Principal bundles and differential structures in noncommutative geometry

Srní, Czech Republic.

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Differential geometry dictionary

	Classical geometry	Quantum geometry
observables	smooth manifold M	associative unital algebra A
differential forms	cotangent bundle $\Lambda^\bullet T^*M$	DGA $\Omega^\bullet = \bigoplus_{k\geq 0} \Omega^k$
symmetry	Lie group action $M \times G \to M$	Hopf algebra coaction $\Delta_A \colon A \to A \otimes H$
principal bundle	$\pi \colon P \xrightarrow{\circlearrowright G} M$ $P \times G \xrightarrow{\cong} P \times_M P$	Quantum principal bundle

(Lost) Literature

We revisit the quantum principal bundle formalism of **Mićo Đurđević**, in particular

- Đurđević, M.: *Geometry of Quantum Principal Bundles II - Extended Version*. Rev. Math. Phys. **9**, 5 (1997) 531-607.
- Đurđević, M.: *Quantum Principal Bundles as Hopf-Galois Extensions*. Preprint arXiv:q-alg/9507022.
- Đurđević, M.: *Quantum Gauge Transformations and Braided Structure on Quantum Principal Bundles*. Preprint arXiv:q-alg/9605010.

A reworked version (including new non-trivial examples) is

- Del Donno, A., Latini, E. and Weber,T.: *On the Đurđević approach to quantum principal bundles*. Preprint arXiv:2404.07944.

1. Differential structures over algebras

2. Hopf–Galois extensions

3. Differential calculi on quantum principal bundles

Differential structures over algebras

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Definition

A first order differential calculus (FODC) (Γ, d) over A is the datum of:

1. an $A-$bimodule Γ;
2. a linear map $\mathrm{d} : A \to \Gamma$ satisfying the Leibniz rule $\mathrm{d}(ab) = (\mathrm{d}a)b + a\mathrm{d}b$ for every $a, b \in A$;
3. a surjectivity condition $\Gamma = A\mathrm{d}A$, i.e. $\Gamma = \mathrm{span}\{a\mathrm{d}b : a, b \in A\}$.

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Definition

A differential calculus on a $\mathbb{K}-$algebra A is a differential graded algebra $(\Omega^\bullet, \wedge, \mathrm{d})$ which is generated in degree zero and such that $\Omega^0 = A$. The former means that

$$\Omega^k = \mathrm{span}_{\mathbb{K}}\{a^0 \mathrm{d}a^1 \wedge \cdots \wedge \mathrm{d}a^k : a^0, \ldots, a^k \in A\}.$$

We call elements of Ω^k differential $k-$forms.

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Theorem

Let $(\Omega^\bullet, \widetilde{\wedge}, \widetilde{\mathrm{d}})$ be any differential calculus on A such that $\Omega^1 = \Gamma$ and $\widetilde{d}|_A = \mathrm{d}$. There exists a surjective morphism $\Gamma^{\wedge} \to \Omega^\bullet$ of differential graded algebras. In particular, $(\Omega^\bullet, \widetilde{\wedge}, \widetilde{\mathrm{d}})$ is a quotient of $(\Gamma^{\wedge}, \wedge, \mathrm{d})$. □

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A first order differential calculus (Γ, d) on a right $H-$comodule algebra (A, Δ_A) is called right $H-$covariant if Γ is a right $H-$covariant $A-$bimodule with right $H-$coaction $\Delta_\Gamma : \Gamma \to \Gamma \otimes H$ such that the differential $\mathrm{d} : A \to \Gamma$ is right $H-$colinear:

$$\Delta_\Gamma \circ \mathrm{d} = (\mathrm{d} \otimes \mathsf{id}) \circ \Delta_A.$$

Similarly for left $H-$covariant and $H-$bicovariant calculi.

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Theorem (Woronowicz)

There is a bijective correspondence

$$\{\textit{left/bi-covariant FODCi on } H\} \iff \{\textit{right ideals} \subseteq \ker \epsilon\}.$$

The calculus is bicovariant if and only if the corresponding ideal $I \subseteq \ker \epsilon$ is $\mathrm{Ad}-$invariant, where $\mathrm{Ad}(h) = h_2 \otimes S(h_1)h_3$. □

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- We call $B \subseteq A$ a Hopf–Galois extension if the *canonical map*
$$\chi : A \otimes_B A \to A \otimes H, \quad a \otimes_B a' \mapsto a\Delta_A(a')$$
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- Given $B \subseteq A$ a Hopf–Galois extension, the inverse of χ is known as the translation map
$$\tau : A \otimes H|_{1 \otimes H} = H \longrightarrow A \otimes_B A.$$

Faithfully flat Hopf–Galois extensions

Principal bundle

$$\begin{tikzcd} P \times G \arrow[r, "\triangleleft"] \arrow[d] & P \arrow[d, two heads] \\ P \times_X P & X = P/G \end{tikzcd}$$

Quantum principal bundle

$$\begin{tikzcd} A \arrow[r, dashed, "\Delta_A"] & A \otimes H \\ B = A^{coH} \arrow[u, hook] & A \otimes_B A \arrow[u, "\chi"'] \end{tikzcd}$$

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A quantum principal bundle is a (faithfully flat) Hopf-Galois extension.

