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Coarse geometric approach to topological phases of matter

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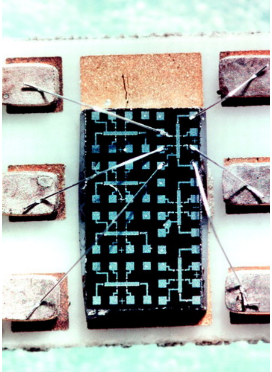


Figure: K. v. Klitzing 2005

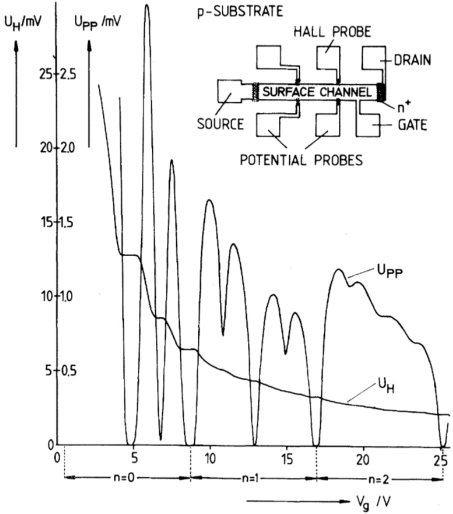


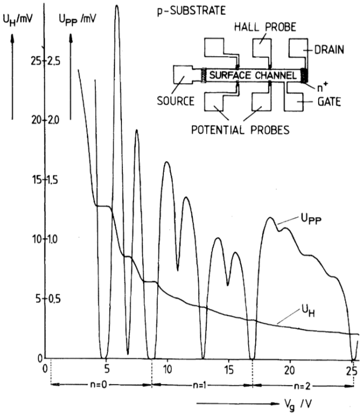
Figure: K. v. Klitzing, Dorda, and Pepper 1980

- U_{pp} vanishes periodically

$\Rightarrow$ Insulating when μ between 2 Landau levels

- U_H has plateaus for the same values of V_g

$$\rho_{xy} = \frac{U_H}{I_y} = \frac{2\pi\hbar}{e^2}\frac{1}{n}$$



Thouless et al. 1982 (TKNN)

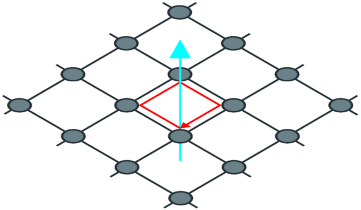
- Square lattice with 2-d non-interacting electron gas and perpendicular magnetic field

$\Rightarrow$ By Bloch-Theorem $\psi_{\mathbf{k}}(\mathbf{x}) = e^{i\mathbf{k}\cdot\mathbf{x}}\, u_{\mathbf{k}}(\mathbf{x}),\ u_{\mathbf{k}}(x)$ periodic

$\Rightarrow$ $k \in \mathbb{R}^2/\mathbb{Z}^2 \cong \mathbb{T}^2$ Brillouin Torus

- Assume Bloch states have band gaps and Fermi energy E_F is in a gap

$$\rho_{xy}^{-1} = \sigma_{xy} = \frac{e^2}{2\pi\hbar} \sum_{\alpha \in \text{occupied Bands}} \mathsf{Ch}_\alpha$$



Freed and Moore 2012, TKNN $\Rightarrow$ Chern number of a vector bundle

$$V_E := \mathsf{span}_{\mathbb{C}}\{\text{Hamiltonian Eigenstates} < E_F\} \longrightarrow \mathbb{T}^2$$

Topological phase

A topological phase is a system mathematically described by a non-trivial topological invariant

- For IQHE non-triviality of $V_E \in K^0(\mathbb{T}^2)$ is fundamental (TKNN)
- Implementing disordered systems we need C^*-algebras Bellissard, Elst, and Schulz- Baldes 1994
- Classification topologial insulators via K-theory by Kitaev 2009
- Other advances using C^*-algs and K-theory:
  - Twisted equivariant K-Theory Freed and Moore 2012
  - Groupoid-C^*-algebras Bourne and Prodan 2017
  - Coarse Geometry Ewert and Meyer 2018, Ludewig 2023 $\rightarrow$ we look at this

