Structure of geodesics for Finsler metrics arising from Riemannian g.o. metrics

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Homogeneous spaces

Let (M, F) be a Finsler manifold,
$G \subset I_0(M)$ be a transitive group of isometries
- *homogeneous Finsler manifold*

Let $p \in M$ be a fixed point, H be the isotropy group at p
- *homogeneous space* $(G/H, F)$

Let $(G/H, F)$ be fixed homogeneous space,
$\mathfrak{g}, \mathfrak{h}$ the Lie algebras of G, H,
$\mathfrak{m}$ a vector space such that $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ and $\mathrm{Ad}(H)\mathfrak{m} \subset \mathfrak{m}$
- *reductive decomposition*

Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ be a fixed reductive decomposition
- the natural identification of $\mathfrak{m} \subset \mathfrak{g}$ and T_pM
(via the natural projection $\pi : G \to G/H$)
- $\mathrm{Ad}(H)$-invariant Minkowski norm F on $\mathfrak{m}$

α_i-type metrics

Definition

Let $(G/H, F)$ be a homogeneous Finsler space with a reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$. Consider the adjoint action of H on $\mathfrak{m}$, which leads to the irreducible decomposition $\mathfrak{m} = \oplus_{i=1}^{s} \mathfrak{m}_i$. Choose symmetric positively definite $\mathrm{Ad}(H)$-invariant bilinear forms α_i on $\mathfrak{m}_i, i = 1 \dots s$ and let y_i be the corresponding projections of a vector $y \in \mathfrak{m}$ onto $\mathfrak{m}_i$.
The Minkowski norm F on $\mathfrak{m}$ and the corresponding homogeneous Finsler metric on G/H is of the α_i-type if there exist a smooth function $f\colon [0,\infty)^s \to \mathbb{R}$ such that

$$F(y) \quad = \quad f(\alpha_1(y_1), \dots, \alpha_s(y_s)), \qquad y \in \mathfrak{m}.$$

- (α_1, α_2)-metrics,
- f-product

Another construction of new Finsler metrics

M.A. Javaloyes and M. Sánchez considered metrics of the type

$$F(y) = \sqrt{L(F_1(y), \ldots, F_k(y), \beta_1(y), \ldots, \beta_l(y))},$$

where F_i are Finsler metrics and β_j are one-forms on M.
The continuous function L must satisfy:
(i) be smooth and positive away from 0,
(ii) be positively homogeneous of degree 2,
(iii) $L_{,i} \geq 0$, for $i = 1, \ldots, k$,
(iv) $Hess(L)$ be positive semi-definite,
(v) $L_{,1} + \cdots + L_{,k} > 0$.
There are many functions which satisfy these conditions,
for example, one can choose $L = \sqrt{g_1} + \cdots + \sqrt{g_k}$.

New Finsler g.o. metrics

We consider homogeneous Finsler metrics of the type

$$F(y) \;=\; \sqrt{L(\sqrt{g_1(y)}, \ldots, \sqrt{g_k(y)})}.$$

We focus on the natural family of <u>positively related</u> initial Riemannian <u>g.o. metrics</u>.
It can be observed that these metrics are particular examples of the α_i-type metrics.
We study the g.o. property of the metric F.

Homogeneous geodesics

Definition

A geodesic γ *in* $(G/H, F)$ *is homogeneous if there is a vector* $X \in \mathfrak{g}$ *such that* $\gamma(t) = \exp(tX)(p)$.
The vector X *is called geodesic vector.*

Definition

A homogeneous space $(G/H, F)$ *is called a Finsler g.o. space, if each geodesic of* $(G/H, F)$ *is homogeneous.*

Definition

Let $(G/H, F)$ *be a g.o. space and* $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ *an* $\mathrm{Ad}(H)$*-invariant decomposition of the Lie algebra* $\mathfrak{g}$.
A geodesic graph is an $\mathrm{Ad}(H)$*-equivariant map* $\xi \colon \mathfrak{m} \to \mathfrak{h}$ *such that* $X + \xi(X)$ *is a geodesic vector for each* $o \neq X \in \mathfrak{m}$.

Positively related homogeneous metrics

Definition

Let G/H be a homogeneous space with a reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$. Consider the $\mathrm{Ad}(H)$-invariant irreducible decomposition $\mathfrak{m} = \oplus_{i=1}^{s} \mathfrak{m}_i$ and let α_i be $\mathrm{Ad}(H)$-invariant scalar products on the respective spaces $\mathfrak{m}_i$. We consider the family of scalar products

$$g(c_1, \dots, c_s) = \sum_{i=1}^{s} c_i \cdot \alpha_i,$$

for any numbers $0 < c_i \in \mathbb{R}$.
This family of scalar products on $\mathfrak{m}$ and corresponding family of Riemannian metrics on G/H will be called scalar products positively related and metrics positively related.

