Winter School Geometry and Physics – Srni 2025

Curvature of quaternionic skew-Hermitian manifolds and bundle constructions

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Almost hypercomplex/quaternionic skew-Hermitian structures

(C.-Gregorovič-Winther – 2022)

• Almost hypercomplex geometries of skew-Hermitian type are geometric structures defined on $4n$-dimensional manifolds M $(n > 1)$. They corresponds to pairs (H, ω), where H is an almost hypercomplex structure and ω is an H-Hermitian almost symplectic form on M.
Equivalently, they correspond to reductions of the frame bundle to the structure group $G = \mathsf{SO}^*(2n)$,

$$\mathsf{SO}^*(2n) = \mathsf{GL}(n, \mathbb{H}) \cap \mathsf{Sp}(4n, \mathbb{R}) \subset \mathsf{GL}(n, \mathbb{H}) \subset \mathsf{GL}(4n, \mathbb{R}) .$$

➘ Then, the triple (M, H, ω) is called an almost hypercomplex skew-Hermitian manifold.

• Almost quaternionic geometries of skew-Hermitian type are geometric structures defined on $4n$-dimensional manifold M $(n > 1)$. They correspond to pairs (Q, ω), where $Q \subset \mathsf{End}(TM)$ is an almost quaternionic structure and ω is a Q-Hermitian almost symplectic form on M.
Equivalently, they correspond to reductions of the frame bundle to the structure group $G = \mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$,

$$\mathsf{SO}^*(2n)\,\mathsf{Sp}(1) := \mathsf{SO}^*(2n) \times_{\mathbb{Z}_2} \mathsf{Sp}(1) = (\mathsf{SO}^*(2n) \times \mathsf{Sp}(1))/\mathbb{Z}_2 .$$

➘ Then, the triple (M, Q, ω) is called an almost quaternionic skew-Hermitian manifold.

Let (M, Q, ω) be an almost quaternionic skew-Hermitian manifold and let $\{I, J, K\}$ be a local admissible basis of the almost quaternionic structure $Q \subset \mathrm{End}(TM)$. We can introduce a globally defined symmetric 4-tensor

$$\Phi := g_I \odot g_I + g_J \odot g_J + g_K \odot g_K \in \Gamma(S^4 T^* M)$$

where

$$g_I := \omega(\cdot\,, I\cdot), \quad g_J := \omega(\cdot\,, J\cdot), \quad g_K := \omega(\cdot\,, K\cdot)$$

❝ The metrics g_I, g_J, g_K provide the analog of the three local 2-forms $\omega_I, \omega_J, \omega_K$ in almost hypercomplex/quaternionic Hermitian geometries.

❝ Φ is the analog of the 4-form Ω induced by $\omega_I, \omega_J, \omega_K$ in almost quaternionic Hermitian geometries: $\Omega = \Sigma_{A=I,J,K}\, \omega_A \wedge \omega_A$.

Adapted connections to $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-structures

An almost qs-H structure (Q,ω) induces a volume form $\omega^{2n} = \mathsf{vol}$. There is a unique minimal $\mathsf{SL}(n,\mathbb{H})\,\mathsf{Sp}(1)$-connection $\nabla^{Q,\mathsf{vol}}$, called the **unimodular Oproiu connection**.

Thm. (C.-Gregorovič-Winther – 2022)
Let (M,ω,Q) be an almost qs-H manifold. Let A^{vol} be the $(1,2)$-tensor field on M defined by

$$\omega(A^{\mathsf{vol}}(X,Y),Z) = \frac{1}{2}(\nabla^{Q,\mathsf{vol}}_X\omega)(Y,Z)\,, \quad \forall\ X,Y,Z \in \Gamma(TM)\,.$$

Then, the connection $\nabla^{Q,\omega} = \nabla^{Q,\mathsf{vol}} + A^{\mathsf{vol}}$ is an almost quaternionic skew-Hermitian connection.

