \quad i \neq j,$$

$$T_i(F_j) = \sum_{t=0}^{-a_{ij}} (-1)^{t-a_{ij}} q_i^{t} (F_i)^{(t)} F_j (F_i)^{(-a_{ij}-t)}, \quad i \neq j,$$

where

$$(X)^{(n)} := X^n / [n]_{q_i}!$$

- In the universal enveloping $U(\mathfrak{g})$ we have root vectors for every root of $\mathfrak{g}$.
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- Let $w = w_{i_1} \dots w_{i_n}$ be a reduced decomposition of the longest element of $\mathcal{W}_{\mathfrak{g}}$, denote by α_i the simple roots of $\mathfrak{g}$. The list

$$\beta_1 = \alpha_{i_1} \quad \beta_k = w_{i_1} \dots w_{i_{k-1}}(\alpha_{i_k})$$

exhausts all the positive roots of $\mathfrak{g}$.

Definition

The elements

$$E_{\beta_k} = T_{i_1} \dots T_{i_{k-1}}(E_{i_k}), \quad F_{\beta_k} = T_{i_1} \dots T_{i_{k-1}}(F_{i_k})$$

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Example

For $\mathfrak{g} = \mathfrak{sl}_3$*, we have only two reduced decompositions of the longest element of the Weyl group:*

$$w_1 w_2 w_1, \quad w_2 w_1 w_2$$

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Correspondingly we have two lists of positive root vectors.

E_{i_1}	E_1	E_2
$T_{i_1}(E_{i_2})$	$[E_1, E_2]_{q^{-1}}$	$[E_2, E_1]_{q^{-1}}$
$T_{i_1} T_{i_2}(E_{i_3})$	E_2	E_1

- For each $\mathfrak{g}$ we can look for a reduced decomposition $w_0 = w_{i_1} \dots w_{i_n}$ such that the corresponding $\{E_{\beta_i}\}_{i=1}^{n}$ span a quantum tangent space in $U_q(\mathfrak{g})$.

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Non-Example

Let $\mathfrak{g} = \mathfrak{sl}_4$, the positive root vectors corresponding to the decomposition

$$w_0 = w_1 w_2 w_3 w_2 w_1 w_2$$

do not span a quantum tangent space, since one has

$$\Delta E_{\beta_3} = E_{\beta_3} \otimes K_1 K_2 K_3 + \nu E_{\beta_4} E_{\beta_6} \otimes E_{\beta_1} K_1 K_3 + 1 \otimes E_{\beta_3}.$$

Example (R. Ó Buachalla–P. Somberg)

Let $\mathfrak{g} = \mathfrak{sl}_{n+1}$, *then the two reduced decompositions*

$$w_1 w_2 \dots w_n w_1 \dots w_{n-1} w_1 \dots w_1 w_2 w_1 \quad \textit{and}$$

$$w_n w_{n-1} \dots w_1 w_n \dots w_2 w_n \dots w_n w_{n-1} w_n$$

give rise to two (isomorphic) tangent spaces on $U_q(\mathfrak{g})$ *spanned by their respective* $\{E_\beta\}$.

$$E_{i,i+k} := E_{\alpha_i+\alpha_{i+1}\dots\alpha_{i+k}} = [E_i, [E_{i+1}, \dots [E_{i+k-1}, E_{i+k}]_{q^\pm}]_{q^\pm} \dots]_{q^\pm}$$

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- *These tangent spaces project to the quantum flag manifolds and the corresponding maximal prolongations of their fodc have classical dimension*

5. Convex orderings and differential calculi

- Denote by Δ^+ the set of positive roots of $\mathfrak{g}$.
- We say that a total ordering $\leq$ on Δ^+ is a **convex ordering** when the following holds:

$$\text{When } \beta,\ \beta',\ \beta+\beta' \in \Delta^+ \Rightarrow \beta \leq \beta+\beta' \leq \beta' \text{ or } \beta' \leq \beta+\beta' \leq \beta$$

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- An ordering on the set of positive roots induces an ordering on the set $\{E_\beta\}$ via $\beta \leq \beta' \Leftrightarrow E_\beta \leq E_{\beta'}$

- Let $\{E_{\beta_k}, F_{\beta_k}\}$ be the set of root vectors of $U_q(\mathfrak{g})$ for a given reduced decomposition of the longest element of $\mathcal{W}_{\mathfrak{g}}$.

- Let us consider partitions

$$\beta_k = \beta'_{i_1} + \cdots + \beta'_{i_s} + \beta''_{j_1} + \cdots + \beta''_{j_r} =: \beta_I + \beta_J$$

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Theorem (A.C., P. Papi)

The coproduct of E_{β_k} and F_{β_k} reads respectively

$$\Delta(E_{\beta_k}) = \sum_{I,J} f(I,J) E_{\beta''_{j_1}} \ldots E_{\beta''_{j_r}} \otimes E_{\beta'_{i_1}} \ldots E_{\beta'_{j_s}} K_{\beta''}.$$

$$\Delta(F_{\beta_k}) = \sum_{I,J} g(I,J) F_{\beta'_{i_1}} \ldots F_{\beta'_{i_r}} K^{-1}_{\beta''} \otimes F_{\beta''_{i_1}} \ldots F_{\beta''_{j_s}}.$$

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- We denote by $\mathbf{i} = i_1 \dots i_n$ our given reduced expression of w_0
- For every F_{β_k} there exists a reduced expression $\mathbf{j} = j_1 \dots j_r$ with $r \leq n$ such that:

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- The terms appearing as left tensor factor are given by $\{F_{j_1}, T_{j_1}(F_{j_2}) \dots\}$.
  We call this elements the **prefix root vectors** of F_{β_k} and denote them by $\{\tilde{F}^k_{\beta_i}\}$.

- We can now describe a quantum tangent space in a completely combinatorial way!
- Let $\overline{w}_k$ be the minimum of the set $\{v \leq s_{i_1} \dots s_{i_k} \mid \beta_k \in N(v)\}$. Fix a reduced decomposition $\overline{w}_k = s^k_{j_1} \dots s^k_{j_r}$.

Conjecture

The quantum root vectors $\{F_{\beta_k}\}$ associated to $\mathbf{i}$ *span a quantum tangent space iff for every k the following hold*

1. $\{\tilde{F}^k_{\beta_i}\} = \{F_{\beta_i}\}$.
2. *If $s^k_{j_i} s^k_{j_{i+1}} = s^k_{j_{i+1}} s^k_{j_i}$ then $\beta^{\mathbf{j}}_i + \beta^{\mathbf{j}}_{i+1} \not\leq \beta_k$ in the usual partial order of positive roots.*
3. *$\overline{w}_k$ does not contain braid relations.*
4. *If $\beta^{\mathbf{j}}_i$ is contained in the support of β_k with multiplicity greater than 1 then $\gamma = \beta_k - \beta^{\mathbf{j}}_i$ cannot be decomposed as the sum of positive roots that follow β_k in the convex ordering induced by* $\mathbf{i}$.

Thank you!