Definition

- $\mathcal{H}$ Hilbert space
- H Hamiltonian, i.e. self-adjoint Operator
- $\mathcal{A} \subset B(\mathcal{H})$ observable algebra, s.t. $f(H) \in \mathcal{A}$ for any $f \in \mathscr{C}_c(\mathbb{R})$

H is an Insulator at energy $E \in \mathbb{R}$ if $E \notin \sigma(H)$ with $p_E := 1_{\mathbb{R}_{\leq E}}(H)$ the spectral projection.
H is a *topological insulator* if $[p_E] \in K_0(\mathcal{A})$ non-trivial.

Definition (Coarse structure, Roe 1993)

Set of *entourages* $\mathcal{C} \subset 2^X \times 2^X$ is a *coarse structure* if

- $\mathsf{diag}(X) \in \mathcal{C}$
- $\mathcal{C}$ is closed under finite unions and taking subsets
- If $U, V \in \mathcal{C}$, then $V \circ U \in \mathcal{C}$ for

$$V \circ U := \{(a, c) \mid \exists b \in X \text{ with } (a, b) \in U, (b, c) \in V\}$$

- If $U \in \mathcal{C}$, then $U^{-1} \in \mathcal{C}$ for

$$U^{-1} := \{(b, a) \mid (a, b) \in U\}$$

Example

Let (X, d) be a metric space. Then define coarse structure $\mathcal{C}_d := \{U_R\}_{R \geq 0}$ with

$$U_R := \{(x, y) \mid d(x, y) \leq R\}$$

Remark

- $(X, d_X, \mathcal{C}_{d_X}) \sim (Y, d_Y, \mathcal{C}_{d_Y})$ *if* $\exists f \colon X \leftrightarrow Y \colon g$ *such that* $B_R(g \circ f(X)) = X$ *and* $B_{R'}(f \circ g(Y)) = Y$ *for* $R, R' > 0$.
- *Notably* $\mathbb{R}^d \sim \mathbb{Z}^d$.

Definition (Coarse cohomology, Roe 1993)

Cohomology theory $HX^{\bullet}(X)$ for proper metric $(M, d, \mathcal{C}_d)$.

$$CX^q(M) := \{f \in \mathscr{C}_{\text{lbb}}(M^{q+1}, \mathbb{R}) \mid \mathsf{supp}(f) \cap B_R(\mathsf{diag}(M^{q+1})) \text{ relative cpt. } \forall R \geq 0\}$$

Boundary maps are Alexander-Spanier boundaries.

Definition (Roe Algebra, Roe 1993)

- $(X, d, \mathcal{C}_d)$ coarse space, $\mathcal{H}$ Hilbert
- Action $\triangleright : \mathscr{C}_0(X) \times \mathcal{H} \to \mathcal{H}$ s.t. $\triangleright(f, -)$ compact iff $f = 0$ ($\mathcal{H}$ is ample X-module).
- $A \in B(\mathcal{H})$ locally compact if $1_K A$ and $A 1_K$ are compact for K bounded
- $A \in B(\mathcal{H})$ finite propagation if there is $R > 0$ s.t. $1_U A 1_V = 0$ whenever $d(U, V) > R$

$\Rightarrow$ Define Roe-Algebra $\mathcal{C}^*_{\text{Roe}}(X) = \{A \mid A \text{ locally compact, finite propagation}\}^{\text{cl}} \subset B(\mathcal{H})$

Proposition

- *$\mathcal{C}^*_{Roe}(X)$ independent of choice of ample $\mathcal{H}$.*
- *If $X \sim Y$ then $HX^{\bullet}(X) \cong HX^{\bullet}(Y)$ and $\mathcal{C}^*_{Roe}(X) \cong \mathcal{C}^*_{Roe}(Y)$*

$\Rightarrow$ *$HX^{\bullet}(\mathbb{R}^d) \cong HX^{\bullet}(\mathbb{Z}^d)$ and $\mathcal{C}^*_{Roe}(\mathbb{R}^d) \cong \mathcal{C}^*_{Roe}(\mathbb{Z}^d)$*