Torsion: $T^{Q,\omega} = T^Q + \delta(A^{\mathsf{vol}})$, where T^Q is the torsion of $\nabla^{Q,\mathsf{vol}}$. The adapted connection $\nabla^{Q,\omega}$ is **torsion-free** if and only if

$$T^Q = 0\,, \quad \text{and} \quad \nabla^{Q,\mathsf{vol}}\omega = 0\,.$$

In other words, $T^{Q,\omega} = 0$ if and only if Q is 1-integrable and $\nabla^{Q,\omega} = \nabla^{Q,\mathsf{vol}}$.

Torsion-free examples

Thm. (C-Gregorovič-Winther – 2022)

The symmetric spaces

$$\mathsf{SO}^*(2n+2)/\,\mathsf{SO}^*(2n)\,\mathsf{U}(1)\,, \quad \mathsf{SU}(2+p,q)/(\mathsf{SU}(2)\,\mathsf{SU}(p,q)\,\mathsf{U}(1))\,, \quad \mathsf{SL}(n+1,\mathbb{H})/(\mathsf{GL}(1,\mathbb{H})\,\mathsf{SL}(n,\mathbb{H}))\,,$$

the latter two being pseudo-Wolf spaces, are the only (up to covering) symmetric spaces K/L with K semisimple, admitting an invariant torsion-free quaternionic skew-Hermitian structure (Q,ω).

•. In this case $\nabla^{Q,\omega}$ coincides with the corresponding canonical connection ∇^0 on K/L.

Key results from holonomy theory.

➟ **Observation:** Let $\mathfrak{g} \subset \mathrm{End}(\mathscr{V})$ be a semisimple and irreducible subalgebra, where $\mathscr{V}$ is some finite-dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$.

• Assume that there is a non-degenerate 2-form Ω on $\mathscr{V}$ such that $\mathfrak{g} \subset \mathfrak{sp}(\mathscr{V}, \Omega)$, where $\mathfrak{sp}(\mathscr{V}, \Omega)$ is the Lie algebra of linear symplectomorphisms of $(\mathscr{V}, \Omega)$.

• By semi-simplicity, there exists a $\mathfrak{g}$-equivariant map

$$\circ : S^2\mathscr{V}^* \cong S^2\mathscr{V} \longrightarrow \mathfrak{g}\,, \quad \circ(x \odot y) = x \circ y\,.$$

• Suppose that for any $A \in \mathfrak{g}$ and some non-zero constant $\kappa \in \mathbb{R}$, the map

$$R_A : \Lambda^2\mathscr{V} \longrightarrow \mathfrak{g}\,, \quad R_A(x, y) := \kappa\, \Omega(x, y)A + x \circ Ay - y \circ Ax\,, \quad x, y \in \mathscr{V}\,,$$

lies in $\mathscr{K}(\mathfrak{g})$.

$$\mathscr{K}(\mathfrak{g}) = \{R \in \bigwedge^2 \mathscr{V}^* \otimes \mathfrak{g} : \mathfrak{S}_{x,y,z} R(x, y)z = 0\,, \text{ for all } x, y, z \in \mathscr{V}\}\,,$$

• Then, the assignment

$$\mathfrak{g} \longrightarrow \mathscr{K}(\mathfrak{g})\,, \quad A \longmapsto R_A$$

is an isomorphism and $\mathfrak{g}$ is an irreducible (symplectic) Berger algebra.

Prop. (Schwachhöfer, Merkulov-Schwachhöfer 1999) For $n \geqslant 2$, the Lie algebra $\mathfrak{g} = \mathfrak{so}^*(2n) \oplus \mathfrak{sp}(1)$ is a (non-symmetric) irreducible Berger algebra, and

$$\mathscr{K}(\mathfrak{g}) \cong \mathfrak{g}\,.$$

- Fix a quaternionic skew-Hermitian manifold (M, Q, ω) endowed with the torsion-free adapted connection $\nabla := \nabla^{Q,\omega}$. We have $\Omega = \omega$ at any point on M and $\mathfrak{g} = \mathfrak{so}^*(2n) \oplus \mathfrak{sp}(1)$.