- The new construction of Finsler metrics F above using positively related metrics gives particular α_i-type metrics.

Positively related g.o. metrics

Conjecture

Consider a family of positively related Riemannian metrics.
If one metric of this family is a g.o. metric,
then all metrics from this family are also g.o. metrics.

Geodesic lemma

Lemma (D. Latifi)

Let $(G/H, F)$ be a homogeneous Finsler space,
$\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ be a reductive decomposition
and g the fundamental tensor on $\mathfrak{m}$ ($g_y(u,v) = \frac{1}{2}\frac{\partial^2 F^2(y+su+tv)}{\partial s \partial t}$).
The vector $Y \in \mathfrak{g}$ is geodesic if and only if it holds

$$g_{Y_\mathfrak{m}}(Y_\mathfrak{m}, [Y, U]_\mathfrak{m}) = 0, \qquad \forall U \in \mathfrak{m}.$$

Fundamental tensor of F

Lemma

Let $g_1, \ldots, g_k$ be homogeneous Riemannian metrics on G/H and let $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ be a reductive decomposition.
Let $F = \sqrt{L(\sqrt{g_1}, \ldots, \sqrt{g_k})}$ on $\mathfrak{m}$, which gives a homogeneous Finsler metric on G/H.
For arbitrary vectors $y, v \in \mathfrak{m}$, the fundamental tensor g of F satisfies the formula

$$g_y(y, v) \;=\; \sum_{j=1}^{k} B_j(y) \cdot g_j(y, v),$$

where the functions $B_j(y)$ are given by

$$B_j(y) \;=\; \frac{L_{,j}}{2\sqrt{g_j(y, y)}} = \frac{L_{,j}}{2\,F_j(y)}, \quad j = 1 \ldots k.$$

Fundamental tensor of F for positively related initial metrics

Let $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ be a reductive decomposition
with the $\mathrm{Ad}(H)$-invariant irreducible decomposition $\mathfrak{m} = \oplus_{i=1}^{s} \mathfrak{m}_i$.
Scalar products g_j have corresponding decompositions

$$g_j = \sum_{i=1}^{s} a_{ji} \cdot \alpha_i, \qquad j = 1 \dots k,$$

where $0 < a_{ji} \in \mathbb{R}$ and α_i are some initial $\mathrm{Ad}(H)$-invariant scalar products on $\mathfrak{m}_i$.

$$\begin{aligned} g_y(y,v) &= \sum_{j=1}^{k} B_j(y) \cdot g_j(y,v) \\ &= \sum_{j=1}^{k} B_j(y) \cdot \sum_{i=1}^{s} a_{ji} \cdot \alpha_i(y,v) = \sum_{i=1}^{s} C_i(y) \cdot \alpha_i(y,v). \end{aligned}$$

Geodesic lemma

Lemma

Let G/H be a homogeneous space with a reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ and the $\mathrm{Ad}(H)$-irreducible decomposition $\mathfrak{m} = \oplus_{i=1}^{s} \mathfrak{m}_i$.
Let g_j be positively related Riemannian metrics
and let $F = \sqrt{L(\sqrt{g_1}, \sqrt{g_2}, \dots, \sqrt{g_k})}$.
The vector $y + \xi(y)$, where $y \in \mathfrak{m}$ and $\xi(y) \in \mathfrak{h}$,
is a geodesic vector for the Finsler metric F if and only if

$$\sum_{i=1}^{s} C_i(y) \cdot \alpha_i\Big(y, [y + \xi(y), U]_{\mathfrak{m}}\Big) \quad = \quad 0, \quad \forall U \in \mathfrak{m}.$$

Main theorem

Theorem

Let G/H be a homogeneous space with a reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ and the $\mathrm{Ad}(H)$-irreducible decomposition $\mathfrak{m} = \oplus_{i=1}^{s} \mathfrak{m}_i$. Let $g_j, j = 1 \dots k$, be from a family of positively related Riemannian metrics on G/H, all of which are g.o. metrics. Then any homogeneous Finsler metric of the type $F = \sqrt{L(\sqrt{g_1}, \sqrt{g_2}, \dots, \sqrt{g_k})}$ on G/H is also a g.o. metric.