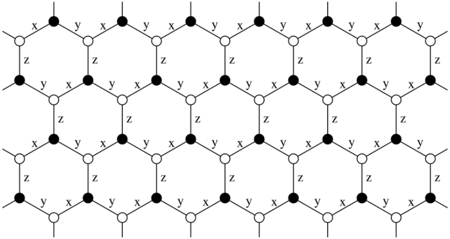
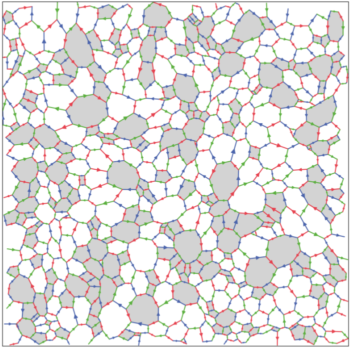


Figure: Kitaev 2006



$$H = -J_x \sum_{x\text{-links}} \sigma_j^x \sigma_k^x - J_y \sum_{y\text{-links}} \sigma_j^y \sigma_k^y - J_z \sum_{z\text{-links}} \sigma_j^z \sigma_k^z,$$

→ See Cassella et al. 2023

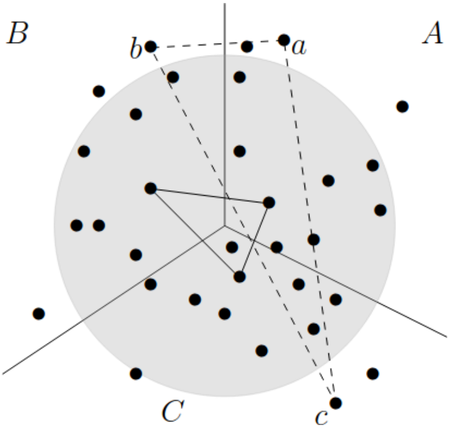
Here $P = \{P_{ij}\}_{i,j\in\mathbb{Z}}$ is the fermi projection.

$$\begin{aligned}\mathrm{Ch} =& \langle A, B, C; P\rangle \\ =& 3\sum_{i\in A}\sum_{j\in B}\sum_{k\in C}(P_{ij}P_{jk}P_{ki} - P_{ik}P_{kj}P_{ji})\end{aligned}$$

- $M \subset \mathbb{R}^d$ discrete, partitioned into $A, B, C \subset M$
- H generic tight-binding Hamiltonian, i.e. H self-adjoint on $L^2(M) \otimes \mathbb{C}^n$.

$\Rightarrow$ Finite propagation is very physical

- For the Fermi projection $P \in \mathcal{A} := \mathcal{C}^*_{\mathrm{Roe}}(M) \otimes M_n(\mathbb{C})$
- $\phi_{A,B,C}(x_1, x_2, x_3) = \sum_{\sigma \in S_3} (-1)^\sigma 1_A(x_{\sigma_0}) 1_B(x_{\sigma_1}) 1_C(x_{\sigma_3})$ is antisymmetric cochain in $CX^2(M)$



Theorem (Ludewig and Thiang 2023)

The chern number $\langle A, B, C; P\rangle$ is given as a pairing of $[P] \in K_0(\mathcal{A})$ with $\chi[\phi_{A,B,C}]$ which is Roe's Connes Character map $\chi\colon HX^{2n}_{anti}(M) \to HC^{2n}(M)$ applied to $[\phi_{A,B,C}]$.

See also Roe 1993, Sec. 4.

- Take-Aways:
  - $\mathcal{C}^*_{\mathrm{Roe}}(M)$ yields a canonical observable algebra on a $\mathcal{C}_0(M)$-module.
  - Manifestly invariant under disorder, as same Roe-Algebra.
  - Many known invariants arise as pairing with Coarse Cohomology.

- My goals?
  - Learn Indextheory
  - Look at concrete constructions of tight binding Hamiltonians in amorphous solids, maybe find some more relations to Coarse Geometry

  ⇒ Correlation length decay can be described by Coarse structure (Elokl and Jones 2024)
  - Understand Systems without Band-Gap but only with mobility gap.

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