Prop. (C.-Cortés-Gregorovič 2024) Let $\circ : S^2\mathscr{V} \longrightarrow \mathfrak{g}$ be a $\mathfrak{g}$-equivariant map such that R_A defined above lies in $\mathscr{K}(\mathfrak{g})$ for all $A \in \mathfrak{g}$. Then $\circ$ decomposes as

$$(x \circ y) = c_1 \cdot (x \circ y)_{\mathfrak{so}^*(2n)} + c_2 \cdot (x \circ y)_{\mathfrak{sp}(1)}\,, \quad \forall\, x, y \in \mathscr{V}\,,$$

where the $\mathfrak{so}^*(2n)$-part $(x \circ y)_{\mathfrak{so}^*(2n)}$ of $x \circ y$ (respectively, the $\mathfrak{sp}(1)$-part $(x \circ y)_{\mathfrak{sp}(1)}$) is given by

$$\begin{aligned} (x \circ y)_{\mathfrak{so}^*(2n)} &= \frac{1}{4}\Big(\omega(x,-)y - \sum_{a=1}^{3} g_a(x,-)J_a y + \omega(y,-)x - \sum_{a=1}^{3} g_a(y,-)J_a x\Big) \in \mathfrak{so}^*(2n)\,, \\ (x \circ y)_{\mathfrak{sp}(1)} &= -\frac{1}{2n}\sum_{a=1}^{3} g_a(x,y)J_a \in \mathfrak{sp}(1)\,, \end{aligned}$$

and $c_1 = 2\kappa \neq 0$, $c_2 = (nc_1)/2 = n\kappa \neq 0$. In particular, the tensors on the right-hand side are independent of the admissible basis $\{J_a\}$ for Q.

The expression of the curvature

Corol. (1) For any $x, y, z \in \mathscr{V} = [\mathsf{E}\,\mathsf{H}] \cong T_m M$, $A \in \mathfrak{so}^*(2n) \oplus \mathfrak{sp}(1)$ and some non-zero $\kappa \in \mathbb{R}$ we have

$$\begin{aligned} R^{Q,\omega}_A(x,y)z \quad = \quad & \kappa\,\omega(x,y)Az + \frac{\kappa}{2}\Big(\omega(x,z)Ay - \sum_{a=1}^{3} g_a(x,z)J_aAy + \omega(Ay,z)x - \sum_{a=1}^{3} g_a(Ay,z)J_ax\Big) \\ & -\frac{\kappa}{2}\Big(\omega(y,z)Ax - \sum_{a=1}^{3} g_a(y,z)J_aAx + \omega(Ax,z)y - \sum_{a=1}^{3} g_a(Ax,z)J_ay\Big) \\ & -\frac{\kappa}{2}\sum_{a=1}^{3}\big(g_a(x,Ay) - g_a(y,Ax)\big)J_az\,. \end{aligned}$$

(2) For any (torsion-free) qs-H manifold (M, Q, ω) we can find a section A of the adjoint bundle $\mathfrak{g}_{\mathscr{Q}} = \mathscr{Q} \times_G \mathfrak{g} \subset T^*M \otimes TM$, such that the curvature $R^{Q,\omega} \in \Gamma(\wedge^2 T^*M \otimes \mathfrak{g}_{\mathscr{Q}})$ of $\nabla^{Q,\omega}$ corresponds to the section $R^{Q,\omega}_A$ of the bundle over M with fiber $\mathscr{K}(\mathfrak{g})$.
(3) The Ricci tensor $\mathsf{Ric}^{Q,\omega}_A(y,z) := \mathsf{Tr}\,\{x \longmapsto R^{Q,\omega}_A(x,y)z\}$ associated to $R^{Q,\omega}_A$ is given by

$$\mathsf{Ric}^{Q,\omega}_A(y,z) = (2n+1)\kappa\,\omega(Ay,z) + 2n\kappa\sum_a g_a(y,z)\,\mathsf{Tr}(J_aA) - \kappa\sum_a \omega(J_aAJ_ay,z)$$

for any $y, z \in \mathscr{V} = [\mathsf{E}\,\mathsf{H}] \cong T_m M$ and $A \in \mathfrak{so}^*(2n) \oplus \mathfrak{sp}(1)$.