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Definition

$\Omega^\bullet(A)$ is a complete differential calculus if the coaction $\Delta_A \colon A \to A \otimes H$ lifts to a morphism

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From now on we always consider complete differential calculi.

Vertical and horizontal forms: the Atiyah sequence

- The space (differential calculus) of vertical forms is defined as

$$\mathfrak{ver}^1(A) := A \otimes \Lambda,$$

where Λ is the space of left coinvariant forms of a given bicovariant FODC on the structure Hopf algebra H.

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Theorem

Let $B \subseteq A$ be a quantum principal bundle with Hopf algebra H, let Γ be a FODC over H and Λ be the corresponding differential graded subalgebra of coinvariant 1-forms of Γ. Let $\Omega^1(A)$ be a complete FODC over A. There is a short-exact sequence of $A-$modules given by

$$0 \longrightarrow \mathfrak{hor}^1(A) \xrightarrow{\iota} \Omega^1(A) \xrightarrow{\pi_{\mathfrak{ver}}} \mathfrak{ver}^1(A) \longrightarrow 0$$

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- The sequence splits by the choice of a connection, giving the total space as a direct sum of vertical and horizontal subspaces.

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$$\begin{tikzcd}
(A \otimes H) \otimes (A \otimes H) \arrow[r, "\mu_{A\otimes H}"] & A \otimes H \arrow[d, "\chi^{-1}"] \\
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- Multiplication on $A \otimes_B A$ is therefore given as

$$\mu_{A\otimes_B A}((a \otimes_B a') \otimes (b \otimes_B b')) = a\sigma(a' \otimes_B b)b',$$

where the Đurđević braiding $\sigma : A \otimes_B A \to A \otimes_B A$ reads

$$\sigma(a \otimes_B a') := a_0 a' \tau(a_1).$$

Complete calculi and graded Hopf–Galois extensions

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$$\chi^\bullet \colon \Omega^\bullet(A \otimes_B A) \to \Omega^\bullet(A) \otimes \Omega^\bullet(H)$$
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Theorem (DDA-Latini-Weber '24)

Let $\Omega^\bullet(A)$ be a complete calculus on QPB $B = A^{\mathrm{co}H} \subseteq A$. We have a **graded Hopf–Galois extension**.

$$\Omega^\bullet(B) = \Omega^\bullet(A)^{co(\Gamma^\wedge)} \subseteq \Omega^\bullet(A)$$

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- The braiding extends to differential forms via

$$\sigma : \Omega^\bullet(A) \otimes_B \Omega^\bullet(A) \to \Omega^\bullet(A) \otimes_B \Omega^\bullet(A)$$
$$\omega \otimes_{\Omega^\bullet(B)} \eta \mapsto (-1)^{|\omega_{[1]}||\eta|} \omega_{[0]} \wedge \eta \wedge \tau^\bullet(\omega_{[1]})$$

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$$\beta\alpha = q\alpha\beta, \quad \gamma\alpha = q\alpha\gamma, \quad \delta\beta = q\beta\delta, \quad \delta\gamma = q\gamma\delta$$
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- $A = \mathcal{O}_q(\mathrm{SU}(2))$ is a right $H-$comodule algebra under the right $H-$coaction

$$\Delta_A : A \to A \otimes H,$$
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The quantum Hopf fibration provided by the above data is a faithfully flat Hopf-Galois extension, i.e. $B \subseteq A$ is a QPB.

Define

$$\Omega^\bullet(A) = A \oplus \underbrace{\Omega^1(A)}_{\mathrm{span}_A\{e^\pm, e^0\}} \oplus \underbrace{\Omega^2(A)}_{\mathrm{span}_A\{e^\pm\wedge e^0, e^+\wedge e^-\}} \oplus \underbrace{\Omega^3(A)}_{\mathrm{span}_A\{e^+\wedge e^-\wedge e^0\}},$$

$$\Omega^\bullet(H) = H \oplus \Omega^1(H), \quad \text{with} \quad \mathrm{d}t \cdot t = q^2 t \cdot \mathrm{d}t.$$

Differential structures over the quantum Hopf fibration

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Theorem (DDA-Latini-Weber '24)

- *The differential calculus $\Omega^\bullet(A) = A \oplus \Omega^1(A) \oplus \Omega^2(A) \oplus \Omega^3(A)$ is complete.*

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- *The braiding $\sigma : \Omega^\bullet(A) \otimes_B \Omega^\bullet(A) \longrightarrow \Omega^\bullet(A) \otimes_B \Omega^\bullet(A)$ is symmetric only on $A \otimes_B A$.*

$\square$

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Thank you for your attention!