Proof. Consider geodesic lemma and an arbitrary fixed vector $y \in \mathfrak{m}$. The values $C_i(y)$ are positive real numbers and the Riemannian metric

$$C_1(y) \cdot \alpha_1 + \dots + C_s(y) \cdot \alpha_s$$

belongs to the family of initial positively related g.o. metrics. Hence there exist a vector $\xi(y)$ which satisfies geodesic lemma. Because $y \in \mathfrak{m}$ was arbitrary, the metric F is also a g.o. metric. □

Example $S^7 = \mathrm{Sp}(2) \cdot \mathrm{U}(1)/\mathrm{Sp}(1) \cdot \mathrm{diag}(\mathrm{U}(1))$

$$\mathfrak{h} = \mathfrak{sp}(1) : \begin{pmatrix} ih_1 & -h_2 - ih_3 & 0 & 0 \\ h_2 - ih_3 & -ih_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

For $\mathfrak{sp}(2) = \mathfrak{sp}(1) + \mathfrak{m}$,

$$\mathfrak{m} : \begin{pmatrix} 0 & 0 & x_1 + ix_2 & -x_3 - ix_4 \\ 0 & 0 & x_3 - ix_4 & x_1 - ix_2 \\ -x_1 + ix_2 & -x_3 - ix_4 & iz_1 & -z_2 - iz_3 \\ x_3 - ix_4 & -x_1 - ix_2 & z_2 - iz_3 & -iz_1 \end{pmatrix}.$$

We extend $\mathfrak{h}$ by one more operator.

Example

$$
\begin{aligned}
y &= x_1 X_1 + \cdots + x_4 X_4 + z_1 Z_1 + z_2 Z_2 + z_3 Z_3, \\
\xi(y) &= \xi_1(y) H_1 + \xi_2(y) H_2 + \xi_3(y) H_3 + \xi_4(y) W.
\end{aligned}
$$

$$
\left(\begin{array}{cccc|c}
x_2 & x_3 & x_4 & -x_2 & (1-2\frac{C_2}{C_1}) z_1 x_2 + (1-2\frac{C_3}{C_1})(z_2 x_3 + z_3 x_4) \\
-x_1 & -x_4 & x_3 & x_1 & -(1-2\frac{C_2}{C_1}) z_1 x_1 + (1-2\frac{C_3}{C_1})(z_2 x_4 - z_3 x_3) \\
x_4 & -x_1 & -x_2 & x_4 & -(1-2\frac{C_2}{C_1}) z_1 x_4 + (1-2\frac{C_3}{C_1})(-z_2 x_1 + z_3 x_2) \\
-x_3 & x_2 & -x_1 & -x_3 & (1-2\frac{C_2}{C_1}) z_1 x_3 - (1-2\frac{C_3}{C_1})(z_2 x_2 + z_3 x_1) \\
0 & 0 & 0 & 2z_3 & 2 z_1 z_3 (\frac{C_2}{C_3} - 1) \\
0 & 0 & 0 & -2z_2 & 2 z_1 z_2 (1 - \frac{C_2}{C_3})
\end{array}\right).
$$

Example

$$K_1 = K_1(y) = \frac{C_2}{C_3} - 2\frac{C_2}{C_1}, \quad K_2 = K_2(y) = 1 - 2\frac{C_3}{C_1}, \quad K_3 = K_3(y) = \frac{C_2}{C_3} - 1.$$

$$\begin{aligned}
\xi_1 &= \frac{K_1 z_1(x_1^2 + x_2^2 - x_3^2 - x_4^2) + 2K_2\left[z_2(x_2x_3 - x_1x_4) + z_3(x_1x_3 + x_2x_4)\right]}{x_1^2 + x_2^2 + x_3^2 + x_4^2},\\
\xi_2 &= \frac{2K_1 z_1(x_2x_3 + x_1x_4) + K_2\left[z_2(x_1^2 - x_2^2 + x_3^2 - x_4^2) + 2z_3(x_3x_4 - x_1x_2)\right]}{x_1^2 + x_2^2 + x_3^2 + x_4^2},\\
\xi_3 &= \frac{2K_1 z_1(x_2x_4 - x_1x_3) + K_2\left[2z_2(x_1x_2 + x_3x_4) + z_3(x_1^2 - x_2^2 - x_3^2 + x_4^2)\right]}{x_1^2 + x_2^2 + x_3^2 + x_4^2},\\
\xi_4 &= K_3 z_1.
\end{aligned}$$

We observe that, if we put $C_i(y) = c_i > 0$,
we obtain formulas for the geodesic graph
of the Riemannian metric $g(c_1, c_2, c_3) = \sum_{i=1}^{3} c_i \alpha_i$.