Thm. (C.-Cortés-Gregorovič 2024) A $4n$-dimensional quaternionic skew-Hermitian manifold (M, Q, ω) with non-degenerate Q-Hermitian Ricci tensor is a quaternionic Kähler locally symmetric space. In particular, if M is simply connected and complete, then (M, Q, ω) is one of the spaces

$$\mathsf{SU}(2+p,q)/(\mathsf{SU}(2)\,\mathsf{SU}(p,q)\,\mathsf{U}(1))\,, \qquad \mathsf{SL}(n+1,\mathbb{H})/(\mathsf{GL}(1,\mathbb{H})\,\mathsf{SL}(n,\mathbb{H}))\,.$$

In this case we have

$$\mathsf{Ric}^{Q,\omega}(X,Y) = 2(n+2)\kappa\,\omega(AX,Y)$$

for any $X, Y \in \Gamma(TM)$, for some $A \in \mathfrak{so}^*(2n)$ and some non-zero constant κ.

Exm.

$$M = K/L = \mathsf{SU}(2+p,q)/(\mathsf{SU}(2)\,\mathsf{SU}(p,q)\,\mathsf{U}(1))$$

- $\mathfrak{k} = \mathfrak{l} \oplus \mathfrak{m}$ symmetric reductive decomposition;
- $\chi(L) \cap \mathsf{Sp}(1) = \mathsf{Sp}(1) \implies K$-invariant quaternionic structure Q on M;
- $g =$ restriction of Killing form on $\mathfrak{m}$ induces a K-invariant quaternionic Kähler metric;
- Let I be the K-invariant complex structure on M, induced by the $\mathsf{U}(1)$-component. Then $I \notin \Gamma(Q)$;

$$A = -\frac{c}{2(n+2)\kappa} I\,, \quad c = \frac{\mathrm{Scal}^g}{4n}\,.$$

The Swann bundle $\hat{M} \to M$ over (M, Q, ω)

• Fix a quaternionic skew-Hermitian manifold $(M^{4n}, Q, \omega, \nabla = \nabla^{Q,\omega} = \nabla^{Q,\mathrm{vol}})$ (torsion-free) with $n > 1$, and let $\pi : \mathcal{Q} \to M$ be the reduced frame bundle
• We consider the **fiber product bundle** over M

$$\hat{M} = S_0 \times_M S = \{(m, (v, s)) \in M \times (S_0 \times S) : m = \pi_{S_0}(v) = \pi_S(s)\} \cong \Delta^*(S_0 \times S),$$

where $\triangle : M \to M \times M$ is the diagonal map and

- $\pi_S : S \to M$ is the principal $\mathsf{SO}(3)$-bundle of admissible frames of the quaternionic structure Q. This satisfies $S \cong \mathcal{Q} / \mathsf{SO}^*(2n)$

- $\pi_{S_0} : S_0 \to M$ is the principal $\mathbb{R}_+$-bundle of positive densities, $S_0 = (\wedge^{4n}(T^*M) \backslash \{0\}) / \mathbb{Z}_2$. This bundle is trivial, i.e., $S_0 \cong M \times \mathbb{R}_+$.

- $S_0 \times S$ is a principal bundle over $M \times M$ with structure group $\mathbb{R}_+ \times \mathsf{SO}(3) \cong \mathbb{H}^\times / \mathbb{Z}_2$

Def. The principal bundle $\hat{M}$ is called the **Swann bundle** over M, and we denote by $\hat{\pi} : \hat{M} \to M$ the bundle projection. The Swann bundle is a principal $\mathbb{H}^\times / \mathbb{Z}_2$-bundle over M.

Goal: *Study the geometry on the total space of $\hat{\pi} : \hat{M} \to M$, induced by the pair (Q, ω) on M.*

➠ Our quaternionic skew-Hermitian connection $\nabla = \nabla^{Q,\omega}$ induces a principal connection on $\mathcal{Q}$ and a principal connection on S, with connection 1-forms

$$\gamma : T\mathcal{Q} \to \mathfrak{so}^*(2n) \oplus \mathfrak{sp}(1)\,, \qquad \theta : TS \to \mathfrak{sp}(1)\,,$$

respectively. We have $\theta = \Sigma_{a=1}^{3} \theta_a e_a$ and $\gamma = \gamma_+ + \gamma_-$, with γ_+ taking values in $\mathfrak{so}^*(2n)$ and $\gamma_- = \theta$ taking values in $\mathfrak{sp}(1)$.

• Consider the connection 1-form $\hat{\theta}$ on $\hat{M}$ with values in $\mathbb{H} \cong \mathbb{R} \oplus \mathfrak{sp}(1)$,

$$\hat{\theta} := \triangle_\sharp^* (\theta_0 \circ pr_{TS_0}, \theta \circ pr_{TS})$$

where $\triangle_\sharp : \hat{M} \to S_0 \times S$ is the canonical bundle morphism induced by $\triangle$.

Prop. (C.-Cortés-Gregorovič 2024) The Swann bundle $\hat{\pi} : \hat{M} \to M$ over a quaternionic-skew Hermitian manifold (M^{4n}, Q, ω), satisfies

$$\hat{M} \cong S \times_{\mathsf{SO}(3)} (\mathbb{R}_+ \times \mathsf{SO}(3)) \cong \mathbb{R}_+ \times S \cong \mathcal{Q} \times_{\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)} (\mathbb{H}^\times / \mathbb{Z}_2)\,.$$

Moreover, the connection 1-forms $\hat{\theta}$ and θ are such that

$$\hat{\theta} = t^{-1}\,\mathrm{d}\,t - \theta = \theta_0 - \sum_{a=1}^{3} \theta_a e_a\,. \quad (*)$$

The canonical hypercomplex structure on $\hat{M}$

Relatively to $\hat{\theta}$ and $\hat{\pi}$ we have the direct sum decomposition

$$T\hat{M} = \hat{\mathscr{V}} \oplus \hat{\mathscr{H}}\,, \quad \hat{\mathscr{V}} := \mathrm{d}\,\hat{\pi}\,, \quad \hat{\mathscr{H}} := \mathrm{Ker}\,\hat{\theta}\,.$$

• Denote by $\widetilde{U}$ the fundamental vector field induced by an element $U \in \mathbb{H} = \mathbb{R} \oplus \mathfrak{so}(3)$ in the Lie algebra of the structure group of $\hat{\pi}$, that is,

$$\widetilde{U}_u = \frac{\mathrm{d}}{\mathrm{d}\,t}\Big|_{t=0} \rho_{\exp tU}(u) \in T_u\hat{M}\,, \quad u = (m, (v, s)) \in \hat{M}\,, \qquad (\rho = \lambda_0 \times \lambda).$$

Set $Z_a := \tilde{e}_a$, for $a = 1, 2, 3$ and $Z_0 := \tilde{e}_0$, where as usual $\{e_1, e_2, e_3\}$ is the basis of $\mathfrak{so}(3) \cong \mathfrak{sp}(1) \cong \mathrm{Im}(\mathbb{H}) \cong \mathbb{R}^3$ and $e_0 = 1 \in \mathbb{R} \cong T_1\,\mathbb{R}_+$.

• We now introduce the triple $\hat{H} := (\hat{I}_1, \hat{I}_2, \hat{I}_3)$ consisting of the endomorphisms $\hat{I}_i : T\hat{M} \to T\hat{M}$

$$\begin{cases} \hat{I}_a Z_0 = -Z_a\,, \quad \hat{I}_a Z_a = Z_0\,, \quad \hat{I}_a Z_b = Z_c\,, \quad \hat{I}_a Z_c = -Z_b, \\ (\hat{I}_a)_u(Y) = (I_a(\hat{\pi}_* Y))^{\hat{h}}_u\,, \quad \forall\, Y \in \bar{\mathscr{H}}_u, \end{cases}$$

where $u = (m, (v, s)) \in \hat{M}$, $s = (I_1, I_2, I_3) \in S$, and the upperscript $\hat{h}$ denotes the horizontal lift with respect to $\hat{\theta}$.

Thm. (C.-Cortés-Gregorovič 2024) Let (M^{4n}, Q, ω) $(n > 1)$ be a quaternionic skew-Hermitian manifold and let $\hat{\pi} : \hat{M} \to M$ be the Swann bundle over M. Then, the almost hypercomplex structure $\hat{H}$ on $\hat{M}$ defined above is **1-integrable**.

Induced SO$^*(2(n+1))$-structures on the Swann bundle $\hat{M}$ over (M, Q, ω)

• **First step:** Description of $\hat{H}$-Hermitian 2-forms on the horizontal part of the decomposition $T\hat{M} = \hat{\mathscr{H}} \oplus \hat{\mathscr{V}}$. These are induced by the scalar 2-form ω downstairs...

Prop. (C.-Cortés-Gregorovič 2024) Let (M, Q, ω) and $\hat{\pi} : \hat{M} \to M$ be as above, and set $G := \mathbb{R}_+ \times \mathsf{SO}(3)$.
(1) The pullback

$$\hat{\omega} := \hat{\pi}^*(\omega) \in \Omega^2(\hat{M})$$

defines a horizontal, G-invariant, closed 2-form on $\hat{M}$, satisfying $\mathscr{L}_Z \hat{\omega} = 0$ for any vertical vector $Z \in \hat{\mathscr{V}}$.
(2) The restriction $\hat{\omega}_{\hat{\mathscr{H}}} := \hat{\omega}|_{\hat{\mathscr{H}} \times \hat{\mathscr{H}}}$ of $\hat{\omega}$ to the horizontal subspace $\hat{\mathscr{H}} = \mathrm{Ker}\, \hat{\theta} \subset T\hat{M}$ is non-degenerate, closed and $\hat{H}$-Hermitian.

• **Second step:** Description of $\hat{H}$-Hermitian 2-forms on the vertical part of the decomposition $T\hat{M} = \hat{\mathscr{H}} \oplus \hat{\mathscr{V}}$.

➟ We consider the $\mathbb{H}$-valued 1-form α on $\hat{M}$

$$\alpha := \frac{1}{t}\hat{\theta} = \frac{1}{t^2}\,\mathrm{d}\,t + \frac{1}{t}\theta \in \Omega^1(\hat{M};\mathbb{H})\,.$$

Obviously, we have

$$\alpha = \alpha_0 + \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$$

for some real-valued 1-forms α_a on $\hat{M}$. Since $\theta = \sum_{a=1}^{3}\theta_a e_a$ we get

$$\alpha_0 = \frac{1}{t^2}\,\mathrm{d}\,t\,,\quad \alpha_a = \frac{1}{t}\theta_a\,,\quad \forall\ a = 1,2,3\,.$$

Prop. (C.-Cortés-Gregorovič 2024) The space of all $\hat{H}$-Hermitian 2-forms β with $\beta|_{\bar{\mathscr{H}}\times\bar{\mathscr{H}}} = 0$ and $\beta|_{\bar{\mathscr{H}}\times\bar{\mathscr{V}}} = 0$, is 3-dimensional. It generated by the 2-forms

$$\beta_a = \alpha_0 \wedge \alpha_a + \alpha_b \wedge \alpha_c$$

for any cyclic permutation of (a,b,c) of $(1,2,3)$.

• Set

$$\tilde{\omega} := \hat{\omega} + \beta$$

where $\hat{\omega} = \hat{\pi}^*(\omega)$, $\beta = \Sigma_{a=1}^{3} f_a\beta_a$ for some smooth functions f_a on $\hat{M}$ with $f_a \neq 0$ for at least one a.

Thm. (C.-Cortés-Gregorovič 2024) If $(M, Q, \omega, \nabla^{Q,\omega})$ is non flat, i.e., $R^{Q,\omega} \neq 0$, then there is **no** non-degenerate $\hat{H}$-Hermitian 2-form β on the Swann bundle $\hat{M}$, i.e., scalar 2-form with respect to $\hat{H}$, such that $\beta|_{\hat{\mathscr{H}}\times\hat{\mathscr{H}}} = 0$, $\beta|_{\hat{\mathscr{H}}\times\hat{\mathscr{V}}} = 0$ and $\mathrm{d}\,\beta = 0$.

➟ Proof (Hints): We have $\Omega = \mathrm{d}\,\theta + \theta\wedge\theta$ as the part of the curvature with values in $\mathfrak{sp}(1)$ (and so the curvature induced by the connection 1-form θ).
• We have $\Omega = \Omega_1 e_1 + \Omega_2 e_2 + \Omega_3 e_3$ and $\beta = f_1\beta_1 + f_2\beta_2 + f_3\beta_3$ for some smooth functions f_i on $\hat{M}$.

$$\begin{aligned} \mathrm{d}\,\beta &= \sum_{a=1}^{3}(\mathrm{d}\,f_a\wedge\beta_a + f_a\,\mathrm{d}\,\beta_a) \\ &= \sum_{a=1}^{3}\mathrm{d}\,f_a\wedge\beta_a - \frac{1}{t}\alpha_0\wedge(f_1\Omega_1 + f_2\Omega_2 + f_3\Omega_3) + \frac{1}{t}\sum_{cycl} f_a(\Omega_b\wedge\alpha_c - \Omega_c\wedge\alpha_b)\,. \end{aligned}$$

From this relation we deduce that $\mathrm{d}\,\beta = 0$ if and only if the following conditions hold:

$$\sum_{a=1}^{3}(\mathrm{d}\,f_a\wedge\beta_a) = 0\,, \quad \sum_{a=1}^{3} f_a\Omega_a = 0\,,$$

$$f_2\Omega_3 - f_3\Omega_2 = 0\,, \quad -f_1\Omega_3 + f_3\Omega_1 = 0\,, \quad f_1\Omega_2 - f_2\Omega_1 = 0\,.$$

Corol. (C.-Cortés-Gregorovič 2024) The pair $(\hat{H}, \tilde{\omega})$ defines a $\mathrm{SO}^*(2(n+1))$-structure on the total space $\hat{M}$ of the Swann bundle over M, which is in general of **type** $\mathscr{X}_{3457}$. If $R^{Q,\omega} \neq 0$, then the $\mathrm{SO}^*(2(n+1))$-structure $(\hat{H}, \tilde{\omega})$ has nontrivial torsion in component $\mathscr{X}_{34}$, i.e., it is not of type $\mathscr{X}_{57}$.

References

- I. Chrysikos, V. Cortés, J. Gregorovič, *Curvature of quaternionic skew-Hermitian manifolds and bundle constructions*, Mathematische Nachrichten, (2024), 1–26.

- I. Chrysikos, J. Gregorovič, H. Winther: *Differential geometry of $SO^*(2n)$-type structures - integrability*, Analysis and Mathematical Physics, Vol 12 (93), (2022), 1–52.

- I. Chrysikos, J. Gregorovič, H. Winther: *Differential geometry of $SO^*(2n)$-type structures*, Annali di Matematica Pura ed Applicata (1923 -), (2022) 201:2603–2662.

Thank you for your attention!